SUPPLEMENT TO "ON QUASI-ALBANESE MAPS"

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ABSTRACT. The proof of Lemma 3.14 in "On quasi-Albanese maps" is insufficient. Here we give a supplementary argument to remedy it.

The proof of [F, Lemma 3.14] is insufficient. We note that [F, Lemma 3.14] plays a crucial role in [F]. Hence we give a supplementary argument here. Throughout this paper, we will freely use the notation in [F]. Let us start with the following lemma, which is well known for abelian varieties.

Lemma 1. *Let G_1 and G_2 be quasi-abelian varieties and let $f\colon G_1 \to G_2$ be a morphism of algebraic varieties with $f(0) = 0$. Then f is a morphism of algebraic groups.*

We do not prove Lemma 1 here. For the details of Lemma 1, see, for example, [NW, Theorem 5.1.37]. Note that [BSU, Theorem 1.2.4], which is called the *Rosenlicht decomposition*, may help the reader understand the proof of [NW, Theorem 5.1.37]. Or, we apply [B, Lemma 5.4.8], which is a kind of rigidity lemma, to the morphism $\varphi\colon G_1 \times G_1 \to G_2$ given by $\varphi(x, y) := f(x + y)$ with $x_0 = y_0 = 0$. Then, $f(x + y) - f(x) - f(y) = 0$, that is, $f(x + y) = f(x) + f(y)$ holds. This means that f is a morphism of algebraic groups.

Lemma 2. *Let G_1 and G_2 be quasi-abelian varieties and let $f\colon G_1 \to G_2$ be a morphism of algebraic varieties such that $f_*\colon H_1(G_1, \mathbb{Z}) \simeq H_1(G_2, \mathbb{Z})$. Then f is an isomorphism.*

Proof of Lemma 2. Without loss of generality, we may assume that $f(0) = 0$ by translation. Then, by Lemma 1, f is a morphism of algebraic groups. Let $p_i\colon \mathbb{C}^{\dim G_i} \to G_i$ be the universal cover of G_i for $i = 1, 2$. Then we can lift $f\colon G_1 \to G_2$ as in the following commutative diagram.

$$\begin{array}{ccc} \mathbb{C}^{\dim G_1} & \xrightarrow{\widetilde{f}} & \mathbb{C}^{\dim G_2} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ G_1 & \xrightarrow[f]{} & G_2 \end{array}$$

Note that every map in the above diagram is a homomorphism of complex Lie groups. By [NW, Proposition 5.1.8], $\pi_1(G_i) \simeq H_1(G_i, \mathbb{Z})$ spans $\mathbb{C}^{\dim G_i}$ as a complex vector space for $i = 1, 2$. Since $f_*\colon H_1(G_1, \mathbb{Z}) \simeq H_1(G_2, \mathbb{Z})$ by assumption, $\widetilde{f}\colon \mathbb{C}^{\dim G_1} \to \mathbb{C}^{\dim G_2}$ is also an isomorphism. This implies that f is an isomorphism. We finish the proof. $\square$

Now let us prove [F, Lemma 3.14].

Lemma 3 ([F, Lemma 3.14])**.** *Let W be a quasi-abelian variety and let*

$$\alpha_W\colon W \to \widetilde{\mathcal{A}}_W$$

be the quasi-Albanese map constructed in [F, Lemma 3.10]*. Then α_W is an isomorphism.*

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Proof of Lemma 3. By [F, Lemma 3.11], α_W is a morphism of algebraic varieties. By [F, Lemma 3.12], we see that

$$(\alpha_W)_* \colon H_1(W, \mathbb{Z}) \to H_1(\widetilde{\mathcal{A}}_W, \mathbb{Z})$$

is an isomorphism. Hence, by Lemma 2, α_W is an isomorphism of algebraic varieties. We finish the proof. $\square$

We make an important remark on Lemma 3.

Remark 4. Lemma 3 (see [F, Lemma 3.14]) easily follows from [F, Theorem 1.1], which is [F, Theorem 3.1]. We note that we use Lemma 3 (see [F, Lemma 3.14]) in the proof of [F, Theorem 3.1] in [F, Section 3]. Hence we have to prove Lemma 3 (see [F, Lemma 3.14]) without using [F, Theorem 3.1].

By Lemma 3 (see [F, Lemma 3.14]), we can apply the explicit description of $\widetilde{\mathcal{A}}_W$ discussed in [F, Section 3] to W since $W \simeq \widetilde{\mathcal{A}}_W$ (see [F, Section 4]). Of course, we can use $\alpha_T \colon T \simeq \widetilde{\mathcal{A}}_T$ in the proof of [F, Lemma 3.15].

Corollary 5. *Let W be a quasi-abelian variety. Then the tangent space of W at 0 is $T_1(W)^*$, that is, the dual vector space of $T_1(W)$.*

Proof of Corollary 5. By construction, the tangent space of $\widetilde{\mathcal{A}}_W$ at 0 is $T_1(W)^*$. By Lemma 3, we have $W \simeq \widetilde{\mathcal{A}}_W$. Thus, the tangent space of W at 0 is $T_1(W)^*$. This is what we wanted. $\square$

Remark 6. The fact stated in Corollary 5 looks nontrivial. We think that it does not directly follow from the definition of quasi-abelian varieties.

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