ON TORIC FOLIATED PAIRS

OSAMU FUJINO AND HIROSHI SATO

Abstract. We discuss lengths of extremal rational curves, Fujita's freeness, and the Kodaira vanishing theorem for log canonical toric foliated pairs.

Contents

1. Introduction

The basics of toric foliations and toric foliated minimal model program were already studied in [Pan], [S, Section 10], [W], [CC], and so on. In [FjS2], we discussed lengths of extremal rational curves for toric foliations on projective $\mathbb{Q}$-factorial toric varieties. This paper is a continuation of [FjS2] and is obviously a generalization of [Fj1]. Throughout this paper, we will work over $\mathbb{C}$, the field of complex numbers. The following theorem is a log canonical generalization of [FjS2, Theorem 1.3] or is a generalization of [Fj1, Theorem 0.1] for toric foliated pairs. Note that our approach in this paper is based on the toric Mori theory (see [R], [M, Chapter 14], [Fj1], and [FjS1]).

Theorem 1.1 (Lengths of extremal rational curves for toric foliated pairs)**.** *Let X be a projective (not necessarily $\mathbb{Q}$-factorial) toric variety and let $(\mathscr{F}, \Delta)$ be a log canonical toric foliated pair on X with* $\operatorname{rank}\mathscr{F} = r$. *Then*

$$l_{(\mathscr{F},\Delta)}(R) := \min_{[C]\in R}\{-(K_{\mathscr{F}} + \Delta)\cdot C\} \leq r + 1$$

holds for every extremal ray R of the Kleiman–Mori cone $\overline{\mathrm{NE}}(X) = \mathrm{NE}(X)$. Moreover, if $l_{(\mathscr{F},\Delta)}(R) > r$ holds for some extremal ray R of $\mathrm{NE}(X)$, *then the contraction morphism $\varphi_R\colon X \to Y$ associated to R is a $\mathbb{P}^r$-bundle over Y. In this case, $\mathscr{F} = \mathscr{T}_{X/Y}$ holds, where $\mathscr{T}_{X/Y}$ is the relative tangent sheaf of $\varphi_R\colon X \to Y$, and the sum of the coefficients of Δ is less than one. In particular, the foliation $\mathscr{F}$ is locally free.*

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We note that we have already treated Theorem 1.1 under the extra assumption that X is $\mathbb{Q}$-factorial and $\Delta = 0$ in [FjS2, Theorem 1.3]. By Theorem 1.1, we have the cone theorem for log canonical toric foliated pairs.

Corollary 1.2 (Cone theorem for toric foliated pairs). *Let $(\mathscr{F}, \Delta)$ be a log canonical toric foliated pair on a projective toric variety X with* $\operatorname{rank}\mathscr{F} = r$. *Then we have*

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \sum_i \mathbb{R}_{\geq 0}[C_i]$$

where C_i is a torus invariant curve with $-(K_{\mathscr{F}} + \Delta) \cdot C_i \leq r + 1$ for every i. Let R be an extremal ray of $\mathrm{NE}(X)$. *Then we can choose C_i such that $-(K_{\mathscr{F}} + \Delta) \cdot C_i \leq r$ with $R = \mathbb{R}_{\geq 0}[C_i]$ unless the associated contraction $\varphi_R \colon X \to Y$ is a $\mathbb{P}^r$-bundle over Y, $\mathscr{F} = \mathscr{T}_{X/Y}$, and the sum of the coefficients of Δ is less than one.*

Corollary 1.2 is almost obvious by Theorem 1.1. It is a generalization of the cone theorem for toric varieties established in [Fj1, Theorem 0.1]. More precisely, if $\operatorname{rank}\mathscr{F} = \dim X$, then Theorem 1.1 and Corollary 1.2 recovers [Fj1, Theorem 0.1]. Fujita's freeness for log canonical toric foliated pairs is an easy consequence of the cone theorem: Corollary 1.2.

Theorem 1.3 (Fujita's freeness for toric foliated pairs). *Let $(\mathscr{F}, \Delta)$ be a log canonical toric foliated pair on a projective toric variety X. Let r denote the rank of $\mathscr{F}$. Let H be a Cartier divisor on X such that $(H - (K_{\mathscr{F}} + \Delta)) \cdot C \geq r + 1$ holds for every torus invariant curve C on X. Then the complete linear system $|H|$ is basepoint-free.*

For toric foliations on smooth projective toric varieties, we have the following statement on Fujita's freeness.

Theorem 1.4 (Fujita's freeness for toric foliations). *Let $\mathscr{F}$ be a toric foliation with* $\operatorname{rank}\mathscr{F} = r$ *on a smooth projective toric variety X. Let A be an ample Cartier divisor on X. Then $|K_{\mathscr{F}} + (r+1)A|$ is basepoint-free. Moreover, $|K_{\mathscr{F}} + rA|$ is basepoint-free unless X has a $\mathbb{P}^r$-bundle structure $\varphi \colon X \to Y$, $\mathscr{F} = \mathscr{T}_{X/Y}$, and $A \cdot \ell = 1$ for a line ℓ in a fiber of $\varphi \colon X \to Y$.*

If $\operatorname{rank}\mathscr{F} = \dim X$ in Theorem 1.4, then it is nothing but the original version of Fujita's freeness for smooth projective toric varieties. Since any ample Cartier divisor on a smooth projective toric variety is very ample, we have the following statement on Fujita's very ampleness.

Theorem 1.5 (Fujita's very ampleness for toric foliations). *Let $\mathscr{F}$ be a toric foliation with* $\operatorname{rank}\mathscr{F} = r$ *on a smooth projective toric variety X. Let A be an ample Cartier divisor on X. Then $|K_{\mathscr{F}} + (r+2)A|$ is very ample. Moreover, $|K_{\mathscr{F}} + (r+1)A|$ is very ample unless X has a $\mathbb{P}^r$-bundle structure $\varphi \colon X \to Y$, $\mathscr{F} = \mathscr{T}_{X/Y}$, and $A \cdot \ell = 1$ for a line ℓ in a fiber of $\varphi \colon X \to Y$.*

Finally, although we do not treat any applications in this paper, we show that the Kodaira vanishing theorem holds for log canonical toric foliated pairs.

Theorem 1.6 (Kodaira's vanishing theorem for toric foliated pairs). *Let $(\mathscr{F}, \Delta)$ be a log canonical toric foliated pair on a projective toric variety X. Let L be a $\mathbb{Q}$-Cartier Weil divisor on X such that $L - (K_{\mathscr{F}} + \Delta)$ is ample. Then $H^i(X, \mathcal{O}_X(L)) = 0$ holds for every positive integer i.*

It is a special case of the vanishing theorems established in [Fj2].

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In this paper, we will use the same notation as in [FjS2]. We will freely use the basic definitions and results in [FjS2]. For the details of toric varieties, see [O1], [O2], [Fl], and [CLS]. For basic definitions and results of the theory of minimal models, see [Fj3] and [Fj4].

2. Preliminaries

In this section, we collect some definitions and results for the reader's convenience. Let us start with the definition of *foliations* on normal algebraic varieties.

Definition 2.1 (Foliations and toric foliations)**.** A *foliation* on a normal algebraic variety X is a saturated subsheaf $\mathscr{F} \subset \mathscr{T}_X$ that is closed under the Lie bracket, where $\mathscr{T}_X$ is the tangent sheaf of X. We note that the *rank* of the foliation $\mathscr{F}$ means the rank of the coherent sheaf $\mathscr{F}$.

We further assume that X is toric. Then a foliation $\mathscr{F}$ on X is called *toric* if the sheaf $\mathscr{F}$ is torus equivariant.

The following result on toric foliations is a starting point of [FjS2] and this paper.

Theorem 2.2 (see [Pan])**.** *Let $X = X(\Sigma)$ be a $\mathbb{Q}$-factorial toric variety with its fan Σ in the lattice $N \simeq \mathbb{Z}^n$. Then there exists a one-to-one correspondence between the set of toric foliations on X and the set of complex vector subspaces $V \subset N_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^n$.*

Let $\mathscr{F}_V$ be the toric foliation associated to a complex vector subspace $V \subset N_{\mathbb{C}}$. Then

$$K_{\mathscr{F}_V} := -c_1(\mathscr{F}_V) = -\sum_{\rho\subset V} D_\rho$$

holds, that is, the first Chern class of $\mathscr{F}_V$ is $\sum_{\rho\subset V} D_\rho$, where D_ρ is the torus invariant prime divisor corresponding to the one-dimensional cone ρ in Σ. In particular, we have

$$K_{\mathscr{F}_V} = K_X + \sum_{\rho\not\subset V} D_\rho.$$

We note that $\operatorname{rank}\mathscr{F} = \dim_{\mathbb{C}} V$.

In this paper, we are mainly interested in the case where X is not $\mathbb{Q}$-factorial.

Remark 2.3. Let $\mathscr{F}$ be a toric foliation on a (not necessarily $\mathbb{Q}$-factorial) toric variety X. Then we can define $K_{\mathscr{F}}$ as follows. We consider the smooth locus X_{sm} of X. Since X_{sm} is a smooth toric variety, we can define $K_{\mathscr{F}}$ on X_{sm} as in Theorem 2.2. Since $\operatorname{codim}_X(X \setminus X_{\mathrm{sm}}) \geq 2$, we can extend it to the whole X and obtain a well-defined torus invariant Weil divisor $K_{\mathscr{F}}$ on X.

For the details of toric foliations, see [Pan], [W] and [CC]. We make a remark for the reader's convenience.

Remark 2.4 (see [CC, Corollary 3.3])**.** Let $\mathscr{F}_V$ be the toric foliation associated to a complex vector subspace $V \subset N_{\mathbb{C}}$. Then D_ρ is $\mathscr{F}_V$-invariant if and only if $\rho \not\subset V$.

Let us introduce the notion of *torically log canonical toric foliated pairs*. We note that if $\mathrm{rank}\mathscr{F} = \dim X$ in Definition 2.5 then it is nothing but the usual definition of toric log canonical pairs.

Definition 2.5 (Torically log canonical toric foliated pairs)**.** A *toric foliated pair* $(\mathscr{F}, \Delta)$ on a toric variety X consists of a toric foliation $\mathscr{F}$ and an effective torus invariant $\mathbb{R}$-divisor Δ on X such that $K_{\mathscr{F}} + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon \widetilde{X} \to X$ be a proper birational toric morphism of toric varieties. Then we can write

$$K_{\widetilde{\mathscr{F}}} + \pi_*^{-1}\Delta = \pi^*(K_{\mathscr{F}} + \Delta) + \sum_E a(E, \mathscr{F}, \Delta)E$$

where $\widetilde{\mathscr{F}}$ is the induced foliation on $\widetilde{X}$ and the sum is over all π-exceptional divisors E. We call $a(E, \mathscr{F}, \Delta)$ the *discrepancy* of E with respect to $(\mathscr{F}, \Delta)$. We put $\iota(E) = 0$ if E is $\widetilde{\mathscr{F}}$-invariant and $\iota(E) = 1$ otherwise. We say that the pair $(\mathscr{F}, \Delta)$ is *torically log canonical* if $a(E, \mathscr{F}, \Delta) \geq -\iota(E)$ for any proper birational *toric* morphism $\pi\colon \widetilde{X} \to X$ and for any π-exceptional prime divisor E on $\widetilde{X}$.

Although the following lemma is easy to prove, it is very important.

Lemma 2.6. *A toric foliated pair* $(\mathscr{F}, \Delta)$ *is* torically log canonical *if and only if* $\mathrm{Supp}\Delta \subset \mathrm{Supp}K_{\mathscr{F}}$ *and the coefficients of* Δ *are in* $[0, 1]$.

We prove Lemma 2.6 for the sake of completeness.

Proof. We assume that $\mathscr{F}$ is the toric foliation associated to a complex vector subspace $V \subset N_{\mathbb{C}}$. Then we have

$$K_{\mathscr{F}} = K_X + \sum_{\rho \not\subset V} D_\rho$$

by Theorem 2.2 and Remark 2.3. We put

$$\Delta = \sum_\rho b_\rho D_\rho$$

with $b_\rho \geq 0$. Hence we have

$$K_{\mathscr{F}} + \Delta = K_X + \sum_{\rho \not\subset V} D_\rho + \sum_\rho b_\rho D_\rho. \tag{2.1}$$

By definition, we can easily see that $(\mathscr{F}, \Delta)$ is torically log canonical if and only if

$$\left(X, \sum_{\rho \not\subset V} D_\rho + \sum_\rho b_\rho D_\rho\right)$$

is log canonical in the usual sense. Thus the pair $(\mathscr{F}, \Delta)$ is torically log canonical if and only if $\mathrm{Supp}\Delta \subset \mathrm{Supp}K_{\mathscr{F}}$ and the coefficients of Δ are in $[0, 1]$. $\square$

Although it is nontrivial, we see that our definition of torically log canonical toric foliated pairs coincides with the log canonicity of toric foliated pairs in the usual sense. In this paper, Definition 2.5 and Lemma 2.6 are sufficient for our purposes.

Theorem 2.7 ([CC, Proposition 4.31])**.** *Let* $(\mathscr{F}, \Delta)$ *be a toric foliated pair. Then* $(\mathscr{F}, \Delta)$ *is torically log canonical if and only if* $(\mathscr{F}, \Delta)$ *is log canonical in the usual sense for foliated pairs.*

Proof. This follows from Lemma 2.6 and [CC, Proposition 4.31]. ☐

For the precise definition of *log canonical foliated pairs*, see, for example, [CC, Definition 4.1]. By Theorem 2.7, we can simply say that a toric foliated pair $(\mathscr{F}, \Delta)$ is log canonical when it is torically log canonical. We close this section with a remark on the minimal model program.

Remark 2.8. Let $(\mathscr{F}, \Delta)$ be a log canonical toric foliated pair on a projective $\mathbb{Q}$-factorial toric variety X. Then we can run the minimal model program with respect to $K_{\mathscr{F}} + \Delta$ (see, for example, [R], [M], [Fj1], and [FjS1]). By (2.1) in the proof of Lemma 2.6, we see that the log canonicity of $(\mathscr{F}, \Delta)$ is preserved by the above minimal model program.

3. Lemmas on projective bundles

In this section, we prepare some lemmas on projective bundles over curves for the proof of Theorem 1.1. Let us start with an easy lemma on projective bundles over a smooth rational curve.

Lemma 3.1. *Let $\pi\colon X \to Y$ be a $\mathbb{P}^r$-bundle over $\mathbb{P}^1$. We write*

$$\pi\colon X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(c_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(c_r)) \to \mathbb{P}^1$$

with $c_1 \leq \cdots \leq c_r$. Note that $\pi\colon X \to Y$ is toric. If there exists an extremal ray R of $\mathrm{NE}(X)$ *such that $K_{X/Y} \cdot R = 0$, then $c_1 = \cdots = c_r = 0$, that is, $X = \mathbb{P}^r \times \mathbb{P}^1$ and π is the second projection.*

Proof. Since $\pi\colon X \to Y$ is a $\mathbb{P}^r$-bundle, we have

$$\mathcal{O}_X(K_{X/Y}) = \pi^*\mathcal{O}_{\mathbb{P}^1}\left(\sum_{i=1}^{r} c_i\right) \otimes \mathcal{O}_X(-(r+1)).$$

Note that $\mathrm{NE}(X)$ is spanned by two extremal rays. One extremal ray corresponds to the projection $\pi\colon X \to Y$. Therefore, $K_{X/Y}$ is negative on it. By assumption, $K_{X/Y} \cdot C \leq 0$ holds for every horizontal torus invariant curve C on X. This implies $c_1 = \cdots = c_r = 0$. Thus $X = \mathbb{P}^r \times \mathbb{P}^1$ and $\pi\colon X \to Y$ is the second projection. We finish the proof. ☐

Lemma 3.2 is a slight generalization of Lemma 3.1.

Lemma 3.2. *Let $\pi\colon X \to Y$ be a $\mathbb{P}^r$-bundle over $\mathbb{P}^1$ and let Δ be a torus invariant horizontal effective $\mathbb{R}$-divisor on X such that every coefficient of Δ is less than one. If there exists an extremal ray R of* $\mathrm{NE}(X)$ *such that $(K_{X/Y} + \Delta) \cdot R = 0$, then $X = \mathbb{P}^r \times \mathbb{P}^1$ and π is the second projection.*

Proof. As in the proof of Lemma 3.1, we write

$$\pi\colon X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(c_0) \oplus \mathcal{O}_{\mathbb{P}^1}(c_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(c_r)) \to \mathbb{P}^1$$

with $0 = c_0 \leq c_1 \leq \cdots \leq c_r$. By assumption, we can write

$$\Delta = \sum_{i=0}^{r} b_i H_i$$

with $b_i \in [0, 1)$ such that

$$\mathcal{O}_X(H_i) \simeq \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-c_i)$$

for every i. Let P be a point of $Y = \mathbb{P}^1$. Then $K_{X/Y} + \Delta$ is $\mathbb{R}$-linearly equivalent to

$$\pi^*\left(\sum_{i=0}^{r}(1-b_i)c_iP\right) + \left(-(r+1) + \sum_{i=0}^{r} b_i\right) H_0.$$

As in the proof of Lemma 3.1, $(K_{X/Y}+\Delta)\cdot C \leq 0$ holds for every horizontal torus invariant curve C on X. This implies that

$$\sum_{i=0}^{r}(1-b_i)c_i \leq 0$$

holds. Hence we obtain $c_0 = c_1 = \cdots = c_r = 0$. This is what we wanted. $\square$

The final lemma in this section is similar to Lemma 3.2 above. However, we note that $\pi\colon X \to Y$ is not toric when $Y \neq \mathbb{P}^1$.

Lemma 3.3. *Let Y be a smooth projective curve and let $\mathcal{L}_i$ be a line bundle on Y for every i. We consider a $\mathbb{P}^r$-bundle $\pi\colon X := \mathbb{P}_Y(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_r) \to Y$ over Y. Let H_i be the horizontal divisor on X corresponding to*

$$\bigoplus_{j=0}^{r} \mathcal{L}_j \to \bigoplus_{j\neq i} \mathcal{L}_j$$

for every i. We put $\Delta = \sum_{i=0}^{r} b_iH_i$ such that $b_i \in [0,1)$ for every i. Assume that there exists an extremal ray R of $\overline{\mathrm{NE}}(X)$ such that $(K_{X/Y}+\Delta)\cdot R = 0$. Then $\deg \mathcal{L}_i = \deg \mathcal{L}_0$ holds for every i. In particular, if $\deg\mathcal{L}_0 = 0$, then $\deg\mathcal{L}_i = 0$ holds for every i. Let C_i be the section of $\pi\colon X \to Y$ corresponding to

$$\bigoplus_{j=0}^{r} \mathcal{L}_j \to \mathcal{L}_i$$

for every i. Then the numerical equivalence class of C_i is in R for every i.

Proof. Note that

$$\mathcal{O}_X(H_i) \simeq \mathcal{O}_X(1) \otimes \pi^*\mathcal{L}_i^{\otimes -1}$$

holds for every i and that

$$\mathcal{O}_X(K_{X/Y}) \simeq \pi^*\left(\bigotimes_{i=0}^{r}\mathcal{L}_i\right) \otimes \mathcal{O}_X(-(r+1)).$$

Hence we can easily check this lemma by modifying the proof of Lemma 3.2 suitably. $\square$

4. Proof of Theorem 1.1

This section is the main part of this paper. Here we give a proof of Theorem 1.1.

Proof of Theorem 1.1. In Step 1, we will prove Theorem 1.1 under the extra assumption that X is $\mathbb{Q}$-factorial. Step 1 is essentially the same as the proof of [FjS2, Theorem 1.3]. Hence we will only explain how to modify it. Then, in Step 2, we will treat the case where X is not $\mathbb{Q}$-factorial. Step 2 is completely new. In our proof in Step 2, we have to treat non-toric varieties.

Step 1. We assume that $\mathscr{F}$ is the toric foliation associated to a complex vector subspace $V \subset N_{\mathbb{C}}$. Then we can write

$$\Delta = \sum_{\rho \subset V} b_\rho D_\rho$$

with $b_\rho \in [0,1]$ and

$$K_{\mathscr{F}} + \Delta = K_X + \sum_{\rho \not\subset V} D_\rho + \sum_{\rho \subset V} b_\rho D_\rho$$

since $(\mathscr{F}, \Delta)$ is log canonical (see Lemma 2.6 and Theorem 2.7). We assume that $l_{(\mathscr{F},\Delta)}(R) > r$ holds. From now, we will only explain how to modify the proof of [FjS2, Theorem 1.3]. Hence we will freely use the same notation as in the proof of [FjS2, Theorem 1.3]. We put $b_\rho = 1$ for $\rho \not\subset V$ and $b_i := b_{\rho_i}$ with $\rho_i := \mathbb{R}_{\geq 0} v_i$ for every i. By changing the order, we may assume that

$$(1-b_1)a_1 \leq \cdots \leq (1-b_n)a_n \leq (1-b_{n+1})a_{n+1}.$$

Then we have

$$-(K_{\mathscr{F}} + \Delta) \cdot C = \sum_{v_i \in V} (1-b_i) V(\langle v_i \rangle) \cdot C > r.$$

By the same argument as in the proof of [FjS2, Theorem 1.3], we obtain

$$a_{n-r+1} v_{n-r+1} + \cdots + a_{n+1} v_{n+1} = 0.$$

Then we see that

$$\begin{aligned} r < -(K_{\mathscr{F}} + \Delta) \cdot V(\mu_{k,n+1}) &\leq \frac{1}{a_{n+1}} \left(\sum_{i=n-r+1}^{n+1} (1-b_i) a_i \right) \frac{\operatorname{mult}(\mu_{k,n+1})}{\operatorname{mult}(\sigma_k)} \\ &\leq (r+1) \frac{\operatorname{mult}(\mu_{k,n+1})}{\operatorname{mult}(\sigma_k)} \end{aligned}$$

holds for every $n - r + 1 \leq k \leq n$. Then the argument in the proof of [FjS2, Theorem 1.3] works without any changes. Thus we obtain that $\varphi_R \colon X \to Y$ is a $\mathbb{P}^r$-bundle and $\mathscr{F} = \mathscr{T}_{X/Y}$. In this case, we can easily check that the sum of the coefficients of Δ is less than one by $l_{(\mathscr{F},\Delta)}(R) > r$.

From now, we may assume that X is not $\mathbb{Q}$-factorial. It is sufficient to prove that $\varphi_R \colon X \to Y$ is a $\mathbb{P}^r$-bundle with $\mathscr{F} = \mathscr{T}_{X/Y}$ under the assumption that $l_{(\mathscr{F},\Delta)}(R) > r$.

Step 2. We take a small projective $\mathbb{Q}$-factorialization $\psi \colon X' \to X$ (see [Fj1, Corollary 5.9]). Let Δ' be the strict transform of Δ and let $\mathscr{F}'$ be the induced foliation on X'. By construction, we have $K_{\mathscr{F}'} + \Delta' = \psi^*(K_{\mathscr{F}} + \Delta)$. Let $\varphi_R \colon X \to Y$ be the contraction morphism associated to R (see [FjS1, Theorem 4.5]). By considering $\varphi_R \circ \psi \colon X' \to Y$, we can find an extremal ray R' of $\mathrm{NE}(X')$ such that $\psi_* R' = R$ and $l_{(\mathscr{F}',\Delta')}(R') > r$. Since X' is $\mathbb{Q}$-factorial, the associated contraction $\varphi_{R'} \colon X' \to Y'$ is a $\mathbb{P}^r$-bundle, $\mathscr{F}'$ is the relative tangent sheaf $\mathscr{T}_{X'/Y'}$, and the sum of the coefficients of Δ' is less than one by Step 1. Note that we can write

$$\varphi_{R'} \colon X' = \mathbb{P}_{Y'}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r) \to Y'$$

with $\mathcal{L}_0 = \mathcal{O}_{Y'}$ since $\varphi_{R'} \colon X' \to Y'$ is toric (see Lemma 6.1 below). We put $E := \operatorname{Exc}(\psi)$, that is, the exceptional locus of ψ. Then E is a torus invariant closed subset of X' with $\operatorname{codim}_{X'} E \geq 2$.

Claim. $E=\varphi_{R'}^{-1}(\varphi_{R'}(E))$.

Proof of Claim. Let Z' be the section of $\varphi_{R'}\colon X'\to Y'$ corresponding to

$$\bigoplus_{i=0}^{r}\mathcal{L}_i\to\mathcal{L}_0.$$

We consider $\psi_{Z'}:=\psi|_{Z'}\colon Z'\to Z:=\psi(Z')$. Then any positive-dimensional fiber of $\psi_{Z'}$ is rationally chain connected since $\psi_{Z'}\colon Z'\to Z$ is toric. Let C be a rational curve in a fiber of $\psi_{Z'}$. Let C' be the normalization of $\varphi_{R'}(C)$. We consider the base change of $\varphi_{R'}\colon X'\to Y'$ by $\mathbb{P}^1\simeq C'\to Y'$. Then, by Lemma 3.2, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^r & \mathbb{P}^r\times C' \arrow[l, "p_1"'] \arrow[r] \arrow[d, "p_2"] \arrow[rr, bend left, "\alpha"] & \varphi_{R'}^{-1}(\varphi_{R'}(C)) \arrow[r, hook] \arrow[d] & X' \arrow[d, "\varphi_{R'}"] \\
 & C' \arrow[r] & \varphi_{R'}(C) \arrow[r, hook] & Y',
\end{tikzcd}$$

where p_1 and p_2 are projections. Thus, the numerical equivalence class of $\alpha(p_1^{-1}(P))$ is independent of $P\in\mathbb{P}^r$. This implies that $\alpha(p_1^{-1}(P))$ is numerically equivalent to C for every $P\in\mathbb{P}^r$. Hence $\psi\colon X'\to X$ contracts $\alpha(p_1^{-1}(P))$ to a point for every $P\in\mathbb{P}^r$. Thus, we obtain $\varphi_{R'}^{-1}(\varphi_{R'}(C))\subset E$. Therefore, we obtain

$$\varphi_{R'}^{-1}(\varphi_{R'}(\mathrm{Exc}(\psi_{Z'})))\subset E. \tag{4.1}$$

If $\psi_Z\colon Z'\to Z$ is not birational, then $\mathrm{Exc}(\psi_{Z'})=Z'$. This implies $X\subset E$ by (4.1). This is obviously a contradiction. Hence we obtain that $\psi_{Z'}\colon Z'\to Z$ is birational. If $\varphi_{R'}^{-1}(\varphi_{R'}(\mathrm{Exc}(\psi_{Z'})))\subsetneq E$, then we can take a curve C such that $\psi(C)$ is a point with

$$C\not\subset\varphi_{R'}^{-1}(\varphi_{R'}(\mathrm{Exc}(\psi_{Z'}))).$$

Let C' be the normalization of $\varphi_{R'}(C)$. We consider the base change of $\varphi_{R'}\colon X'\to Y'$ by $C'\to Y'$. Then we have the following commutative diagram:

$$\begin{tikzcd}
X'_{C'} \arrow[r, "\beta"'] \arrow[d] \arrow[rr, bend left, "\alpha"] & \varphi_{R'}^{-1}(\varphi_{R'}(C)) \arrow[r, hook] \arrow[d] & X' \arrow[d, "\varphi_{R'}"] \\
C' \arrow[r] & \varphi_{R'}(C) \arrow[r, hook] & Y',
\end{tikzcd} \tag{4.2}$$

where $X'_{C'}\to C'$ is the base change of $X'\to Y'$ by $C'\to Y'$. By construction, we can take a curve $C^\flat$ on $X'_{C'}$ such that $\beta(C^\flat)=C$. Since $X'_{C'}$ is a $\mathbb{P}^r$-bundle over C', the Picard number of $X'_{C'}$ is two. Moreover, $X'_{C'}$ has two nontrivial contractions $X'_{C'}\to C'$ and $\psi\circ\alpha\colon X'_{C'}\to W$, where W is the normalization of $(\psi\circ\alpha)(X'_{C'})$. Thus $\overline{\mathrm{NE}}(X'_{C'})$ has an extremal ray Q spanned by $C^\flat$ since $C^\flat$ is contracted by $\psi\circ\alpha$. Note that $C^\flat\cdot(K_{X'_{C'}/C'}+\Delta'_{C'})=0$, where $K_{X'_{C'}/C'}+\Delta'_{C'}=\alpha^*(K_{X'/Y'}+\Delta')$. By Lemma 3.3, we can take a curve $C^\sharp$ on $X'_{C'}$ such that the numerical equivalence class of $C^\sharp$ is in Q and $\beta(C^\sharp)=\varphi_{R'}^{-1}(\varphi_{R'}(C))\cap Z'$. Thus the curve $\varphi_{R'}^{-1}(\varphi_{R'}(C))\cap Z'$ is contracted by ψ, that is, $\varphi_{R'}^{-1}(\varphi_{R'}(C))\cap Z'\subset\mathrm{Exc}(\psi_{Z'})$. Then

$$\varphi_{R'}^{-1}(\varphi_{R'}(C))\cap Z'\subset\mathrm{Exc}(\psi_{Z'})\subset\varphi_{R'}^{-1}(\varphi_{R'}(\mathrm{Exc}(\psi_{Z'}))).$$

Hence we have

$$C \subset \varphi_{R'}^{-1}(\varphi_{R'}(\mathrm{Exc}(\psi_{Z'}))).$$

This is a contradiction. This implies that

$$E = \varphi_{R'}^{-1}(\varphi_{R'}(\mathrm{Exc}(\psi_{Z'}))).$$

Therefore, we have the desired equality $E = \varphi_{R'}^{-1}(\varphi_{R'}(E))$. We finish the proof of Claim. $\square$

We put $\mathcal{L}_{i,Z'} := (\varphi_{R'}^* \mathcal{L}_i)|_{Z'}$ for every i. Let C be any curve on Z' such that $\psi_{Z'}(C)$ is a point. Let C' be the normalization of $\varphi_{R'}(C)$. We consider the commutative diagram (4.2) as before. Then we can check that $\mathcal{L}_{i,Z'} \cdot C = 0$ for every i by applying Lemma 3.3 to $X'_{C'} \to C'$. This implies that there exists a line bundle $\mathcal{L}_{i,Z}$ on Z such that $\mathcal{L}_{i,Z'} = \psi_{Z'}^* \mathcal{L}_{i,Z}$ holds for every i since $\psi_{Z'} \colon Z' \to Z$ is a projective birational toric morphism. Hence, $\mathcal{L}_{i,Z'}|_C$ is a trivial line bundle for every i. Then $X'_{C'} \to C'$, which is the base change of $\varphi_{R'} \colon X' \to Y'$ by $C' \to Y'$, is the second projection

$$\mathbb{P}^r \times C' = \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \cdots \oplus \mathcal{O}_{C'}) \to C'.$$

In particular, we obtain that $\Delta' \cdot C^\dagger = 0$ holds for every curve $C^\dagger$ on X' such that $\psi(C^\dagger)$ is a point.

We consider $\psi_{Z'} \circ (\varphi_{R'}|_{Z'})^{-1} \colon Y' \to Z$. By the above observation, for any point $x \in X$, we see that $(\psi_{Z'} \circ (\varphi_{R'}|_{Z'})^{-1} \circ \varphi_{R'})(\psi^{-1}(x))$ is a point. Therefore, there exists a morphism $X \to Z$ and we have the following commutative diagram.

$$\begin{tikzcd}
X' \arrow[rr, "\psi"] \arrow[d, "\varphi_{R'}"'] & & X \arrow[d] \\
Y' \arrow[r, "\simeq"] & Z' \arrow[r, "\psi_{Z'}"'] & Z
\end{tikzcd}$$

By this commutative diagram and the observation before, we see that every fiber of $X \to Z$ is contracted to a point by φ_R. Thus $\varphi_R \colon X \to Y$ factors through Z. Since the relative Picard number of $\varphi_R \colon X \to Y$ is one, Z is isomorphic to Y. Hence we have the following commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "\psi"] \arrow[d, "\varphi_{R'}"'] & X \arrow[d, "\varphi_R"] \\
Y' \arrow[r, "\psi_{Y'}"'] & Y
\end{tikzcd}$$

and we see that $\mathcal{L}_i = \psi_{Y'}^* \mathcal{M}_i$ holds for some line bundle $\mathcal{M}_i$ on Y for every i.

We put $X'' := \mathbb{P}_Y(\mathcal{M}_0 \oplus \cdots \oplus \mathcal{M}_r)$. Then $\varphi_{R'} \colon X' \to Y'$ is the base change of $X'' \to Y$ by $\psi_{Y'} \colon Y' \to Y$. We put $\rho \colon X' \to X''$. Then $K_{X'/Y'} + \Delta' = \rho^*(K_{X''/Y} + \Delta'')$ with $\Delta'' := \rho_* \Delta'$. By construction, $K_{X'/Y'} + \Delta' = \psi^*(K_{X/Y} + \Delta)$. We set $B := (\psi_{Y'} \circ \varphi_{R'})(E)$. Then B is a closed subset of Y with $\mathrm{codim}_Y B \geq 2$. By Claim and the construction of $\psi_{Y'}$, we have $E = (\psi_{Y'} \circ \varphi_{R'})^{-1}(B)$. Thus we obtain the following commutative diagram:

$$\begin{tikzcd}
X' \setminus E \arrow[r, "\sim", "\psi"'] \arrow[d, "\varphi_{R'}"'] & X \setminus \psi(E) \arrow[d, "\varphi_R"] \\
Y' \setminus \varphi_{R'}(E) \arrow[r, "\sim", "\psi_{Y'}"'] & Y \setminus B.
\end{tikzcd}$$

Note that $\mathrm{codim}_{X'}E \geq 2$, $\mathrm{codim}_{Y'}\varphi_{R'}(E) \geq 2$, and $\mathrm{codim}_X\psi(E) \geq 2$. By the above diagram, we can easily check that X'' and X are isomorphic in codimension one. By construction again, $-(K_{X/Y} + \Delta)$ is ample over Y and $-(K_{X''/Y} + \Delta'')$ is also ample over Y. Hence, X is isomorphic to X'' over Y. This is what we wanted.

We finish the proof of Theorem 1.1. □

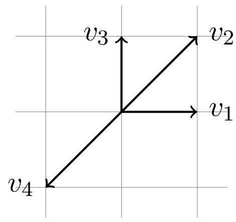
Remark 4.1. The authors do not know how to prove Theorem 1.1 using only toric geometry when X is not $\mathbb{Q}$-factorial.

We close this section with an example, which shows that the estimate in Theorem 1.1 is sharp.

Example 4.2. We consider $N = \mathbb{Z}^2$. We put

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } v_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Let us consider the fan Σ consisting of $\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$, $\mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3$, $\mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_4$, $\mathbb{R}_{\geq 0}v_4 + \mathbb{R}_{\geq 0}v_1$, and their faces.



Then the toric variety $X := X(\Sigma)$ is a $\mathbb{P}^1$-bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ over $\mathbb{P}^1$. We put $\rho_i := \mathbb{R}_{\geq 0}v_i$ and $D_i := D_{\rho_i}$ for every i. Then the Kleiman–Mori cone is spanned by $[D_2]$ and $[D_1] = [D_3]$, that is,

$$\mathrm{NE}(X) = \mathbb{R}_{\geq 0}[D_2] + \mathbb{R}_{\geq 0}[D_3].$$

Let V be the complex vector subspace of $N_{\mathbb{C}}$ spanned by v_2. Let $\mathscr{F}_V$ be the associated toric foliation on X. Then $\mathrm{rank}\mathscr{F}_V = 1$ and $K_{\mathscr{F}_V} = -D_2 - D_4$ (see Theorem 2.2). Similarly, let W be the complex vector subspace of $N_{\mathbb{C}}$ spanned by v_1 and let $\mathscr{F}_W$ be the associated toric foliation on X. Then $K_{\mathscr{F}_W} = -D_1$ and $\mathrm{rank}\mathscr{F}_W = 1$ (see Theorem 2.2). We can directly check that

$$\begin{cases} -K_{\mathscr{F}_V} \cdot D_2 = -1 \\ -K_{\mathscr{F}_V} \cdot D_3 = 2 \end{cases}$$

and

$$\begin{cases} -K_{\mathscr{F}_W} \cdot D_2 = 1 \\ -K_{\mathscr{F}_W} \cdot D_3 = 0. \end{cases}$$

Note that $\mathbb{R}_{\geq 0}[D_3]$ corresponds to the $\mathbb{P}^1$-bundle structure of X and that $\mathbb{R}_{\geq 0}[D_2]$ gives a blow-down $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^2$.

5. Proofs of Corollary 1.2, Theorems 1.3, 1.4, 1.5, and 1.6

In this section, we prove the results in Section 1.

Proof of Corollary 1.2. It is well known that $\mathrm{NE}(X)$ is spanned by torus invariant curves on X (see, for example, [R], [M], [Fj1], and [FjS1]). In particular, it is a rational polyhedral cone. The statement on lengths of extremal rational curves follows from Theorem 1.1. We finish the proof. □

Theorems 1.3 and 1.4 are easy consequences of the cone theorem: Corollary 1.2.

Proof of Theorem 1.3. By Corollary 1.2, $H \cdot R \geq 0$ for every extremal ray R of $\mathrm{NE}(X)$, that is, H is a nef Cartier divisor on X. This implies that $|H|$ is basepoint-free. □

Proof of Theorem 1.4. This follows from Corollary 1.2. More precisely, we can check the nefness of $K_{\mathscr{F}} + (r+1)A$ and $K_{\mathscr{F}} + rA$ under the given assumptions as in the proof of Theorem 1.3. □

We learned the following example from Professor Fanjun Meng.

Example 5.1. For any positive integer n, we can construct a smooth projective surface Y and an ample Cartier divisor H on Y such that $|nH|$ is not basepoint-free (see [L, Examples 5.1.18 and 5.2.1]). We take an elliptic curve E and a point P of E. We put $X := Y \times E$ and $A := p_1^*H + p_2^*P$, where $p_1 \colon X \to Y$ and $p_2 \colon X \to E$ are projections. Then A is an ample Cartier divisor on X such that $|nA|$ is not basepoint-free. We consider $\pi := p_1 \colon X \to Y$ and $\mathscr{F} := \mathscr{T}_{X/Y}$, that is, $\mathscr{F}$ is the relative tangent sheaf of $\pi \colon X \to Y$. Then the canonical bundle $\mathcal{O}_X(K_{\mathscr{F}})$ of $\mathscr{F}$ is trivial. In this case, $|K_{\mathscr{F}} + nA|$ is not basepoint-free. Hence we cannot formulate Fujita's freeness conjecture for foliations naively.

Theorem 1.5 is obvious by Theorem 1.4.

Proof of Theorem 1.5. Since A is an ample Cartier divisor on a smooth projective toric variety X, A is very ample (see, for example, [O2, Corollary 2.15]). Hence we have the desired statement by Theorem 1.4. □

We finally prove the Kodaira vanishing theorem for log canonial toric foliated pairs.

Proof of Theorem 1.6. We assume that $\mathscr{F}$ is the toric foliation associated to a complex vector subspace $V \subset N_{\mathbb{C}}$. Then we can write

$$\Delta = \sum_{\rho \subset V} b_\rho D_\rho$$

with $b_\rho \in [0,1]$ and

$$K_{\mathscr{F}} + \Delta = K_X + \sum_{\rho \not\subset V} D_\rho + \sum_{\rho \subset V} b_\rho D_\rho$$

since $(\mathscr{F}, \Delta)$ is log canonical (see Lemma 2.6). By assumption,

$$L - (K_{\mathscr{F}} + \Delta) = L - \left(K_X + \sum_{\rho \not\subset V} D_\rho + \sum_{\rho \subset V} b_\rho D_\rho \right)$$

is ample. By perturbing the coefficients, we can construct an effective $\mathbb{Q}$-divisor Δ' on X such that

$$\mathrm{Supp}\Delta' = \mathrm{Supp}\left(\sum_{\rho \not\subset V} D_\rho + \sum_{\rho \subset V} b_\rho D_\rho \right),$$

every coefficient of Δ' is less than one, and $L-(K_X+\Delta')$ is still ample. In this setting, by [Fj2, Corollary 1.7], we obtain

$$0=H^i(X,\mathcal{O}_X(K_X+\lceil L-(K_X+\Delta')\rceil))=H^i(X,\mathcal{O}_X(L))$$

for every positive integer i. This is what we wanted. $\square$

6. Appendix: Toric projective bundles

In this appendix, we give a proof of the following well-known result (see [O1, p.41 Remark]) for the sake of completeness. To the best knowledge of the authors, we do not find it in the standard literature.

Lemma 6.1 (Toric projective bundles, [O1, p.41 Remark])**.** *Let $\varphi\colon X\to Y$ be a toric morphism of toric varieties such that $\varphi\colon X\to Y$ is a $\mathbb{P}^r$-bundle. Then $X\simeq\mathbb{P}_Y(\mathcal{L}_0\oplus\cdots\oplus\mathcal{L}_r)$ for some line bundles $\mathcal{L}_0,\dots,\mathcal{L}_r$ on Y and $\varphi\colon X\to Y$ is isomorphic to the projection $\pi\colon\mathbb{P}_Y(\mathcal{L}_0\oplus\cdots\oplus\mathcal{L}_r)\to Y$.*

Proof. Since X is a $\mathbb{P}^r$-bundle over Y, we can write $X=\mathbb{P}_Y(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on Y. We take a torus invariant Cartier divisor H on X such that $\mathcal{O}_X(H)\simeq\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$. Then we have $\varphi_*\mathcal{O}_X(H)\simeq\mathcal{E}$. Thus, by replacing $\mathcal{E}$ with $\varphi_*\mathcal{O}_X(H)$, we may assume that $\mathcal{E}$ is a toric vector bundle on Y, that is, the torus action on Y lifts to an action on $\mathcal{E}$ and it is linear on the fibers. Let U be any affine toric open subset U of Y. Then it is not difficult to see that $\mathcal{E}|_U$ is isomorphic to $\mathcal{O}_U^{\oplus r+1}$ as a toric vector bundle on U (see, for example, [Pay, Proposition 2.2]). Therefore, the restriction of $\varphi\colon X\to Y$ to U is isomorphic to the second projection $\mathbb{P}^r\times U\to U$. Let $h\colon(N,\Sigma)\to(N',\Sigma')$ be a map of fans corresponding to $\varphi\colon X\to Y$. Let N'' be the kernel of $h\colon N\to N'$. Without loss of generality, we may assume that $N=N''\oplus N'$. We fix a $\mathbb{Z}$-basis $\{n''_1,\dots,n''_r\}$ of N''. Since $\varphi\colon X\to Y$ is isomorphic to the second projection $\mathbb{P}^r\times U\to U$ for any affine toric open subset U of Y, we can lift any cone $\sigma'\in\Sigma'$ to a cone $\sigma\in\Sigma$. Hence we can find Σ'-linear support functions $h_1,\dots,h_r$ such that the map $N'_{\mathbb{R}}\to N_{\mathbb{R}}=N''_{\mathbb{R}}\oplus N'_{\mathbb{R}}$ given by $y\mapsto(\sum_{i=1}^r h_i(y)n''_i,y)$ defines the desired lifts of cones. Let $\mathcal{L}_i$ be the line bundle on Y defined by the Σ'-linear support function h_i for every i. Then, by construction, we can check that $X\simeq\mathbb{P}_Y(\mathcal{O}_Y\oplus\mathcal{L}_1\oplus\cdots\oplus\mathcal{L}_r)$ and $\varphi\colon X\to Y$ is isomorphic to the projection $\pi\colon\mathbb{P}_Y(\mathcal{O}_Y\oplus\mathcal{L}_1\oplus\cdots\oplus\mathcal{L}_r)\to Y$ (see Remark 6.2 below). We finish the proof. $\square$

Remark 6.2 (see [Par, p.124, Remark.(2)])**.** The minus sign in [O2, p.59] needs to be deleted.

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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: `fujino@math.kyoto-u.ac.jp`

Department of Applied Mathematics, Faculty of Sciences, Fukuoka University, 8-19-1, Nanakuma, Jonan-ku, Fukuoka 814-0180, Japan

Email address: `hirosato@fukuoka-u.ac.jp`