A REMARK ON TORIC FOLIATIONS

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Abstract. If a toric foliation on a projective $\mathbb{Q}$-factorial toric variety has an extremal ray whose length is longer than the rank of the foliation, then the associated extremal contraction is a projective space bundle and the foliation is the relative tangent sheaf of the extremal contraction.

Contents

1. Introduction

Let us start with the definition of *foliations* on normal algebraic varieties.

Definition 1.1 (Foliations and toric foliations)**.** A *foliation* on a normal algebraic variety X is a nonzero saturated subsheaf $\mathscr{F} \subset \mathscr{T}_X$ that is closed under the Lie bracket, where $\mathscr{T}_X$ is the tangent sheaf of X. We note that the *rank* of the foliation $\mathscr{F}$ means the rank of the coherent sheaf $\mathscr{F}$.

We further assume that X is toric. Then a foliation $\mathscr{F}$ on X is called *toric* if the sheaf $\mathscr{F}$ is torus equivariant.

The following result on toric foliations is a starting point of this paper.

Theorem 1.2 (see [P])**.** *Let $X = X(\Sigma)$ be a $\mathbb{Q}$-factorial toric variety with its fan Σ in the lattice $N \simeq \mathbb{Z}^n$. Then there exists a one-to-one correspondence between the set of toric foliations on X and the set of complex vector subspaces $V \subset N_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^n$.*

Let $\mathscr{F}_V$ be the toric foliation associated to a complex vector subspace $V \subset N_{\mathbb{C}}$ (here, we should remark that the rank of $\mathscr{F}_V$ is $\dim_{\mathbb{C}} V$). Then

$$K_{\mathscr{F}_V} := -c_1(\mathscr{F}_V) = -\sum_{\rho \subset V} D_\rho$$

holds, that is, the first Chern class of $\mathscr{F}_V$ is $\sum_{\rho \subset V} D_\rho$, where D_ρ is the torus invariant prime divisor corresponding to the one-dimensional cone ρ in Σ. In particular, we have

$$K_{\mathscr{F}_V} = K_X + \sum_{\rho \not\subset V} D_\rho.$$

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For the basics of toric foliations, see also [CC] and [W]. By [FjS], we see that we can run the minimal model program with respect to $K_{\mathscr{F}}$ for any foliation $\mathscr{F}$ on a projective $\mathbb{Q}$-factorial toric variety X. For more details on the toric foliated minimal model program, see [CC] and [W]. In this paper, we establish:

Theorem 1.3 (Main Theorem)**.** *Let X be a projective $\mathbb{Q}$-factorial toric variety and let $\mathscr{F}$ be a toric foliation of rank r on X. Then*

$$l_{\mathscr{F}}(R) := \min_{[C]\in R}\{-K_{\mathscr{F}}\cdot C\} \leq r+1$$

holds for every extremal ray R of $\overline{\mathrm{NE}}(X)=\mathrm{NE}(X)$. Moreover, if $l_{\mathscr{F}}(R) > r$ holds for some extremal ray R of $\mathrm{NE}(X)$, then the contraction morphism $\varphi_R\colon X\to Y$ associated to R is a $\mathbb{P}^r$-bundle over Y. In this case, $\mathscr{F}=\mathscr{T}_{X/Y}$ holds, where $\mathscr{T}_{X/Y}$ is the relative tangent sheaf of $\varphi_R\colon X\to Y$. In particular, $\mathscr{F}$ is locally free.

We note that we call $l_{\mathscr{F}}(R)$ the *length* of an extremal ray R with respect to the foliation $\mathscr{F}$. We will use Reid's description of the toric extremal contraction morphisms in [R] (see also [M, Chapter 14]) for the proof of Theorem 1.3. This paper can be seen as a continuation of [Fj1] (see also [Fj2]).

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2. Preliminaries on toric varieties

Let $N\simeq\mathbb{Z}^n$ be a lattice of rank n. A toric variety $X(\Sigma)$ is associated to a *fan* Σ, a collection of convex cones $\sigma\subset N_{\mathbb{R}}:=N\otimes_{\mathbb{Z}}\mathbb{R}$ satisfying:

- Each convex cone σ is a rational polyhedral cone in the sense that there are finitely many $n_1,\ldots,n_s\in N\subset N_{\mathbb{R}}$ such that
$$\sigma=\{r_1n_1+\cdots+r_sn_s;\ r_i\geq 0\}=:\langle n_1,\ldots,n_s\rangle,$$
and it is strongly convex in the sense that
$$\sigma\cap -\sigma=\{0\}.$$
- Each face τ of a convex cone $\sigma\in\Sigma$ is again an element in Σ.
- The intersection of two cones in Σ is a face of each.

The *dimension* $\dim\sigma$ of a cone σ is the dimension of the linear space $\mathbb{R}\sigma=\sigma+(-\sigma)$ spanned by σ. We define the sublattice N_σ of N generated (as a subgroup) by $\sigma\cap N$ as follows:

$$N_\sigma:=\sigma\cap N+(-\sigma\cap N).$$

If σ is a k-dimensional simplicial cone, and $v_1,\ldots,v_k$ are the first lattice points along the edges of σ, then $\sigma=\langle v_1,\ldots,v_k\rangle$ holds. The *multiplicity* of σ is defined to be the *index* of the lattice generated by the $\{v_1,\ldots,v_k\}$ in the lattice N_σ;

$$\mathrm{mult}(\sigma):=[N_\sigma:\mathbb{Z}v_1+\cdots+\mathbb{Z}v_k].$$

We note that the affine toric variety $X(\sigma)$ associated to the cone σ is smooth if and only if $\mathrm{mult}(\sigma)=1$. We also note that a toric variety $X(\Sigma)$ is $\mathbb{Q}$-factorial if and only if each cone $\sigma\in\Sigma$ is simplicial (see e.g. [M, Lemma 14-1-1]).

The *star* of a cone $\tau \in \Sigma$ can be defined abstractly as the set of cones σ in Σ that contain τ as a face. Such cones σ are determined by their images in $N(\tau) := N/N_\tau$, that is, by

$$\overline{\sigma} := (\sigma + (N_\tau)_{\mathbb{R}})\,/(N_\tau)_{\mathbb{R}} \subset N(\tau)_{\mathbb{R}}.$$

These cones $\{\overline{\sigma}; \tau \prec \sigma\}$ form a fan in $N(\tau)$, and we denote this fan by $\mathrm{Star}(\tau)$. We set $V(\tau) = X(\mathrm{Star}(\tau))$, that is, the toric variety associated to the fan $\mathrm{Star}(\tau)$. It is well known that $V(\tau)$ is an $(n-k)$-dimensional closed toric subvariety of $X(\Sigma)$, where $\dim \tau = k$. If $\dim V(\tau) = 1$ (resp. $n-1$), then we call $V(\tau)$ a *torus invariant curve* (resp. *torus invariant divisor*). For the details about the correspondence between τ and $V(\tau)$, see [Fl, 3.1 Orbits].

3. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3.

Proof of Theorem 1.3. We assume that the toric variety X is associated to a fan Σ, which is a collection of convex cones in $N \simeq \mathbb{Z}^n$ as explained in Section 2. In particular, $\dim X = n$. It is well known that every extremal ray of $\overline{\mathrm{NE}}(X) = \mathrm{NE}(X)$ is spanned by a torus invariant curve (see e.g. [M, Theorem 14-1-4]). Let R be an extremal ray of $\mathrm{NE}(X)$. If $l_{\mathscr{F}}(R) \le r$ holds, then there is nothing to prove. Therefore, we assume that $-K_{\mathscr{F}} \cdot C > r$ holds for every torus invariant curve C with $[C] \in R$. We further assume that C corresponds to an $(n-1)$-dimensional cone $W = \langle v_1, \dots, v_{n-1}\rangle \in \Sigma$, where $v_1, \dots, v_{n-1}$ are primitive vectors. Let $v_n, v_{n+1} \in N$ be the two primitive vectors such that they together with W generate the two n-dimensional cones $\sigma, \sigma' \in \Sigma$, respectively. As usual, we can write

$$a_1 v_1 + \cdots + a_{n-1}v_{n-1} + a_n v_n + a_{n+1} v_{n+1} = 0 \tag{3.1}$$

such that a_i is an integer for every i with $\gcd(a_1, \dots, a_{n+1}) = 1$ and $a_n, a_{n+1} > 0$. We should remark that for a 1-dimensional cone $\langle v\rangle \in \Sigma$, where $v \in N$ is a primitive vector, we have the following formula for the intersection number of $D_v := V(\langle v \rangle)$ with C (see e.g. [CLS, Proposition 6.4.4]):

$$D_v \cdot C = \begin{cases} 0 & \cdots \quad v \notin \{v_1, \dots, v_{n+1}\} \\ \dfrac{a_i \mathrm{mult}(W)}{a_n \mathrm{mult}(\sigma)} & \cdots \quad v = v_i \text{ for } 1 \le i \le n \\ \dfrac{\mathrm{mult}(W)}{\mathrm{mult}(\sigma')} & \cdots \quad v = v_{n+1} \end{cases}$$

In this setting, [M, Proposition 14-1-5 (i)] says that for $1 \le i \le n-1$ with $a_i > 0$, we have

$$\langle \{v_1, \dots, v_n\} \setminus \{v_i\}\rangle \in \Sigma$$

and

$$[V(\langle \{v_1, \dots, v_n\} \setminus \{v_i\}\rangle)] \in R.$$

Thus, we may assume that

$$a_1 \le \cdots \le a_n \le a_{n+1}$$

by changing the order. In particular, the above formula tells us that $D \cdot C \le 1$ for any torus invariant divisor D on X. Since we have

$$-K_{\mathscr{F}} \cdot C = \sum_{v_i \in V} V(\langle v_i \rangle) \cdot C > r,$$

we obtain $1 \leq i_1 < i_2 < \cdots < i_r < i_{r+1} \leq n+1$ such that

$$v_{i_1}, v_{i_2}, \dots, v_{i_r}, v_{i_{r+1}} \in V.$$

Since the rank of $\mathscr{F}$ is r, we obtain $\dim_{\mathbb{C}} V = r$ and

$$\begin{aligned} V &= \mathbb{R}\langle v_{i_1}, v_{i_2}, \dots, v_{i_r}, v_{i_{r+1}}\rangle \otimes_{\mathbb{R}} \mathbb{C} \\ &= \mathbb{R}\langle v_{i_1}, v_{i_2}, \dots, v_{i_r}\rangle \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned}$$

In particular, we have $v_i \notin V$ for every $i \notin \{i_1, i_2, \dots, i_r, i_{r+1}\}$. Then $a_i = 0$ holds in (3.1) for every $i \notin \{i_1, i_2, \dots, i_r, i_{r+1}\}$. Thus, $\{i_1, i_2, \dots, i_r, i_{r+1}\} = \{n-r+1, n-r+2, \dots, n, n+1\}$ holds and (3.1) becomes

$$a_{n-r+1}v_{n-r+1} + \cdots + a_{n+1}v_{n+1} = 0. \tag{3.2}$$

We define n-dimensional cones

$$\sigma_i := \langle v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}\rangle \in \Sigma$$

for $n-r+1 \leq i \leq n+1$. We put $\mu_{i,j} = \sigma_i \cap \sigma_j \in \Sigma$ for $i \neq j$. We note that

$$\begin{aligned} r < -K_{\mathscr{F}} \cdot V(\mu_{k,n+1}) &\leq \frac{1}{a_{n+1}} \left(\sum_{i=n-r+1}^{n+1} a_i \right) \frac{\operatorname{mult}(\mu_{k,n+1})}{\operatorname{mult}(\sigma_k)} \\ &\leq (r+1) \frac{\operatorname{mult}(\mu_{k,n+1})}{\operatorname{mult}(\sigma_k)} \end{aligned} \tag{3.3}$$

holds for every $n-r+1 \leq k \leq n$. By definition, we know that

$$\frac{\operatorname{mult}(\sigma_k)}{\operatorname{mult}(\mu_{k,n+1})}$$

is a positive integer. Hence (3.3) implies that

$$\operatorname{mult}(\mu_{k,n+1}) = \operatorname{mult}(\sigma_k)$$

holds for every $n-r+1 \leq k \leq n$. Therefore, a_k divides a_{n+1} for every $n-r+1 \leq k \leq n$. By (3.3), we obtain the following claim. Though the proof is completely similar to the proof of the claim in [Fj1, Proposition 2.9], we describe it for the sake of completeness.

Claim.

$$a_{n-r+1} = \cdots = a_{n+1} = 1.$$

Proof of Claim. Suppose that $a_{n-r+1} \neq a_{n+1}$. Since

$$v_{n-r+1} = -\frac{1}{a_{n-r+1}} \sum_{i=n-r+2}^{n+1} a_i v_{n+1}$$

is a primitive vector, $a_{n-r+2} \neq a_{n+1}$ also holds. Namely,

$$\frac{a_{n-r+1}}{a_{n+1}}, \ \frac{a_{n-r+2}}{a_{n+1}} \leq \frac{1}{2},$$

and this contradicts (3.3). □

Thus, (3.2) is nothing but

$$v_{n-r+1} + \cdots + v_{n+1} = 0.$$

Since this equality says that $v_i = -v_{n+1}$ in $N/N_{\mu_{i,n+1}}$ for every $n-r+1 \leq i \leq n$, v_i generates $N/N_{\mu_{i,n+1}}$, that is, we have an isomorphism

$$\mathbb{Z}v_i \xrightarrow{\sim} N/N_{\mu_{i,n+1}}. \tag{3.4}$$

Let v be any element of N. Then, by (3.4), we can find $b_{n-r+1}, \ldots, b_n \in \mathbb{Z}$ such that

$$v - (b_{n-r+1}v_{n-r+1} + \cdots + b_n v_n) \in N_{\langle v_1,\ldots,v_{n-r}\rangle}.$$

This implies that $\{v_{n-r+1}, \ldots, v_n\}$ spans $N_{\langle v_{n-r+1},\ldots,v_n\rangle}$ and that there exists a splitting $N = N_{\langle v_{n-r+1},\ldots,v_n\rangle} \oplus N_{\langle v_1,\ldots,v_{n-r}\rangle}$. The natural projection map

$$N \to N/N_{\langle v_{n-r+1},\ldots,v_n\rangle}$$

and the fan Σ define a fan Σ_Y in $N/N_{\langle v_{n-r+1},\ldots,v_n\rangle}$. Then we obtain a toric extremal contraction morphism of fibering type

$$\varphi_R \colon X = X(\Sigma) \to Y := Y(\Sigma_Y).$$

For the details of the above description of toric extremal contractions, see e.g. [M, Corollary 14-2-2]. Since $\{v_{n-r+1}, \ldots, v_n\}$ spans $N_{\langle v_{n-r+1},\ldots,v_n\rangle}$,

$$v_{n-r+1} + \cdots + v_{n+1} = 0,$$

and there exists a splitting

$$N = N_{\langle v_{n-r+1},\ldots,v_n\rangle} \oplus N_{\langle v_1,\ldots,v_{n-r}\rangle},$$

the extremal contraction $\varphi_R \colon X \to Y$ is a $\mathbb{P}^r$-bundle (see e.g. [Fl, Exercise. (Fiber bundles) on page 41]). Hence, we can easily check that $\mathscr{F} = \mathscr{T}_{X/Y}$ (see e.g. [P, Proposition 3.1.6]) and that $l_{\mathscr{F}}(R) = r+1$ holds under the assumption that $l_{\mathscr{F}}(R) > r$. Thus we obtain all the desired properties. We finish the proof. □

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