ON SPECIAL TERMINATION (PRIVATE NOTE)

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Abstract. In this short note, we will give a precise proof of [S2, 2.2, 2.3], namely, Special Termination modulo the log MMP for lower dimensional varieties.

This note is a supplement of [S2, §2 Special termination]. We will give a precise proof of [S2, 2.2, 2.3]. Step 2 in the proof below seems to be slightly different form [S2, 2.2]. See also [S2, 2.4 Remark]. However, the final result is same. So there is no problem for applications.

The following is the main theorem of this note.

Theorem 1 (Special Termination). *We assume that the log MMP for $\mathbb{Q}$-factorial dlt pairs holds in dimension $\leq n-1$. Let X be a normal n-fold and B an effective $\mathbb{R}$-divisor such that (X, B) is dlt. Assume that X is $\mathbb{Q}$-factorial. Consider a sequence of log flips starting from $(X, B) = (X_0, B_0)$:*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots,$$

where $\phi_i : X_i \to Z_i$ is a contraction of an extremal ray R_i with $(K_{X_i} + B_i) \cdot R_i < 0$, and $\phi_i^+ : X_i^+ = X_{i+1} \to Z_i$ is the log flip. Then after finitely many flips, flipping locus (and thus the flipped locus) is disjoint from $\lfloor B_i \rfloor$.

Definition 2. A curve C on X_i is called *flipping* (resp. *flipped*) if $\phi_i(C)$ (resp. $\phi_{i-1}^+(C)$) is a point.

Remark 3. Let S be a center of log canonical singularities (CLC, for short) of the pair (X, B). Then (S, B_S) is also dlt, where $K_S + B_S = (K_X + B)|_S$. We note that B_S is defined by applying adjunction $\dim X - \dim S$ times repeatedly. For the details of dlt pairs, see [F, Chapter 1].

Definition 4. Let $\varphi : (X, B) \to (X', B')$ be a morphism between two dlt pairs. We call φ a *crepant isomorphism* if φ is an isomorphism and $K_X + B = \varphi^*(K_{X'} + B')$.

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This note was written for my private usage. So, it may contain some errors.

Remark 5. In [F, Definition 1.5], we introduced *B-birational maps* and so on. However, I give up using B-birational etc. since there are many B-$\cdots$.

Definition 6. Let $f : V \to W$ be a birational contraction with $\dim V \geq 2$. We say that f is type (S) if f is an isomorphism in codimension one. We say that f is type (D) if f contracts at least one divisor. Let

$$V \xrightarrow{f} W \xleftarrow{g} U$$

be a pair of birational contractions. We call this type (SD) if f is type (S) and g is type (D). We can define (SS), (DS), and (DD) similarly.

Definition 7. Let $B = \sum b_j B^j$ be the irreducible decomposition. We define a finite set of positive numbers $\mathbf{b} = \{b_j\}$. We define

$$\mathcal{S}(\mathbf{b}) := \left\{ 1 - \frac{1}{m} + \sum \frac{r_j b_j}{m} \;\middle|\; m \in \mathbb{Z}_{>0}, r_j \in \mathbb{Z}_{\geq 0} \right\}.$$

Definition 8 ([FA, 7.5.1 Definition])**.** Let S be a CLC of the dlt pair (X, B). we define

$$d_{\mathbf{b}}(S, B_S) := \sum_{\alpha \in \mathcal{S}(\mathbf{b})} \sharp \left\{ E \;\middle|\; a(E, S, B_S) < -\alpha, \mathrm{Center}_S(E) \not\subset \lfloor B_S \rfloor \right\}.$$

This is a precise version of *difficulty*. It is obvious that $d_{\mathbf{b}}(S, B_S) < \infty$. We note that $(S \setminus \lfloor B_S \rfloor, B_S)$ is klt and [FA, 7.4.4 Lemma].

Proof of Theorem 1. **Step 1.** *After finitely many flips, the flipping locus contains no CLC's.*

Proof. We note that the number of CLC's is finite. If the flipping locus contains a CLC, then the number of CLC's decreases by [FA, (2.28)]. □

So we can assume that the flipping locus contains no CLC's of the pair (X_i, B_i) for every i.

On this assumption, $\varphi_i : X_i \dashrightarrow X_{i+1}$ induces a birational map $\varphi_i|_{S_i} : S_i \dashrightarrow S_{i+1}$, where S_i is a CLC of (X_i, B_i) and S_{i+1} is the corresponding CLC of (X_{i+1}, B_{i+1}). We will omit the subscript $|_{S_i}$ if there is no danger of confusion.

Remark 9. By adjunction, we have

$$a(E, S_i, B_{S_i}) \leq a(E, S_{i+1}, B_{S_{i+1}}),$$

for every valuation E. In particular,

$$\mathrm{totaldiscrep}(S_i, B_{S_i}) \leq \mathrm{totaldiscrep}(S_{i+1}, B_{S_{i+1}})$$

for every i.

Step 2. *Assume that $\varphi_i : X_i \dashrightarrow X_{i+1}$ induces a crepant isomorphism on every $(d-1)$-dimensional CLC for all i. Then after finitely many flips, φ_i induces a crepant isomorphism on every d-dimensional CLC.*

Remark 10. It is obvious that φ_i induces a crepant isomorphism on every 0-dimensional CLC. When $d = 1$, Step 2 is obvious by [FA, 7.4.4 Lemma] and Remark 9.

So we can assume that $d \geq 2$.

Remark 11. Let (S_i, B_{S_i}) be a CLC. Assume that $\varphi_i : (S_i, B_{S_i}) \to (S_{i+1}, B_{S_{i+1}})$ is a crepant isomorphism. Then S_i contains no flipping curves and S_{i+1} contains no flipped curves. This is obvious by the negativity lemma.

Proposition 12. *The inequality $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$ holds. Moreover, if $S_i \to T_i \leftarrow S_i^+ = S_{i+1}$ is type (SD) or (DD), then $d_{\mathbf{b}}(S_i, B_{S_i}) > d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$, where T_i is the normalization of $\phi_i(S_i)$. We note that there exists a $\phi_i^+|_{S_{i+1}}$-exceptional divisor E on S_{i+1}. By adjunction and the negativity lemma,*

$$a(E, S_i, B_{S_i}) < a(E, S_{i+1}, B_{S_{i+1}}) = -\alpha$$

for some $\alpha \in \mathcal{S}(\mathbf{b})$. Therefore, after finitely many flips, $S_i \to T_i \leftarrow S_{i+1}$ is type (SS) or (DS).

Proof. See [FA, 7.5.3 Lemma, 7.4.3 Lemma]. We note that φ_i is a crepant isomorphism on $\lfloor B_i \rfloor$ by the assumption. Therefore,

$$\mathrm{Center}_{S_i}(E) \subset \lfloor B_{S_i} \rfloor \text{ if and only if } \mathrm{Center}_{S_{i+1}}(E) \subset \lfloor B_{S_{i+1}} \rfloor.$$

More precisely, if $\mathrm{Center}_{S_i}(E)$ (resp. $\mathrm{Center}_{S_{i+1}}(E)$) is contained in $\lfloor B_{S_i} \rfloor$ (resp. $\lfloor B_{S_{i+1}} \rfloor$), then φ_i is an isomorphism at the generic point of $\mathrm{Center}_{S_i}(E)$ (resp. $\mathrm{Center}_{S_{i+1}}(E)$) by the negativity lemma. Therefore, we obtain $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$ by Remark 9. □

So we can assume that every step is type (SS) or (DS). Let $f : S_0^0 \to S_0$ be a $\mathbb{Q}$-factorial dlt model, that is, $(S_0^0, B_{S_0^0})$ is $\mathbb{Q}$-factorial and dlt such that $K_{S_0^0} + B_{S_0^0} = f^*(K_{S_0} + B_{S_0})$. Applying the log MMP to $S_0^0 \to T_0$, we obtain a sequence of divisorial contractions and log flips over T_0

$$S_0^0 \dashrightarrow S_0^1 \dashrightarrow \cdots,$$

and finally a relative log minimal model $S_0^{k_0}$.

Since $S_1 \to T_0$ is the lc model of $S_0^0 \to S_0 \to T_0$, we have a unique natural morphism $g : S_0^{k_0} \to S_1$. We note that $K_{S_0^{k_0}} + B_{S_0^{k_0}} = g^*(K_{S_1} +$

B_{S_1}). Applying the log MMP to $S_1^0 := S_0^{k_0} \to S_1 \to T_1$ over T_1, we obtain a sequence

$$S_1^0 \dashrightarrow \cdots \dashrightarrow S_1^{k_1} \to S_2.$$

by the same reason. Applying the log MMP to $S_2^0 := S_1^{k_1} \to S_2 \to T_2$. Repeating this procedure, we obtain a sequence of log flips and divisorial contractions. This sequence terminates by the log MMP. Thus we obtain the required results.

Remark 13. In Step 2, we obtain no information about flipping curves that are not contained in $\lfloor B_i \rfloor$ but intersect with $\lfloor B_i \rfloor$.

Step 3. *After finitely many flips, we can assume that $\lfloor B_i \rfloor$ contains no flipping curves and no flipped curves by Step 2. If the flipping locus intersects $\lfloor B_i \rfloor$, then there exists a flipping curve C such that $C \cdot \lfloor B_i \rfloor > 0$. Note that X_i is $\mathbb{Q}$-factorial. Then $\lfloor B_{i+1} \rfloor$ intersects with every flipped curve negatively. So $\lfloor B_{i+1} \rfloor$ contains a flipped curve. This is a contradiction.*

Therefore, we finished the proof of Theorem 1. □

Remark 14. Let (X, B) be a dlt pair. Then (S, B_S) is an sdlt pair in the sense of [F, Definition 1.1], where $S = \lfloor B \rfloor$. By the definition of sdlt pairs, the normalization of (S, B_S) is dlt. We also note that every irreducible component of S is normal. Therefore, we can apply inductive treatments for dlt pairs. This inductive structure is quite non-trivial. It is very difficult to prove it from the original definition of dlt pairs (see [S1], [FA, (2.13.3)]).

References

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