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Subadjunction for quasi-log canonical pairs and its applications

by

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Abstract

We establish a kind of subadjunction formula for quasi-log canonical pairs. As an application, we prove that a connected projective quasi-log canonical pair whose quasi-log canonical class is anti-ample is simply connected and rationally chain connected. We also supplement the cone theorem for quasi-log canonical pairs. More precisely, we prove that every negative extremal ray is spanned by a rational curve. Finally, we treat the notion of Mori hyperbolicity for quasi-log canonical pairs.

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§1. Introduction

Let (X, Δ) be a projective log canonical pair and let W be a minimal log canonical center of (X, Δ). Then we can find an effective $\mathbb{R}$-divisor Δ_W on W such that

$$(K_X + \Delta)|_W \sim_{\mathbb{R}} K_W + \Delta_W$$

and that (W, Δ_W) is a kawamata log terminal pair. This is a famous subadjunction formula for minimal log canonical centers (see [Ka2, Theorem 1] and [FG, Theorem 1.2]) and has already played a very important role in the theory of minimal models for higher-dimensional algebraic varieties. Hence, it is very natural and interesting to consider some useful generalizations. In this paper, we prove a kind of subadjunction formula for (not necessarily minimal) qlc centers of quasi-log canonical pairs. We note that the notion of quasi-log canonical pairs is a generalization of

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that of log canonical pairs. Then we discuss several powerful applications of our new subadjunction formula for quasi-log canonical pairs.

The main purpose of this paper is to establish the following theorem, which we call *subadjunction* for qlc strata. Theorem 1.1 is a generalization of [F6, Corollary 1.10], where we treat only minimal qlc centers. Our proof heavily depends on the structure theorem for normal irreducible quasi-log canonical pairs established in [F6, Theorem 1.7]. We note that it is a consequence of some deep results of the theory of variations of mixed Hodge structure discussed in [FF]. Therefore, Theorem 1.1 is highly nontrivial.

Theorem 1.1 (Subadjunction for qlc strata). *Let $[X, \omega]$ be a quasi-log canonical pair and let W be a qlc stratum of $[X, \omega]$. Let $\nu : W^\nu \to W$ be the normalization. Assume that W^ν is quasi-projective and H is an ample $\mathbb{R}$-divisor on W^ν. Then there exists a boundary $\mathbb{R}$-divisor Δ on W^ν such that*

$$K_{W^\nu} + \Delta \sim_{\mathbb{R}} \nu^*(\omega|_W) + H$$

and that

$$\mathrm{Nklt}(W^\nu, \Delta) = \nu^{-1}\,\mathrm{Nqklt}(W, \omega|_W),$$

where $\mathrm{Nklt}(W^\nu, \Delta)$ *denotes the non-klt locus of* (W^ν, Δ)*. More precisely, the equality*

$$\nu_*\mathcal{J}(W^\nu, \Delta) = \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}$$

holds, where $\mathcal{J}(W^\nu, \Delta)$ is the multiplier ideal sheaf of (W^ν, Δ) and $\mathcal{I}_{\mathrm{Nqklt}(W,\omega|_W)}$ is the defining ideal sheaf of $\mathrm{Nqklt}(W, \omega|_W)$ *on W. Furthermore, if $[X, \omega]$ has a $\mathbb{Q}$-structure and H is an ample $\mathbb{Q}$-divisor on W^ν, then we can make Δ a $\mathbb{Q}$-divisor on W^ν such that*

$$K_{W^\nu} + \Delta \sim_{\mathbb{Q}} \nu^*(\omega|_W) + H$$

holds.

We give two remarks in order to help the reader understand Theorem 1.1.

Remark 1.2. In Theorem 1.1, $[W, \omega|_W]$ naturally becomes a quasi-log canonical pair by adjunction (see [F4, Theorem 6.3.5 (i)]) and $\mathrm{Nqklt}(W, \omega|_W)$ denotes the union of all qlc centers of $[W, \omega|_W]$. By adjunction again (see [F4, Theorem 6.3.5 (i)]),

$$[\mathrm{Nqklt}(W, \omega|_W), \omega|_{\mathrm{Nqklt}(W,\omega|_W)}]$$

becomes a quasi-log canonical pair.

Remark 1.3. By [FLh1, Theorem 1.1], we know that $[W^\nu, \nu^*(\omega|_W)]$ naturally has a quasi-log canonical structure. In the proof of Theorem 1.1, we see that the equality

$$\mathcal{J}(W^\nu, \Delta) = \mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))}$$

holds, where $\mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))}$ is the defining ideal sheaf of $\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))$, the union of all qlc centers of $[W^\nu, \nu^*(\omega|_W)]$, on W.

By combining Theorem 1.1 with [F3, Theorem 1.2], we can easily obtain:

Corollary 1.4 (Subadjunction for slc strata). *Let (X, Δ) be a quasi-projective semi-log canonical pair and let W be an slc stratum of (X, Δ). Let $\nu : W^\nu \to W$ be the normalization and let H be an ample $\mathbb{R}$-divisor on W^ν. Then there exits a boundary $\mathbb{R}$-divisor $\Delta^\dagger$ on W^ν such that*

$$K_{W^\nu} + \Delta^\dagger \sim_{\mathbb{R}} \nu^*\left((K_X + \Delta)|_W\right) + H$$

and that

$$\mathrm{Nklt}(W^\nu, \Delta^\dagger) = \nu^{-1}E,$$

where E is the union of all slc centers of (X, Δ) that are strictly contained in W and $\mathrm{Nklt}(W^\nu, \Delta^\dagger)$ denotes the non-klt locus of $(W^\nu, \Delta^\dagger)$. More precisely, the equality

$$\nu_*\mathcal{J}(W^\nu, \Delta^\dagger) = \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}$$

holds, where $\omega := K_X + \Delta$, $\mathcal{J}(W^\nu, \Delta^\dagger)$ is the multiplier ideal sheaf of $(W^\nu, \Delta^\dagger)$, and $\mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}$ is the defining ideal sheaf of $\mathrm{Nqklt}(W, \omega|_W)$ on W. Note that $[X, \omega]$ naturally becomes a quasi-log canonical pair and that $[W, \omega|_W]$ has a quasi-log canonical structure induced from the natural quasi-log canonical structure of $[X, \omega]$ by adjunction. Furthermore, if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and H is an ample $\mathbb{Q}$-divisor on W^ν, then we can make $\Delta^\dagger$ a $\mathbb{Q}$-divisor on W^ν such that

$$K_{W^\nu} + \Delta^\dagger \sim_{\mathbb{Q}} \nu^*\left((K_X + \Delta)|_W\right) + H$$

holds.

Corollary 1.4 is a very powerful generalization of [Ka2, Theorem 1]. We give a small remark on Corollary 1.4 for the reader's convenience.

Remark 1.5 (see Remark 4.1). If (X, Δ) is log canonical, equivalently, X is normal, in Corollary 1.4, then it is sufficient to assume that W^ν is quasi-projective. We do not need to assume that X is quasi-projective when X is normal in Corollary 1.4.

As an application of Theorem 1.1, we can prove:

Theorem 1.6 (Simply connectedness of qlc Fano pairs). *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $-\omega$ is ample and that X is connected. Then X is simply connected, that is, the topological fundamental group of X is trivial.*

Theorem 1.6, which is a generalization of [FLw, Theorem 0.2], completely confirms a conjecture raised by the author (see [F5, Conjecture 1.3] and Remark 2.10). We can also prove:

Theorem 1.7 (Rationally chain connectedness of qlc Fano pairs). *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $-\omega$ is ample and that X is connected. Then X is rationally chain connected. This means that for arbitrary closed points $x_1, x_2 \in X$ there exists a connected curve $C \subset X$ which contains x_1 and x_2 such that every irreducible component of C is rational.*

Of course, Theorem 1.7 is a generalization of [FLw, Corollary 2.5] by [F3, Theorem 1.2] (see also Theorem 2.9) and adjunction for quasi-log canonical pairs (see [F4, Theorem 6.3.5 (i)]).

From now on, we discuss the cone theorem for quasi-log canonical pairs. Let $[X, \omega]$ be a quasi-log canonical pair and let $\pi : X \to S$ be a projective morphism between schemes. Then it is well known that the cone theorem

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega \geq 0} + \sum R_j$$

holds, where R_j's are the ω-negative extremal rays of the relative Kleiman–Mori cone $\overline{NE}(X/S)$. For the details, see [F4, Theorem 6.7.4]. As an application of Theorem 1.1, we obtain:

Theorem 1.8 (Lengths of extremal rational curves). *Each ω-negative extremal ray R_j is spanned by an integral (possibly singular) rational curve C_j on X such that $\pi(C_j)$ is a point and that $0 < -\omega \cdot C_j \leq 2 \dim X$.*

Theorem 1.8 is a generalization of [Ka1, Theorem 1]. Note that Theorem 1.8 depends on [Ka1, Theorem 1]. More generally, we have:

Theorem 1.9 (Mori hyperbolicity, see [S, Theorems 1.2 and 6.5] and Theorem 7.7). *In Theorem 1.8, the curve C_j can be so taken that there exist a qlc stratum W of $[X, \omega]$ and a non-constant morphism $f : \mathbb{A}^1 \to W \setminus \mathrm{Nqklt}(W, \omega|_W)$ such that $C_j \cap (W \setminus \mathrm{Nqklt}(W, \omega|_W))$ contains $f(\mathbb{A}^1)$.*

We note that Theorem 7.7 is obviously a generalization of [S, Theorems 1.2 and 6.5]. By the proof of Theorem 1.9, we obtain:

Theorem 1.10 (Cone theorem for semi-log canonical pairs, see [F3, Theorem 1.19]). *Let (X, Δ) be a semi-log canonical pair and let $\pi : X \to S$ be a projective morphism onto a scheme S. Then, for each $(K_X + \Delta)$-negative extremal ray R, we can find an slc stratum W of (X, Δ), a non-constant morphism $f : \mathbb{A}^1 \to X$, and a possibly singular rational curve C whose numerical equivalence class spans R such that $f(\mathbb{A}^1) \subset C \cap (W \setminus E)$ holds with $0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$, where E is the union of all slc centers of (X, Δ) that are strictly contained in W.*

In [F7], we will treat the cone theorem for quasi-log schemes which are not necessarily quasi-log canonical.

We summarize the contents of this paper. In Section 2, we recall some basic definitions. In Section 3, we review some important result in [F6], which is the main ingredient of this paper. In Section 4, we prove Theorem 1.1 and Corollary 1.4, that is, subadjunction for qlc strata and slc strata, respectively. In Section 5, we explain how to modify the arguments in [FLw] to prove Theorems 1.6 and 1.7. In Section 6, we discuss lengths of extremal rational curves for qlc pairs (see Theorem 1.8). In Section 7, we treat the notion of Mori hyperbolicity for quasi-log canonical pairs.

Conventions. We work over $\mathbb{C}$, the complex number field, throughout this paper. A *scheme* means a separated scheme of finite type over $\mathbb{C}$. A *variety* means an integral scheme, that is, an irreducible and reduced separated scheme of finite type over $\mathbb{C}$. Let $f : Y \to X$ be a proper birational morphism between varieties. Then $\mathrm{Exc}(f)$ denotes the *exceptional locus* of f. We freely use the basic notation of the minimal model program as in [F2] and [F4]. For the details of the theory of quasi-log schemes, see [F4, Chapter 6]. For the details of semi-log canonical pairs, we recommend the reader to see [F3] and [Kl2].

§2. Preliminaries

In this section, let us briefly recall some basic definitions. For the details, see [F2], [F4], and [Kl1]. We also recommend the reader to see [F6, Section 2] for the theory of quasi-log schemes.

Let us explain singularities of pairs and some related definitions.

Definition 2.1 (Singularities of pairs). A *normal pair* (X, Δ) consists of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a projective birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_*\left(\sum_E a(E,X,\Delta)E\right)=-\Delta,$$

where E runs over prime divisors on Y. We call $a(E,X,\Delta)$ the *discrepancy* of E with respect to (X,Δ). Note that we can define the discrepancy $a(E,X,\Delta)$ for any prime divisor E over X by taking a suitable resolution of singularities of X. If $a(E,X,\Delta)\geq -1$ (resp. >-1) for every prime divisor E over X, then (X,Δ) is called *sub log canonical* (resp. *sub kawamata log terminal*). We further assume that Δ is effective. Then (X,Δ) is called *log canonical* and *kawamata log terminal* if it is sub log canonical and sub kawamata log terminal, respectively.

Let (X,Δ) be a log canonical pair. If there exists a projective birational morphism $f:Y\to X$ from a smooth variety Y such that both $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f)\cup \mathrm{Supp}\, f^{-1}_*\Delta$ are simple normal crossing divisors on Y and that $a(E,X,\Delta)>-1$ holds for every f-exceptional divisor E on Y, then (X,Δ) is called *divisorial log terminal* (*dlt*, for short).

Let (X,Δ) be a normal pair. If there exist a projective birational morphism $f\colon Y\to X$ from a normal variety Y and a prime divisor E on Y such that (X,Δ) is sub log canonical in a neighborhood of the generic point of $f(E)$ and that $a(E,X,\Delta)=-1$, then $f(E)$ is called a *log canonical center* of (X,Δ).

Definition 2.2 (Operations for $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors). Let X be an equidimensional reduced scheme. Note that X is not necessarily regular in codimension one. Let D be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor), that is, D is a finite formal sum $\sum_i d_iD_i$, where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is a real number (resp. a rational number) for every i such that $D_i\neq D_j$ for $i\neq j$. We put

$$D^{<c}=\sum_{d_i<c}d_iD_i,\quad D^{\leq c}=\sum_{d_i\leq c}d_iD_i,\quad D^{=1}=\sum_{d_i=1}D_i,\quad \text{and}\quad \lceil D\rceil=\sum_i\lceil d_i\rceil D_i,$$

where c is any real number and $\lceil d_i\rceil$ is the integer defined by $d_i\leq\lceil d_i\rceil<d_i+1$. Similarly, we put

$$D^{>c}=\sum_{d_i>c}d_iD_i\quad\text{and}\quad D^{\geq c}=\sum_{d_i\geq c}d_iD_i$$

for any real number c. Moreover, we put $\lfloor D\rfloor=-\lceil -D\rceil$ and $\{D\}=D-\lfloor D\rfloor$.

Let D be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor) as above. We call D a *subboundary* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) if $D=D^{\leq 1}$ holds. When D is effective and $D=D^{\leq 1}$ holds, we call D a *boundary* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor).

Let Δ_1 and Δ_2 be $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) divisors on X. Then $\Delta_1\sim_{\mathbb{R}}\Delta_2$ (resp. $\Delta_1\sim_{\mathbb{Q}}\Delta_2$) means that Δ_1 is $\mathbb{R}$-linearly (resp. $\mathbb{Q}$-linearly) equivalent to Δ_2.

In this paper, we need the notion of *multiplier ideal sheaves*. Although it is well known, we recall it here for the reader's convenience.

Definition 2.3 (Multiplier ideal sheaves and non-lc ideal sheaves). Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution with

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

such that $\operatorname{Supp}\Delta_Y$ is a simple normal crossing divisor on Y. We put

$$\mathcal{J}(X, \Delta) = f_*\mathcal{O}_Y(-\lfloor\Delta_Y\rfloor).$$

Then $\mathcal{J}(X, \Delta)$ is an ideal sheaf on X and is known as the *multiplier ideal sheaf* associated to the pair (X, Δ). It is independent of the resolution $f : Y \to X$. The closed subscheme $\operatorname{Nklt}(X, \Delta)$ defined by $\mathcal{J}(X, \Delta)$ is called the *non-klt locus* of (X, Δ). It is obvious that (X, Δ) is kawamata log terminal if and only if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$. Similarly, we put

$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_*\mathcal{O}_X(-\lfloor\Delta_Y\rfloor + \Delta_Y^{=1})$$

and call it the *non-lc ideal sheaf* associated to the pair (X, Δ). We can check that it is independent of the resolution $f : Y \to X$. The closed subscheme $\operatorname{Nlc}(X, \Delta)$ defined by $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ is called the *non-lc locus* of (X, Δ). It is obvious that (X, Δ) is log canonical if and only if $\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = \mathcal{O}_X$.

By definition, the natural inclusion

$$\mathcal{J}(X, \Delta) \subset \mathcal{J}_{\mathrm{NLC}}(X, \Delta)$$

always holds. Therefore, we have

$$\operatorname{Nlc}(X, \Delta) \subset \operatorname{Nklt}(X, \Delta).$$

For the details of $\mathcal{J}(X, \Delta)$ and $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$, see [F1], [F2, Section 7], and [L, Chapter 9].

Definition 2.4 (Semi-log canonical pairs). Let X be an equidimensional scheme which satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor on X such that no irreducible component of $\operatorname{Supp}\Delta$ is contained in the singular locus of X and that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is a *semi-log canonical* pair if (X^ν, Δ_{X^ν}) is log canonical in the usual sense, where $\nu : X^\nu \to X$ is the normalization of X and $K_{X^\nu} + \Delta_{X^\nu} = \nu^*(K_X + \Delta)$, that is, Δ_{X^ν} is the sum of the inverse images of Δ and the conductor of X. An *slc center* of (X, Δ) is the ν-image of a log canonical center of (X^ν, Δ_{X^ν}). An *slc* *stratum* of (X, Δ) means either an slc center of (X, Δ) or an irreducible component of X.

We need the notion of *globally embedded simple normal crossing pairs* for the theory of quasi-log schemes described in [F4, Chapter 6].

Definition 2.5 (Globally embedded simple normal crossing pairs). Let Z be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that Z and B have no common irreducible components and that the support of $Z + B$ is a simple normal crossing divisor on M. In this situation, $(Z, B|_Z)$ is called a *globally embedded simple normal crossing pair*.

Let us quickly recall the definition of *quasi-log canonical pairs*.

Definition 2.6 (Quasi-log canonical pairs). Let X be a scheme and let ω be an $\mathbb{R}$-Cartier divisor (or an $\mathbb{R}$-line bundle) on X. Let $f : Z \to X$ be a proper morphism from a globally embedded simple normal crossing pair (Z, Δ_Z). If the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Z(\lceil -(\Delta_Z^{<1})\rceil)$$

is an isomorphism, Δ_Z is a subboundary $\mathbb{R}$-divisor, and $f^*\omega \sim_{\mathbb{R}} K_Z + \Delta_Z$ holds, then

$$(X, \omega, f : (Z, \Delta_Z) \to X)$$

is called a *quasi-log canonical pair* (*qlc pair*, for short). If there is no danger of confusion, we simply say that $[X, \omega]$ is a qlc pair. We usually call ω the *quasi-log canonical class* of $[X, \omega]$.

We say that $(X, \omega, f : (Z, \Delta_Z) \to X)$ or $[X, \omega]$ has a $\mathbb{Q}$-*structure* if Δ_Z is a $\mathbb{Q}$-divisor, ω is a $\mathbb{Q}$-Cartier divisor (or a $\mathbb{Q}$-line bundle), and $f^*\omega \sim_{\mathbb{Q}} K_Z + \Delta_Z$ holds in the above definition.

We can define *qlc Fano pairs* as follows.

Definition 2.7 (Qlc Fano pairs). Let $[X, \omega]$ be a projective qlc pair such that $-\omega$ is ample. Then we simply say that $[X, \omega]$ is a *qlc Fano pair*.

The notion of *qlc strata* plays a crucial role in the theory of quasi-log schemes.

Definition 2.8 (Qlc strata and qlc centers). Let $(X, \omega, f : (Z, \Delta_Z) \to X)$ be a quasi-log canonical pair as in Definition 2.6. Let $\nu : Z^\nu \to Z$ be the normalization. We put

$$K_{Z^\nu} + \Theta = \nu^*(K_Z + \Delta_Z),$$

that is, Θ is the sum of the inverse images of Δ_Z and the singular locus of Z. Then (Z^ν, Θ) is sub log canonical. Let W be a log canonical center of (Z^ν, Θ)

or an irreducible component of Z^ν. Then $f \circ \nu(W)$ is called a *qlc stratum* of $(X, \omega, f : (Z, \Delta_Z) \to X)$. If there is no danger of confusion, we simply call it a qlc stratum of $[X, \omega]$. If C is a qlc stratum of $[X, \omega]$ but is not an irreducible component of X, then C is called a *qlc center* of $[X, \omega]$. The union of all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega)$ (see [F4, Notation 6.3.10]). It is important that

$$[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}]$$

naturally has a quasi-log canonical structure induced from $(X, \omega, f : (Z, \Delta_Z) \to X)$ by adjunction (see [F4, Theorem 6.3.5 (i)]).

We recall the main result of [F3], which makes the theory of quasi-log schemes (see [F4, Chapter 6]) useful for the study of semi-log canonical pairs.

Theorem 2.9 ([F3, Theorem 1.2]). *Let (X, Δ) be a quasi-projective semi-log canonical pair. Then $[X, K_X + \Delta]$ becomes a quasi-log canonical pair such that W is an slc stratum of (X, Δ) if and only if W is a qlc stratum of $[X, K_X + \Delta]$.*

For the details of Theorem 2.9, we recommend the reader to see [F3].

Remark 2.10. By combining Theorem 2.9 with adjunction for quasi-log canonical pairs (see [F4, Theorem 6.3.5 (i)]), we see that any union V of slc strata of a given quasi-projective semi-log canonical pair (X, Δ) becomes a quasi-log canonical pair, that is, $[V, (K_X + \Delta)|_V]$ is a quasi-log canonical pair.

We collect some basic properties of qlc strata for the reader's convenience.

Proposition 2.11 (Basic properties of qlc strata). *Let $[X, \omega]$ be a quasi-log canonical pair. Then its qlc strata have the following nice properties.*

(i) *there is a unique minimal (with respect to the inclusion) qlc stratum through a given point,*
(ii) *the minimal qlc stratum at a given point is normal at that point, and*
(iii) *the intersection of two qlc strata is a union of qlc strata.*

If X is additionally a connected projective scheme and $-\omega$ is ample, that is, $[X, \omega]$ is a connected qlc Fano pair, then

(iv) *any union of qlc strata of $[X, \omega]$ is connected, and*
(v) *there is a unique minimal qlc stratum of $[X, \omega]$, which is normal.*

Sketch of Proof of Proposition 2.11. For (i), (ii), and (iii), see [F4, Theorem 6.3.11]. For (iv), it is sufficient to show that $H^0(V, \mathcal{O}_V) = \mathbb{C}$ for any union V of qlc strata of $[X, \omega]$. Since $-\omega$ is ample, we have $H^1(X, \mathcal{I}_V) = 0$, where $\mathcal{I}_V$ is the defining ideal sheaf of V on X, by [F4, Theorem 6.3.5 (ii)]. Therefore, the surjection

$$\mathbb{C} = H^0(X, \mathcal{O}_X) \to H^0(V, \mathcal{O}_V) \to 0$$

implies $H^0(V, \mathcal{O}_V) = \mathbb{C}$. Finally, we note that (v) is a direct consequence of (i), (ii), (iii) and (iv). □

For the details of the theory of quasi-log schemes, we recommend the reader to see [F4, Chapter 6].

We close this section with the definition of *rationally chain connected schemes*.

Definition 2.12 (Rationally chain connected schemes). A projective scheme X is *rationally chain connected* if and only if for arbitrary closed points $x_1, x_2 \in X$ there exists a connected curve $C \subset X$ which contains x_1 and x_2 such that every irreducible component of C is rational.

For the details of rationally chain connected schemes and various related topics, see [Kl1].

§3. Quick review of [F6]

In this section, we quickly look at the structure theorem for normal irreducible quasi-log canonical pairs obtained in [F6]. Theorem 3.2 is the main ingredient of this paper.

Let us recall the definition of *potentially nef* divisors in order to explain Theorem 3.2.

Definition 3.1 (Potentially nef divisors). Let X be a normal variety and let D be a divisor on X. If there exist a completion $\overline{X}$ of X, that is, $\overline{X}$ is a normal complete variety and contains X as a dense Zariski open set, and a nef divisor $\overline{D}$ on $\overline{X}$ such that $D = \overline{D}|_X$, then D is called a *potentially nef* divisor on X. A finite $\mathbb{R}_{>0}$-linear (resp. $\mathbb{Q}_{>0}$-linear) combination of potentially nef divisors is called a *potentially nef* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor).

For the basic properties of potentially nef divisors, we recommend the reader to see [F6, Section 2].

The following theorem will play a crucial role in the theory of quasi-log schemes (see [F6], [FLh2], and [FLh3]).

Theorem 3.2 (Structure theorem for normal irreducible quasi-log canonical pairs, see [F6, Theorem 1.7]). *Let $[X, \omega]$ be a quasi-log canonical pair such that X is a normal variety. Then there exists a projective birational morphism $p : X' \to X$ from a smooth quasi-projective variety X' such that*

$$K_{X'} + B_{X'} + M_{X'} = p^*\omega,$$

where $B_{X'}$ is a subboundary $\mathbb{R}$-divisor such that $\operatorname{Supp} B_{X'}$ is a simple normal crossing divisor and that $B_{X'}^{<0}$ is p-exceptional, and $M_{X'}$ is a potentially nef $\mathbb{R}$-divisor on X'. Furthermore, we can make $B_{X'}$ satisfy $p(B_{X'}^{=1}) = \mathrm{Nqklt}(X, \omega)$.

We further assume that $[X, \omega]$ has a $\mathbb{Q}$-structure. Then we can make $B_{X'}$ and $M_{X'}$ $\mathbb{Q}$-divisors in the above statement.

In [F6], we introduce the notion of *basic slc-trivial fibrations*, which is a kind of canonical bundle formula for reducible schemes. Then we prove some fundamental properties by using the theory of variations of mixed Hodge structure on cohomology with compact support (see [FF] and [FFS]). Theorem 3.2 (see [F6, Theorem 1.7]) is an application of the main result of [F6], that is, [F6, Theorem 1.2].

§4. Subadjunction for qlc pairs

In this section, we prove Theorem 1.1, which is a direct consequence of Theorem 3.2. We note that Corollary 1.4 follows from Theorems 1.1 and 2.9 (see [F3, Theorem 1.2]).

Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1. By adjunction (see [F4, Theorem 6.3.5 (i)]), $[W, \omega|_W]$ is a quasi-log canonical pair. By [FLh1, Theorem 1.1], we see that $[W^\nu, \nu^*(\omega|_W)]$ naturally becomes a quasi-log canonical pair such that $\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W)) = \nu^{-1}\mathrm{Nqklt}(W, \omega|_W)$ holds. More precisely, we obtain that the equality

$$\nu_*\mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))} = \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}$$

holds. Note that $\mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}$ is the defining ideal sheaf of $\mathrm{Nqklt}(W, \omega|_W)$ on W and $\mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))}$ is that of $\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))$ on W^ν. By Theorem 3.2, there is a projective birational morphism $p : W' \to W^\nu$ from a smooth quasi-projective variety W' such that

$$K_{W'} + B_{W'} + M_{W'} = p^*\nu^*(\omega|_W),$$

where $B_{W'}$ is a subboundary $\mathbb{R}$-divisor on W' whose support is a simple normal crossing divisor, $B_{W'}^{<0}$ is p-exceptional, $M_{W'}$ is a potentially nef $\mathbb{R}$-divisor on W', and $p(B^{=1}_{W'}) = \mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))$. We may further assume that there is an effective p-exceptional divisor F on W' such that $-F$ is p-ample and that $\mathrm{Supp}\, F \cup \mathrm{Supp}\, B_{W'}$ is contained in a simple normal crossing divisor on W'. Then $p^*H - \varepsilon F + M_{W'}$ is semi-ample for any $0 < \varepsilon \ll 1$. We take a general effective $\mathbb{R}$-divisor G on W' such that $G \sim_{\mathbb{R}} p^*H - \varepsilon F + M_{W'}$ with $0 < \varepsilon \ll 1$, $\mathrm{Supp}\, G \cup \mathrm{Supp}\, B_{W'} \cup \mathrm{Supp}\, F$ is contained in a simple normal crossing divisor on W', and $\lfloor (B_{W'} + \varepsilon F + G)^{\geq 1} \rfloor = B^{=1}_{W'}$. Then we have

$$\begin{aligned} K_{W'} + B_{W'} + M_{W'} + p^*H &= K_{W'} + B_{W'} + \varepsilon F + p^*H - \varepsilon F + M_{W'} \\ &\sim_{\mathbb{R}} K_{W'} + B_{W'} + \varepsilon F + G. \end{aligned}$$

We put $\Delta := p_*(B_{W'} + \varepsilon F + G)$. By construction, $K_{W^\nu} + \Delta \sim_{\mathbb{R}} \nu^*(\omega|_W) + H$. Let $\mathcal{J}(W^\nu, \Delta)$ be the multiplier ideal sheaf of (W^ν, Δ). Then $\mathcal{J}(W^\nu, \Delta) = p_*\mathcal{O}_{W'}(-\lfloor B_{W'} + \varepsilon F + G \rfloor)$ by definition (see Definition 2.3). Since the effective part of $-\lfloor B_{W'} + \varepsilon F + G \rfloor$ is p-exceptional, we obtain

$$\begin{aligned} \mathcal{J}(W^\nu, \Delta) &= p_*\mathcal{O}_{W'}(-\lfloor B_{W'} + \varepsilon F + G \rfloor) \\ &= p_*\mathcal{O}_{W'}(-\lfloor (B_{W'} + \varepsilon F + G)^{\geq 1} \rfloor) \\ &= p_*\mathcal{O}_{W'}(-B^{=1}_{W'}) \\ &= \mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))}. \end{aligned} \tag{4.1}$$

As we saw above, by [FLh1, Theorem 1.1], we have the equality

$$\nu_*\mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))} = \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}. \tag{4.2}$$

Therefore, we obtain

$$\nu_*\mathcal{J}(W^\nu, \Delta) = \nu_*\mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))} = \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}$$

by (4.1) and (4.2). Thus we get

$$\mathcal{J}(W^\nu, \Delta) = \mathcal{I}_{\mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W))} = \nu^{-1}\mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)} \cdot \mathcal{O}_{W^\nu}.$$

This implies that

$$\mathrm{Nklt}(W^\nu, \Delta) = \mathrm{Nqklt}(W^\nu, \nu^*(\omega|_W)) = \nu^{-1}\, \mathrm{Nqklt}(W, \omega|_W)$$

holds. This is what we wanted.

When $[X, \omega]$ has a $\mathbb{Q}$-structure, we can make $B_{W'}$ and $M_{W'}$ $\mathbb{Q}$-divisors by Theorem 3.2. Then it is easy to see that we can make Δ a $\mathbb{Q}$-divisor on W^ν such that $K_{W^\nu} + \Delta \sim_{\mathbb{Q}} \nu^*(\omega|_W) + H$ if H is an ample $\mathbb{Q}$-divisor and $[X, \omega]$ has a $\mathbb{Q}$-structure by the above construction of Δ. □

Corollary 1.4 easily follows from Theorems 1.1 and 2.9 (see [F3, Theorem 1.2]).

Proof of Corollary 1.4. By Theorem 2.9 (see [F3, Theorem 1.2]), $[X, K_X+\Delta]$ has a natural quasi-log canonical structure which is compatible with the original semi-log canonical structure of (X, Δ). Then, by adjunction (see [F4, Theorem 6.3.5 (i)]), $[W, (K_X+\Delta)|_W]$ is quasi-log canonical such that $\mathrm{Nqklt}(W, (K_X+\Delta)|_W) = E$ (see Remark 2.10). By Theorem 1.1, we can take an effective $\mathbb{R}$-divisor $\Delta^\dagger$ on W^ν such that

$$K_{W^\nu} + \Delta^\dagger \sim_{\mathbb{R}} \nu^* \left((K_X + \Delta)|_W\right) + H$$

and that $\mathrm{Nklt}(W^\nu, \Delta^\dagger) = \nu^{-1}E$. More precisely,

$$\nu_* \mathcal{J}(W^\nu, \Delta^\dagger) = \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}$$

holds. Of course, by Theorem 1.1, we can make $\Delta^\dagger$ a $\mathbb{Q}$-divisor with

$$K_{W^\nu} + \Delta^\dagger \sim_{\mathbb{Q}} \nu^* \left((K_X + \Delta)|_W\right) + H$$

if $K_X + \Delta$ and H are both $\mathbb{Q}$-divisors. □

Remark 4.1. In Corollary 1.4, if (X, Δ) is log canonical, that is, X is normal, then we do not need the assumption that X is quasi-projective. This is because $[X, \omega]$, where $\omega := K_X+\Delta$, always has a natural quasi-log canonical structure that is compatible with the original log canonical structure of (X, Δ) (see [F4, 6.4.1]). We do not need the quasi-projectivity of X to construct the quasi-log canonical structure on $[X, \omega]$ when X is normal. When X is not normal in Corollary 1.4, we need the quasi-projectivity of X to use Theorem 2.9 (see [F3, Theorem 1.2]).

§5. On qlc Fano pairs

In this short section, we explain how to modify the arguments in [FLw] to prove Theorems 1.6 and 1.7. Since this section is independent of the other sections, the reader can skip it if he or she is not interested in qlc Fano pairs.

We prepare an important lemma, which is an easy application of Theorem 1.1.

Lemma 5.1 (see [FLw, Lemmas 2.3 and 2.6]). *Let W be a qlc stratum of a connected qlc Fano pair $[X, \omega]$ and let E be the union of all qlc strata that are strictly contained in W. We take the normalization $\nu : W^\nu \to W$ of W. Let H be an ample Cartier divisor on X and let ε be a sufficiently small positive real number. Then there exists a boundary $\mathbb{R}$-divisor Δ on W^ν such that $K_{W^\nu} + \Delta \sim_{\mathbb{R}}$*

$\nu^*((\omega+\varepsilon H)|_W)$, $\mathrm{Nklt}(W^\nu, \Delta) = \nu^{-1}E$, *and* $-(K_{W^\nu}+\Delta)$ *is ample. We note that* $\mathrm{Nklt}(W^\nu, \Delta)$ *is connected since* $-(K_{W^\nu}+\Delta)$ *is ample.*

Proof. We note that $-(\omega+\varepsilon H)$ is ample for any sufficiently small positive real number ε. By Theorem 1.1, we can take a boundary $\mathbb{R}$-divisor Δ on W^ν with $K_{W^\nu}+\Delta \sim_{\mathbb{R}} \nu^*((\omega+\varepsilon H)|_W)$ and $\mathrm{Nklt}(W^\nu, \Delta) = \nu^{-1}E$. By the Nadel vanishing theorem (see [F4, Theorem 3.4.2]), we have $H^1(W^\nu, \mathcal{J}(W^\nu, \Delta)) = 0$, where $\mathcal{J}(W^\nu, \Delta)$ is the multiplier ideal sheaf of (W^ν, Δ), since $-(K_{W^\nu}+\Delta)$ is ample. This implies that $\mathrm{Nklt}(W^\nu, \Delta) = \nu^{-1}E$ is connected. □

By the standard argument in the recent developments of the theory of higher-dimensional minimal models, we have the following lemma.

Lemma 5.2. *Let V be a normal projective variety and let Δ be a boundary $\mathbb{R}$-divisor on V such that $-(K_V+\Delta)$ is ample. Then we can take a boundary $\mathbb{Q}$-divisor Δ' on V such that $-(K_V+\Delta')$ is ample and that the equality $\mathcal{J}(V, \Delta') = \mathcal{J}(V, \Delta)$ holds, where $\mathcal{J}(V, \Delta)$ (resp. $\mathcal{J}(V, \Delta')$) is the multiplier ideal sheaf of (V, Δ) (resp. (V, Δ')). Moreover, we can choose Δ' such that* $\mathrm{mult}_P \Delta' = \mathrm{mult}_P \Delta$ *holds for any prime divisor P on V with* $\mathrm{mult}_P \Delta \in \mathbb{Q}$.

Proof. By slightly perturbing the coefficients of Δ, we get a boundary $\mathbb{Q}$-divisor Δ' with the desired properties. We leave the details as an exercise for the reader. □

Since many results were formulated and stated only for $\mathbb{Q}$-divisors in the literature, Lemma 5.2 is useful and helpful.

By Lemmas 5.1 and 5.2, the proof of [FLw, Corollary 2.5 and Theorem 2.7] works with some minor modifications.

Sketch of Proof of Theorems 1.6 and 1.7. Let W_0 be the unique minimal qlc stratum of $[X, \omega]$ (see Proposition 2.11 (v)). Then we can take a boundary $\mathbb{Q}$-divisor Δ_0 on W_0 such that (W_0, Δ_0) is kawamata log terminal and $-(K_{W_0}+\Delta_0)$ is ample by Lemmas 5.1 and 5.2. Thus it is well known that W_0 is rationally (chain) connected and simply connected (see, for example, [FLw, Corollary 2.4]).

Let W be any qlc stratum of $[X, \omega]$. By Lemmas 5.1 and 5.2, the proof of [FLw, Corollary 2.5] works with some minor modifications. Therefore, we obtain that X is rationally chain connected. Hence we get Theorem 1.7.

By Lemmas 5.1 and 5.2 again, we can easily see that the proof of [FLw, Theorem 2.7] works with some minor changes. Hence we see that X is simply connected. This is Theorem 1.6. □

§6. Lengths of extremal rational curves for qlc pairs

In this section, we prove Theorem 1.8, which is a generalization of [Ka1, Theorem 1]. Our proof of Theorem 1.8 below heavily depends on [F4, Theorem 4.6.7].

Let us start the proof of Theorem 1.8. Although we will treat a more general result in Section 7, the proof of Theorem 1.8 plays a crucial role.

Proof of Theorem 1.8. Let $\varphi_{R_j} : X \to Y$ be the extremal contraction associated to R_j (see [F4, Theorems 6.7.3 and 6.7.4]). By replacing $\pi : X \to S$ with $\varphi_{R_j} : X \to Y$, we may assume that $-\omega$ is π-ample. We take a qlc stratum W of $[X, \omega]$ such that $\pi : \mathrm{Nqklt}(W, \omega|_W) \to \pi\left(\mathrm{Nqklt}(W, \omega|_W)\right)$ is finite and that $\pi : W \to \pi(W)$ is not finite. It is sufficient to find a rational curve C on W such that $\pi(C)$ is a point and that $0 < -(\omega|_W) \cdot C \leq 2 \dim W \leq 2 \dim X$. Therefore, by replacing $\pi : X \to S$ with $\pi : W \to S$, we may assume that X is irreducible and that $\pi : \mathrm{Nqklt}(X, \omega) \to \pi\left(\mathrm{Nqklt}(X, \omega)\right)$ is finite. Let $\nu : X^\nu \to X$ be the normalization. Then, by [FLh1, Theorem 1.1], $[X^\nu, \nu^*\omega]$ naturally becomes a quasi-log canonical pair with $\mathrm{Nqklt}(X^\nu, \nu^*\omega) = \nu^{-1}\mathrm{Nqklt}(X, \omega)$. Therefore, by replacing $\pi : X \to S$ with $\pi \circ \nu : X^\nu \to S$, we may assume that X is a normal variety such that $\pi : \mathrm{Nqklt}(X, \omega) \to \pi\left(\mathrm{Nqklt}(X, \omega)\right)$ is finite. In this situation, all we have to do is to find a rational curve C on X such that $\pi(C)$ is a point and that $0 < -\omega \cdot C \leq 2 \dim X$. Without loss of generality, we may assume that X and S are quasi-projective by shrinking S suitably. Let H be an ample Cartier divisor on X. By Theorem 1.1, we can construct a boundary $\mathbb{R}$-divisor Δ_ε on X such that $K_X + \Delta_\varepsilon \sim_{\mathbb{R}} \omega + \varepsilon H$ and that $\mathrm{Nklt}(X, \Delta_\varepsilon) = \mathrm{Nqklt}(X, \omega)$ for every positive real number ε. Note that $\mathrm{Nlc}(X, \Delta_\varepsilon) \subset \mathrm{Nklt}(X, \Delta_\varepsilon) = \mathrm{Nqklt}(X, \omega)$, where $\mathrm{Nlc}(X, \Delta_\varepsilon)$ denotes the non-lc locus of (X, Δ_ε) as in Definition 2.3. Therefore, $\pi : \mathrm{Nlc}(X, \Delta_\varepsilon) \to \pi\left(\mathrm{Nlc}(X, \Delta_\varepsilon)\right)$ is finite. We assume that ε is sufficiently small such that $-(\omega + \varepsilon H)$ is π-ample. Then, by the cone theorem for (X, Δ_ε), we can find a rational curve C_ε on X such that $\pi(C_\varepsilon)$ is a point and that $0 < -(K_X + \Delta_\varepsilon) \cdot C_\varepsilon \leq 2 \dim X$ (see [F2, Theorem 1.1] and [F4, Theorem 4.6.7]). We take an ample $\mathbb{Q}$-divisor A on X such that $-(\omega + A)$ is π-ample. We take $\{\varepsilon_i\}_{i=0}^{\infty}$ such that $\lim_{i \to 0} \varepsilon_i = 0$, ε_i is a positive real number, and $-(\omega + A + \varepsilon_i H)$ is π-ample for every i. As we saw above, we can take a rational curve C_i on X such that $\pi(C_i)$ is a point and that $0 < -(\omega + \varepsilon_i H) \cdot C_i \leq 2 \dim X$ for every i. Note that

$$0 < A \cdot C_i = ((\omega + \varepsilon_i H + A) - (\omega + \varepsilon_i H)) \cdot C_i < 2 \dim X.$$

It follows that the curves C_i belong to a bounded family. Thus, possibly passing to a subsequence, we may assume that $C_i = C$ is constant. Therefore, we get

$$0 < -\omega \cdot C = \lim_{i \to \infty} -(\omega + \varepsilon_i H) \cdot C = \lim_{i \to \infty} -(\omega + \varepsilon_i H) \cdot C_i \leq 2 \dim X.$$

This is what we wanted. □

Remark 6.1. We expect that the estimate $\leq 2\dim X$ should be replaced by $\leq \dim X + 1$ in Theorem 1.8 (see [F4, Remark 4.6.3]).

We give a remark on the proof of [F4, Theorem 4.6.7], which was used in the proof of Theorem 1.8 above.

Remark 6.2. In the proof of [F4, Theorem 4.6.7], the author claims that π is an isomorphism in a neighborhood of $\mathrm{Nlc}(X, \Delta)$ by replacing $\pi : X \to S$ with the extremal contraction $\varphi_R : X \to Y$ over S. However, it is not correct. In general, π is not necessarily an isomorphism around $\mathrm{Nlc}(X, \Delta)$ (see Example 6.3 below). By replacing $\pi : X \to S$ with $\varphi_R : X \to Y$, we can assume that $\pi : \mathrm{Nlc}(X, \Delta) \to \pi(\mathrm{Nlc}(X, \Delta))$ is finite. Note that all we need in the proof of [F4, Theorem 4.6.7] is the fact that π contracts no curves in $\mathrm{Nlc}(X, \Delta)$. Therefore, the proof of [F4, Theorem 4.6.7] works without any modifications.

Example 6.3. We put $X = \mathbb{P}^1$, $\pi : X \to S = \operatorname{Spec} \mathbb{C}$, and $\Delta = \frac{3}{2}P$, where P is a point of $X = \mathbb{P}^1$. Then $-(K_X + \Delta)$ is π-ample and $\rho(X/S) = 1$. Of course, π is not an isomorphism around $P = \mathrm{Nlc}(X, \Delta)$.

We close this section with an important remark.

Remark 6.4. The proof of [F4, Theorem 4.6.7] needs Mori's bend and break technique to create rational curves (see [F4, Remark 4.6.4]). Therefore, we need the mod p reduction technique for the proof of Theorem 1.8. We note that we take a dlt blow-up (see [F4, Theorem 4.4.21]) in the proof of [F4, Theorem 4.6.7]. This means that Theorem 1.8 depends on the minimal model program mainly due to [BCHM].

§7. Mori hyperbolicity for quasi-log canonical pairs

In this final section, we generalize the main result of [S] for quasi-log canonical pairs. We note that in [S] the notion of *crepant log structures*, which is a very special case of that of quasi-log schemes, plays a crucial role. On the other hand, we can directly treat highly singular reducible schemes by the framework of quasi-log schemes (see [F4, Chapter 6]) and basic slc-trivial fibrations (see [F6]). This is the main difference between [S] and our approach here.

Let us start with the following key result.

Proposition 7.1 ([S, Proposition 5.2]). *Let $\pi : X \to S$ be a projective morphism from a normal $\mathbb{Q}$-factorial variety X onto a scheme S. Let $\Delta = \sum_i d_i D_i$ be an* *effective* $\mathbb{R}$*-divisor on* X*, where the* D_i*'s are the distinct prime components of* Δ *for all* i*, such that*

$$\left(X, \Delta' := \sum_{d_i<1} d_i D_i + \sum_{d_i \geq 1} D_i\right)$$

is dlt. Assume that $(K_X + \Delta)|_{\mathrm{Nklt}(X,\Delta)}$ *is nef over* S*. Then* $K_X + \Delta$ *is nef over* S *or there exists a non-constant morphism* $f : \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(X, \Delta)$ *such that* $\pi \circ f(\mathbb{A}^1)$ *is a point.*

More precisely, the curve C*, the closure of* $f(\mathbb{A}^1)$ *in* X*, is a* (*possibly singular*) *rational curve with*

$$0 < -(K_X + \Delta) \cdot C \leq 2 \dim X.$$

This is one of the most important results of [S]. We give a detailed proof for the sake of completeness.

Proof of Proposition 7.1. Note that $\mathrm{Nklt}(X, \Delta)$ coincides with $(\Delta')^{=1} = \lfloor \Delta' \rfloor$, $\Delta^{\geq 1}$, and $\lfloor \Delta \rfloor$ set theoretically because (X, Δ') is dlt by assumption. It is sufficient to construct a non-constant morphism $f : \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(X, \Delta)$ such that $\pi \circ f(\mathbb{A}^1)$ is a point with the desired properties when $K_X + \Delta$ is not nef over S. By shrinking S suitably, we may assume that S and X are both quasi-projective. By the cone and contraction theorem (see [F2, Theorem 1.1]), we can take a $(K_X + \Delta)$-negative extremal ray R of $\overline{NE}(X/S)$ and the associated extremal contraction morphism $\varphi := \varphi_R : X \to Y$ over S since $(K_X + \Delta)|_{\mathrm{Nklt}(X,\Delta)}$ is nef over S. Note that $(K_X + \Delta^{<1}) \cdot R < 0$ and $(K_X + \Delta') \cdot R < 0$ hold because $(K_X + \Delta)|_{\mathrm{Nklt}(X,\Delta)}$ is nef over S. Since $(X, \Delta^{<1})$ is kawamata log terminal and $-(K_X + \Delta^{<1})$ is φ-ample, we get $R^i \varphi_* \mathcal{O}_X = 0$ for every $i > 0$ by the relative Kawamata–Viehweg vanishing theorem (see [F4, Corollary 5.7.7]). By construction, $\varphi : \mathrm{Nklt}(X, \Delta) \to \varphi(\mathrm{Nklt}(X, \Delta))$ is finite. We have the following short exact sequence

$$0 \to \mathcal{O}_X(-\lfloor \Delta' \rfloor) \to \mathcal{O}_X \to \mathcal{O}_{\lfloor \Delta' \rfloor} \to 0.$$

Since $-\lfloor \Delta' \rfloor - (K_X + \{\Delta'\}) = -(K_X + \Delta')$ is φ-ample and $(X, \{\Delta'\})$ is kawamata log terminal, $R^i \varphi_* \mathcal{O}_X(-\lfloor \Delta' \rfloor) = 0$ holds for every $i > 0$ by the relative Kawamata–Viehweg vanishing theorem again (see [F4, Corollary 5.7.7]). Therefore,

$$0 \to \varphi_* \mathcal{O}_X(-\lfloor \Delta' \rfloor) \to \mathcal{O}_Y \to \varphi_* \mathcal{O}_{\lfloor \Delta' \rfloor} \to 0$$

is exact. This implies that $\mathrm{Supp} \lfloor \Delta' \rfloor = \mathrm{Supp}\, \Delta^{\geq 1}$ is connected in a neighborhood of any fiber of φ.

Case 1. Assume that φ is a Fano contraction, that is, $\dim Y < \dim X$. Then we see that $\Delta^{\geq 1}$ is φ-ample and that $\dim Y = \dim X - 1$. Note that $\mathrm{Supp}\, \Delta^{\geq 1}$ is finite over Y. In this situation, we can easily see that every general fiber is $\mathbb{P}^1$ by $R^1\varphi_*\mathcal{O}_X = 0$. Moreover, any general fiber intersects $\Delta^{\geq 1}$ in at most one point by the connectedness of $\operatorname{Supp}\Delta^{\geq 1}$ discussed above. Therefore, we can find a non-constant morphism $f : \mathbb{A}^1 \to X \setminus \operatorname{Nklt}(X, \Delta)$ such that $\pi \circ f(\mathbb{A}^1)$ is a point and that $0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$ holds, where C is the closure of $f(\mathbb{A}^1)$ in X.

Case 2. Assume that φ is a birational contraction and that the exceptional locus $\operatorname{Exc}(\varphi)$ of φ is disjoint from $\operatorname{Nklt}(X, \Delta)$. In this situation, we can find a rational curve C in a fiber of φ with $0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$ by the cone theorem (see [F2, Theorem 1.1]). It is obviously disjoint from $\operatorname{Nklt}(X, \Delta)$. Therefore, we can take a non-constant morphism $f : \mathbb{A}^1 \to X \setminus \operatorname{Nklt}(X, \Delta)$ such that the closure of $f(\mathbb{A}^1)$ is C.

Case 3. Assume that φ is a birational contraction and that $\operatorname{Exc}(\varphi) \cap \operatorname{Nklt}(X, \Delta) \neq \emptyset$. In this situation, as in Case 1, we see that $\Delta^{\geq 1}$ is φ-ample and that $\dim \varphi^{-1}(y) \leq 1$ for every $y \in Y$. By taking a complete intersection of general hypersurfaces of Y and its inverse image, we can reduce the problem to the case where $\varphi(\operatorname{Exc}(\varphi)) =: P$ is a point. Then $R^1\varphi_*\mathcal{O}_X = 0$ implies that every irreducible component of $\varphi^{-1}(P)$ is $\mathbb{P}^1$. We take any irreducible component C of $\varphi^{-1}(P)$. By the connectedness of $\operatorname{Supp}\Delta^{\geq 1}$ discussed above, C intersects $\Delta^{\geq 1}$ in at most one point. Therefore, we can get a non-constant morphism $f : \mathbb{A}^1 \to X \setminus \operatorname{Nklt}(X, \Delta)$ such that $f(\mathbb{A}^1) \subset C \cap (X \setminus \operatorname{Nklt}(X, \Delta))$. By applying the cone theorem (see [F2, Theorem 1.1]) to $\varphi : X \to Y$, we may assume that $0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$.

Therefore, we get the desired statement. □

Let us recall the following useful lemma, which is a kind of dlt blow-ups. Here we need the minimal model theory mainly due to [BCHM].

Lemma 7.2 ([S, Theorem 3.4]). *Let X be a normal quasi-projective variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then we can construct a projective birational morphism $g : Y \to X$ from a normal $\mathbb{Q}$-factorial variety Y with the following properties.*

(i) $K_Y + \Delta_Y := g^*(K_X + \Delta)$,

(ii) *the pair*

$$\left(Y, \Delta'_Y := \sum_{d_i < 1} d_i D_i + \sum_{d_i \geq 1} D_i\right)$$

is dlt, where $\Delta_Y = \sum_i d_i D_i$ is the irreducible decomposition of Δ_Y,

(iii) *every g-exceptional prime divisor is a component of $(\Delta'_Y)^{=1}$, and*

(iv) $g^{-1}\operatorname{Nklt}(X,\Delta)$ *coincides with* $\operatorname{Nklt}(Y,\Delta_Y)$ *and* $\operatorname{Nklt}(Y,\Delta'_Y)$ *set theoretically.*

Sketch of Proof of Lemma 7.2. It is well known that there exists a dlt blow-up $\alpha : Z \to X$ with $K_Z + \Delta_Z := \alpha^*(K_X + \Delta)$ satisfying (i), (ii), and (iii) (see [F4, Theorem 4.4.21]). Note that $(Z, \Delta_Z^{<1})$ is a $\mathbb{Q}$-factorial kawamata log terminal pair. We take a minimal model $(Z', \Delta_{Z'}^{<1})$ of $(Z, \Delta_Z^{<1})$ over X by [BCHM].

$$\begin{array}{ccc} Z & \overset{\varphi}{\dashrightarrow} & Z' \\ & {}_{\alpha}\searrow \quad \swarrow_{\alpha'} & \\ & X & \end{array}$$

Then $K_{Z'} + \Delta_{Z'}^{<1} \sim_{\mathbb{R}} -\Delta_{Z'}^{\geq 1} + \alpha'^*(K_X + \Delta)$ is nef over X. Of course, we put $\Delta_{Z'} = \varphi_*\Delta_Z$. We take a dlt blow-up $\beta : Y \to Z'$ of $(Z', \Delta_{Z'}^{<1} + \operatorname{Supp}\Delta_{Z'}^{\geq 1})$ again (see [F4, Theorem 4.4.21]) and put $g := \alpha' \circ \beta : Y \to X$. It is not difficult to see that this birational morphism $g : Y \to X$ with $K_Y + \Delta_Y := g^*(K_X + \Delta)$ satisfies the desired properties. It is obvious that $g^{-1}\operatorname{Nklt}(X,\Delta)$ contains the support of $\beta^*\Delta_{Z'}^{\geq 1}$. Since $-\beta^*\Delta_{Z'}^{\geq 1}$ is nef over X, we see that $\beta^*\Delta_{Z'}^{\geq 1}$ coincides with $g^{-1}\operatorname{Nklt}(X,\Delta)$ set theoretically. For the details, see the proof of [S, Theorem 3.4]. □

By combining Proposition 7.1 with Lemma 7.2, we obtain:

Corollary 7.3 ([S, Corollary 5.3]). *Let X be a normal variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi : X \to S$ be a projective morphism onto a scheme S. Assume that there is no non-constant morphism $f : \mathbb{A}^1 \to X \setminus \operatorname{Nklt}(X,\Delta)$ such that $\pi \circ f(\mathbb{A}^1)$ is a point. Then $K_X + \Delta$ is nef over S if and only if $(K_X + \Delta)|_{\operatorname{Nklt}(X,\Delta)}$ is nef over S.*

Proof. If $K_X + \Delta$ is nef over S, then $(K_X + \Delta)|_{\operatorname{Nklt}(X,\Delta)}$ is obviously nef over S. Therefore, it is sufficient to construct a non-constant morphism $f : \mathbb{A}^1 \to X \setminus \operatorname{Nklt}(X,\Delta)$ such that $\pi \circ f(\mathbb{A}^1)$ is a point under the assumption that $(K_X + \Delta)|_{\operatorname{Nklt}(X,\Delta)}$ is nef over S and that $K_X + \Delta$ is not nef over S. By shrinking S suitably, we may assume that X and S are both quasi-projective. By Lemma 7.2, we can construct a projective birational morphism $g : Y \to X$ from a normal $\mathbb{Q}$-factorial variety Y satisfying (i), (ii), and (iv) in Lemma 7.2. Let us consider $\pi \circ g : Y \to S$. Note that $K_Y + \Delta_Y$ is not nef over S since $K_Y + \Delta_Y = g^*(K_X + \Delta)$. It is obvious that $(K_Y + \Delta_Y)|_{\operatorname{Nklt}(Y,\Delta_Y)}$ is nef over S by (iv) because so is $(K_X + \Delta)|_{\operatorname{Nklt}(X,\Delta)}$. Therefore, by Proposition 7.1, we have a non-constant morphism $h : \mathbb{A}^1 \to Y \setminus \operatorname{Nklt}(Y,\Delta_Y)$ such that $(\pi \circ g) \circ h(\mathbb{A}^1)$ is a point. By Proposition 7.1, we have $0 < -(K_Y + \Delta_Y) \cdot C \leq 2\dim Y = 2\dim X$, where C is the closure of $h(\mathbb{A}^1)$ in Y. Since $K_Y + \Delta_Y = h^*(K_X + \Delta)$ holds, g does not contract C to a point. This implies that $f := g \circ h : \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(X, \Delta)$ is a desired non-constant morphism such that $\pi \circ f(\mathbb{A}^1)$ is a point by (iv). □

We introduce the notion of *open qlc strata* in order to state the main result of this section (see Theorem 7.5 below).

Definition 7.4 (Open qlc strata). Let W be a qlc stratum of a quasi-log canonical pair $[X, \omega]$. We put

$$U := W \setminus \bigcup_{W'} W',$$

where W' runs over qlc strata of $[X, \omega]$ strictly contained in W, and call it the *open qlc stratum* of $[X, \omega]$ associated to W.

The following theorem is the main result of this section, which is a generalization of [S, Theorem 1.1] (see also [LZ]).

Theorem 7.5 (cf. [S, Theorem 1.1]). *Let $[X, \omega]$ be a quasi-log canonical pair and let $\pi : X \to S$ be a projective morphism onto a scheme S. Assume that for all open qlc strata U of $[X, \omega]$ there is no non-constant morphism $f : \mathbb{A}^1 \to U$ such that $\pi \circ f(\mathbb{A}^1)$ is a point. Then ω is nef over S.*

Proof. We divide the proof into five small steps.

Step 1. We use induction on $\dim X$. If $\dim X = 0$, then the statement is obvious.

Step 2. Let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition. Then X_i is a qlc stratum of $[X, \omega]$ for every $i \in I$. By adjunction (see [F4, Theorem 6.3.5 (i)]), $[X_i, \omega|_{X_i}]$ is a quasi-log canonical pair for every $i \in I$. We note that the qlc strata of $[X_i, \omega|_{X_i}]$ are exactly the qlc strata of $[X, \omega]$ contained in X_i (see [F4, Theorem 6.3.5 (i)]). Therefore, by replacing $[X, \omega]$ with $[X_i, \omega|_{X_i}]$, we may assume that X is irreducible.

Step 3. By adjunction (see [F4, Theorem 6.3.5 (i)]), $[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}]$ becomes a quasi-log canonical pair whose qlc strata are exactly the qlc strata of $[X, \omega]$ contained in $\mathrm{Nqklt}(X, \omega)$. Therefore, by induction, $\omega|_{\mathrm{Nqklt}(X,\omega)}$ is nef over S. Therefore, it is sufficient to prove that ω is nef over S under the assumption that $\omega|_{\mathrm{Nqklt}(X,\omega)}$ is nef over S.

Step 4. We take the normalization $\nu : X^\nu \to X$. Then, by [FLh1, Theorem 1.1], $[X^\nu, \nu^*\omega]$ naturally becomes a quasi-log canonical pair such that $\nu^{-1}\mathrm{Nqklt}(X, \omega) = \mathrm{Nqklt}(X^\nu, \nu^*\omega)$ holds. We note that ω is nef over S if and only if so is $\nu^*\omega$.

Step 5. We assume that ω is not nef over S. Without loss of generality, we may assume that S is quasi-projective by shrinking S suitably. Therefore, X and X^ν are both quasi-projective. We take an ample $\mathbb{Q}$-divisor H on X^ν such that $\nu^*\omega+H$ is not nef over S. By Theorem 1.1, we can take a boundary $\mathbb{R}$-divisor Δ on X^ν such that $K_{X^\nu}+\Delta \sim_{\mathbb{R}} \nu^*\omega + H$ and that $\mathrm{Nklt}(X^\nu, \Delta) = \mathrm{Nqklt}(X^\nu, \nu^*\omega) = \nu^{-1}\mathrm{Nqklt}(X, \omega)$. Thus $(K_{X^\nu}+\Delta)|_{\mathrm{Nklt}(X^\nu,\Delta)}$ is ample over S. Hence it is obviously nef over S. Since $K_{X^\nu}+\Delta$ is not nef over S, there exists a non-constant morphism $f : \mathbb{A}^1 \to X^\nu \setminus \mathrm{Nklt}(X^\nu, \Delta)$ such that $(\pi \circ \nu) \circ f(\mathbb{A}^1)$ is a point by Corollary 7.3. Thus $\nu \circ f : \mathbb{A}^1 \to X \setminus \mathrm{Nqklt}(X, \omega)$ is a non-constant morphism such that $\pi \circ (\nu \circ f)(\mathbb{A}^1)$ is a point. This is a contradiction because $X \setminus \mathrm{Nqklt}(X, \omega)$ is an open qlc stratum of $[X, \omega]$. Therefore, ω is nef over S.

This is what we wanted. □

As an obvious corollary of Theorem 7.5, we have:

Corollary 7.6. *Let $[X, \omega]$ be a projective quasi-log canonical pair. Assume that $[X, \omega]$ is Mori hyperbolic, that is, for any open qlc stratum U, there is no non-constant morphism $f : \mathbb{A}^1 \to U$. Then ω is nef.*

We give a slight generalization of the cone theorem for quasi-log canonical pairs. For log canonical pairs, it is nothing but [S, Theorem 1.2]. Of course, Theorem 7.5 can be seen as a special case of Theorem 7.7.

Theorem 7.7 (Cone theorem for quasi-log canonical pairs). *Let $[X, \omega]$ be a quasi-log canonical pair and let $\pi : X \to S$ be a projective morphism onto a scheme S. Then we have*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega \geq 0} + \sum_j R_j,$$

where

(i) *R_j is spanned by a rational curve C_j such that $\pi(C_j)$ is a point with*

$$0 < -\omega \cdot C_j \leq 2 \dim X,$$

and

(ii) *there exists an open qlc stratum U of $[X, \omega]$ such that $C_j \cap U$ contains the image of a non-constant morphism $f : \mathbb{A}^1 \to U$.*

Sketch of Proof of Theorem 7.7. In this proof, we only explain how to modify the proof of Theorem 1.8. So we will use the same notation as in the proof of Theorem 1.8. By construction, $(K_X+\Delta_\varepsilon)|_{\mathrm{Nklt}(X,\Delta_\varepsilon)}$ is obviously nef over S since $\pi : \mathrm{Nqklt}(X, \omega) \to \pi(\mathrm{Nqklt}(X, \omega))$ is finite and $\mathrm{Nklt}(X, \Delta_\varepsilon) = \mathrm{Nqklt}(X, \omega)$. Therefore, by the proof of Corollary 7.3 (see also Proposition 7.1), we can take a non-constant morphism $f_\varepsilon : \mathbb{A}^1 \to X \setminus \mathrm{Nqklt}(X, \omega)$ such that C_ε, the closure of $f_\varepsilon(\mathbb{A}^1)$, is a rational curve on X such that $\pi(C_\varepsilon)$ is a point and that $0 < -(K_X + \Delta_\varepsilon) \cdot C_\varepsilon \leq 2 \dim X$. As in the proof of Theorem 1.8, we finally get a rational curve C_j spanning R_j with the desired properties. □

We close this section with a sketch of the proof of Theorem 1.10.

Sketch of Proof of Theorem 1.10. We note that the cone and contraction theorem holds for semi-log canonical pairs by [F3, Theorem 1.19]. Let R be a $(K_X + \Delta)$-negative extremal ray. By replacing $\pi : X \to S$ with the extremal contraction $\varphi_R : X \to Y$ over S associated to R and shrinking S suitably, we may assume that $-(K_X + \Delta)$ is π-ample and that X and S are quasi-projective. Then, by Theorem 2.9, $[X, K_X + \Delta]$ naturally becomes a quasi-log canonical pair such that V is a qlc stratum of $[X, K_X + \Delta]$ if and only if it is an slc stratum of (X, Δ). By the above sketch of the proof of Theorem 7.7, we can check that there exists a possibly singular rational curve C with the desired properties. □

Alternatively, in Theorem 1.10, we can take the normalization of X and reduce the problem to the case where (X, Δ) is log canonical. Then we can apply [S, Theorems 1.2 and 6.5] to find a desired rational curve C.

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