SIMPLE CONNECTEDNESS OF FANO LOG PAIRS WITH SEMI-LOG CANONICAL SINGULARITIES

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Abstract. We show that any union of slc strata of a Fano log pair with semi-log canonical singularities is simply connected. In particular, Fano log pairs with semi-log canonical singularities are simply connected, which confirms a conjecture of the first author.

Introduction

Fano manifolds are complex projective manifolds whose canonical class is anti-ample. It is known that every Fano manifold is simply connected. Indeed, there are at least three independent proofs of this fact, and we refer to [Tak00] and [Fuj14b, Section 6] for more details. In birational geometry, the notion of Fano manifolds is generalized to that of Fano log pairs which allows singularities coming up naturally in the minimal model program (MMP). The rational chain connectedness of Fano log pairs with more and more general (up to log canonical) singularities is then established in [Cam92, KMM92, Zha06, HM07], and this implies that the (topological) fundamental group of the variety is finite. On the other hand, one shows that the *algebraic* fundamental group of a Fano log pair is trivial by vanishing theorems. Combining these two facts, the simple connectedness of Fano log pairs with log canonical singularities follows ([Fuj17b, Theorem 6.1]).

The class of semi-log canonical singularities incorporates the non-normal counterpart of log-canonical singularities. They appear on the varieties at the boundaries of the compactifications of moduli spaces (see [HK10, Part III], [Kol13a, Kol13b]). Even more general singularities, namely the quasi-log canonical singularities, are introduced in the inductive treatment of the MMP ([Amb03]). This class of singularities allows to put log pairs with semi-log canonical singularities and their slc strata, or any union thereof, on the equal footing. The fundamental theorems in the MMP, especially the vanishing theorems, are now available in this context ([Fuj17a]). As a consequence, quasi-log canonical varieties with anti-ample quasi-log canonical class has trivial algebraic fundamental group ([Fuj17b, Corollary 1.2]). This leads to the following

Conjecture 0.1 ([Fuj17b], Conjecture 1.3). *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $-\omega$ is ample. Then X is simply connected.*

In this paper we confirm the conjecture for an important special case. It is an answer to [Fuj14b, Problem 3.6].

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Theorem 0.2 (see Theorem 2.7)**.** *Any union of slc strata of a Fano log pair (X, Δ) with semi-log canonical singularities is simply connected.*

As a corollary, we solve [Fuj17b, Conjecture 1.4]:

Corollary 0.3. *Fano log pairs with semi-log canonical singularities are simply connected.*

A key ingredient of the proof is a subadjunction formula for slc strata (Lemma 2.3). In particular, this makes the minimal slc stratum into a Fano log pair with Kawamata log terminal singularities, so it is simply connected (Corollary 2.4).

What also follows is that any union of slc strata of a Fano log pair with semi-log canonical singularities are rationally chain connected (Corollary 2.5). However, for non-normal varieties rational chain connectedness does not imply the finiteness of the fundamental group as in the normal case (consider for example a rational curve with nodes [Fuj17b, Remark 6.2]).

Thus we need to invoke [HM07, Corollary 1.4] and the van Kampen theorem to show that the natural homomorphism of fundamental groups induced by the inclusion of the minimal slc stratum into a union of slc strata is surjective. In this way the required simple connectedness is proved.

Conventions: We work over $\mathbb{C}$, the complex number field, throughout this paper. A *scheme* means a separated scheme of finite type over $\mathbb{C}$. We freely use the standard notation of the MMP as in [Fuj17a]. If $f\colon X \to Y$ is a continuous map between two path-connected topological spaces, we omit the base points for the fundamental groups in the induced homomorphism $\pi_1(X) \to \pi_1(Y)$, which will be harmless for the arguments in this paper. When we treat a Fano log pair (X, Δ) with semi-log canonical singularities, we always assume that X is connected.

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1. Preliminary

A log pair (X, Δ) consists of an equi-dimensional demi-normal scheme X together with an effective $\mathbb{R}$-divisor Δ on X, such that Δ does not contain any irreducible components of the non-normal locus of X and $K_X + \Delta$ is $\mathbb{R}$-Cartier. We note that a scheme X is demi-normal if it satisfies Serre's S_2 condition and if its codimension one points are either regular points or nodes ([Kol13b, Denition 5.1]).

Definition 1.1. A projective log pair (X, Δ) is Fano if $-(K_X + \Delta)$ is ample, or put another way, if $K_X + \Delta$ is anti-ample.

Let (X, Δ) be a normal log pair. Let $f\colon Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$ has a simple normal crossing support, where $\mathrm{Exc}(f)$ is the exceptional locus of f and $f_*^{-1}\Delta$ is the strict transform of Δ on Y. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy* of E_i with respect to (X, Δ). We say that (X, Δ) is *log canonical* (resp. *Kawamata log terminal*) if $a_i \geq -1$ (resp. $a_i > -1$) for every i. We use abbreviations *lc* and *klt* for log canonical and Kawamata log terminal respectively.

If (X, Δ) is a normal log pair (resp. an lc pair) and if there exist a resolution $f\colon Y \to X$ and a prime divisor E on Y such that $a(E, X, \Delta) \leq -1$ (resp. $a(E, X, \Delta) = -1$) then $f(E)$ is called an *Nklt center* (resp. *an lc center*) of (X, Δ). The *Nklt locus* of (X, Δ), denoted by $\mathrm{Nklt}(X, \Delta)$, is the union of all Nklt centers. An *lc stratum* of an lc pair (X, Δ) means either an lc center or an irreducible component of X.

Definition 1.2. A log pair (X, Δ) is said to have *semi-log canonical* (*slc*) singularities if $(\bar{X}, \Delta_{\bar{X}})$ has log canonical singularities, where $\nu\colon \bar{X} \to X$ is the normalization and $K_{\bar{X}} + \Delta_{\bar{X}} = \nu^*(K_X + \Delta)$. An *slc center* of (X, Δ) is the image of an lc center of $(\bar{X}, \Delta_{\bar{X}})$. An *slc stratum* of (X, Δ) means either an slc center of (X, Δ) or an irreducible component of X.

Note that an slc stratum is irreducible by definition.

Let $D = \sum_i d_i D_i$ be an $\mathbb{R}$-divisor, where D_i is a prime divisor and $d_i \in \mathbb{R}$ for every i such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad D^{=1} = \sum_{d_i = 1} D_i, \quad \text{and} \quad \lceil D \rceil = \sum_i \lceil d_i \rceil D_i,$$

where $\lceil d_i \rceil$ is the integer defined by $d_i \leq \lceil d_i \rceil < d_i + 1$.

Let Z be a simple normal crossing divisor on a smooth variety M and B an $\mathbb{R}$-divisor on M such that Z and B have no common irreducible components and that the support of $Z + B$ is a simple normal crossing divisor on M. In this situation, $(Z, B|_Z)$ is called a *globally embedded simple normal crossing pair*.

Let us quickly look at the definition of qlc pairs for the reader's convenience.

Definition 1.3 (qlc pairs)**.** Let X be a scheme and ω an $\mathbb{R}$-Cartier divisor (or an $\mathbb{R}$-line bundle) on X. Let $f : Z \to X$ be a proper morphism from a globally embedded simple normal crossing pair (Z, Δ_Z). If the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Z(\lceil -(\Delta_Z^{<1}) \rceil)$ is an isomorphism, $\Delta_Z = \Delta_Z^{\leq 1}$, and $f^*\omega \sim_{\mathbb{R}} K_Z + \Delta_Z$, then $[X, \omega]$ is called a *quasi-log canonical pair* (*qlc pair*, for short).

Remark 1.4. We can define *qlc centers*, *qlc strata*, $\mathrm{Nqklt}(X, \omega)$, and so on, for a qlc pair $[X, \omega]$, which are counterparts of (s)lc centers, (s)lc strata, and $\mathrm{Nklt}(X, \Delta)$, respectively. In the situation of Definition 1.3, C is a qlc stratum of $[X, \omega]$ if and only if C is the f-image of some slc stratum of $(Z, \Delta_Z^{=1})$. The subvariety C is a qlc center of $[X, \omega]$ if and only if C is a qlc stratum of $[X, \omega]$ but is not an irreducible component of X. The union of all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega)$. For the details, see [Fuj17a, Definitions 6.2.2, 6.2.8, 6.2.9, and Notation 6.3.10].

We refer to [Fuj17a, Chapter 6] for the theory of quasi-log schemes.

Theorem 1.5 (see [Fuj14a, Theorem 1.2])**.** *Let (X, Δ) be a quasi-projective log pair with semi-log canonical singularities. Then $[X, \omega]$, where $\omega = K_X + \Delta$, has a qlc structure which is compatible with the original slc structure of (X, Δ). This means that C is an slc center* (*resp. slc stratum*) *of (X, Δ) if and only if C is a qlc center* (*resp. qlc stratum*) *of $[X, \omega]$. In particular, any union of slc strata of (X, Δ) is qlc by adjuction.*

For the details of Theorem 1.5, see [Fuj14a]. By this theorem, we can apply the theory of quasi-log schemes in [Fuj17a, Chapter 6] to log pairs with semi-log canonical singularities.

Let (X,Δ) be a quasi-projective log pair with semi-log canonical singularities. Then its slc strata have some nice properties ([Fuj17a, Theorem 6.3.11]):

(a) there is a unique minimal slc stratum through a given point;
(b) the minimal slc stratum at a given point is normal at that point;
(c) the intersection of two slc strata is a union of slc strata.

If (X,Δ) is additionally Fano, that is, X is projective and $-(K_X+\Delta)$ is ample, then

(d) any union of slc strata of (X,Δ) is connected;
(e) there is a unique minimal slc stratum of (X,Δ), which is normal.

For (d) it suffices to show that $H^0(W,\mathcal{O}_W)=\mathbb{C}$ for any union W of slc strata, which follows from the vanishing $H^1(X,\mathcal{I}_W)=0$ ([Fuj14a, Theorem 1.11]), and (e) is a direct consequence of (a), (b) and (d).

2. Proof

For the proof of Theorem 0.2, we may assume that Δ is a $\mathbb{Q}$-divisor by perturbing Δ. Therefore, for simplicity, we assume that every divisor is a $\mathbb{Q}$-divisor from now on. Let us start with the following easy lemma.

Lemma 2.1. *Let $f: X\to Y$ be a surjective morphism between connected normal projective varieties. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X,Δ) is lc and is klt over the generic point of Y. Assume that $K_X+\Delta\sim_{\mathbb{Q}} f^*D$ for some $\mathbb{Q}$-Cartier divisor D on Y. Then we can construct an effective $\mathbb{Q}$-divisor Δ_Y on Y such that $K_Y+\Delta_Y\sim_{\mathbb{Q}} D$ and $\operatorname{Nklt}(Y,\Delta_Y)\subset f(\operatorname{Nklt}(X,\Delta))$.*

Proof. Let

$$f: X\xrightarrow{g} Z\xrightarrow{h} Y$$

be the Stein factorization of $f: X\to Y$. By the theory of lc-trivial fibrations (see [Amb04, Theorem 0.2] and [Amb05, Theorem 3.3]), there exist a proper birational morphism $\sigma: Z'\to Z$ from a smooth projective variety Z', a $\mathbb{Q}$-divisor $B_{Z'}$ on Z', and a nef $\mathbb{Q}$-divisor $M_{Z'}$ on Z' with the following properties:

(i) $\sigma^*h^*D\sim_{\mathbb{Q}} K_{Z'}+B_{Z'}+M_{Z'}$,
(ii) the support of $B_{Z'}$ is a simple normal crossing divisor on Z', $B_{Z'}=B_{Z'}^{\leqq 1}$, and $B_{Z'}=B_{Z'}^{<1}$ outside $\sigma^{-1}(g(\operatorname{Nklt}(X,\Delta)))$, and
(iii) there exist a proper surjective morphism $p: Z'\to Z''$ onto a normal projective variety Z'' and a nef and big $\mathbb{Q}$-divisor $M_{Z''}$ on Z'' such that $M_{Z'}\sim_{\mathbb{Q}} p^*M_{Z''}$.

Then we can take an effective $\mathbb{Q}$-divisor $G_{Z'}$ such that $G_{Z'}\sim_{\mathbb{Q}} M_{Z'}$ and that $K_Z+\Delta_Z$ is klt outside $g(\operatorname{Nklt}(X,\Delta))$, where $\Delta_Z=\sigma_*(B_{Z'}+G_{Z'})$. We note that $K_Z+\Delta_Z\sim_{\mathbb{Q}} h^*D$ by construction. Then the proof of [FG12, Lemma 1], applied to $h: Z\to Y$, gives an effective $\mathbb{Q}$-divisor Δ_Y on Y such that $K_Y+\Delta_Y\sim_{\mathbb{Q}} D$ and that $\operatorname{Nklt}(Y,\Delta_Y)\subset h(\operatorname{Nklt}(Z,\Delta_Z))\subset f(\operatorname{Nklt}(X,\Delta))$. □

Remark 2.2. In Lemma 2.1, it is easy to construct an effective $\mathbb{Q}$-divisor Δ_Y on Y such that $K_Y+\Delta_Y\sim_{\mathbb{Q}} D+A$ with $\operatorname{Nklt}(Y,\Delta_Y)\subset f(\operatorname{Nklt}(X,\Delta_X))$, where A is any ample $\mathbb{Q}$-divisor on Y (see also [Fuj99, Theorem 1.2]). We can prove the above weaker statement without using Ambro's deep result ([Amb05, Theorem 3.3]). The nefness of $M_{Z'}$ (see [Amb04, Theorem 0.2]), which is much simpler than [Amb05, Theorem 3.3], is sufficient. For the proof of Theorem 0.2, we can replace Δ with $\Delta+\varepsilon H$, where H is a general very ample effective divisor on X and ε is a sufficiently small positive rational number. Therefore, we can prove Theorem 0.2 without using [Amb05, Theorem 3.3].

As an application of Lemma 2.1, we have the following lemma.

Lemma 2.3. *Let W be an slc stratum of a projective log pair (X, Δ) with semi-log canonical singularities, and let E be the union of all slc strata that are strictly contained in W. Let $\nu\colon \bar{W} \to W$ be the normalization. Then there is an effective $\mathbb{Q}$-divisor $B_{\bar{W}}$ on $\bar{W}$ such that $K_{\bar{W}} + B_{\bar{W}} \sim_{\mathbb{Q}} (K_X + \Delta)|_{\bar{W}}$ and $\mathrm{Nklt}(\bar{W}, B_{\bar{W}}) \subset \nu^{-1}(E)$. Moreover, if (X, Δ) is additionally Fano, then $\mathrm{Nklt}(\bar{W}, B_{\bar{W}})$ is connected.*

Proof. Let $\mu\colon \bar{X} \to X$ be the normalization of X. Let $\bar{X}_i$ be an irreducible component of $\bar{X}$ that contains an irreducible component V of $\mu^{-1}(W)$. Let $\Delta_{\bar{X}_i}$ be the effective $\mathbb{Q}$-divisor defined by $K_{\bar{X}_i} + \Delta_{\bar{X}_i} = (K_X + \Delta_X)|_{\bar{X}_i}$. Then $(\bar{X}_i, \Delta_{\bar{X}_i})$ has log canonical singularities.

Let $f\colon (Y, \Delta_Y) \to (\bar{X}_i, \Delta_{\bar{X}_i})$ be a $\mathbb{Q}$-factorial dlt blow-up such that $K_Y + \Delta_Y = f^*(K_{\bar{X}_i} + \Delta_{\bar{X}_i})$ (see, for example, [Fuj17a, Theorem 4.4.21]). There is an lc stratum S of (Y, Δ_Y) dominating V. We take S to be a minimal such lc stratum. Then S is normal, and if Δ_S is the effective $\mathbb{Q}$-divisor on S defined by adjunction $K_S + \Delta_S = (K_Y + \Delta_Y)|_S$ then (S, Δ_S) is again a dlt pair.

Let $\sigma\colon \bar{V} \to V$ be the normalization. Then the morphism $\mu|_V \circ \sigma\colon \bar{V} \to W$ factors through a morphism $\bar{\mu}\colon \bar{V} \to \bar{W}$. Since S is normal, the morphism $S \to V$ factors through a morphism $g\colon S \to \bar{V}$. Thus we have a commutative diagram

$$\begin{array}{ccc} S & & \\ {\scriptstyle g}\downarrow & \searrow & \\ \bar{V} & \xrightarrow{\sigma} & V \\ {\scriptstyle \bar{\mu}}\downarrow & & \downarrow{\scriptstyle \mu|_V} \\ \bar{W} & \xrightarrow{\nu} & W \end{array}$$

where the morphisms in the lower square are all finite.

By the choice of S, the log pair (S, Δ_S) has klt singularities over the generic point of $\bar{W}$. By applying Lemma 2.1 to $\bar{\mu} \circ g : S \to \bar{W}$, we can take an effective $\mathbb{Q}$-divisor $B_{\bar{W}}$ on $\bar{W}$ such that $K_{\bar{W}} + B_{\bar{W}} \sim_{\mathbb{Q}} (K_X + \Delta)|_{\bar{W}}$ and that the following inclusions

$$\mathrm{Nklt}(\bar{W}, B_{\bar{W}}) \subset \bar{\mu} \circ g(\mathrm{Nklt}(S, \Delta_S)) \subset \nu^{-1}(E)$$

hold.

If $-(K_X + \Delta)$ is ample, then so is $-(K_{\bar{W}} + B_{\bar{W}})$. Therefore, by the Nadel vanishing theorem (see, for example, [Fuj17a, Theorem 3.4.2]), we obtain

$$H^i(\bar{W}, \mathcal{J}(\bar{W}, B_{\bar{W}})) = 0$$

for any $i > 0$, where $\mathcal{J}(\bar{W}, B_{\bar{W}})$ is the multiplier ideal sheaf of $(\bar{W}, B_{\bar{W}})$. It follows from the long exact sequence of cohomology that the natural restriction map

$$H^0(\bar{W}, \mathcal{O}_{\bar{W}}) \to H^0(\mathrm{Nklt}(\bar{W}, B_{\bar{W}}), \mathcal{O}_{\mathrm{Nklt}(\bar{W}, B_{\bar{W}})})$$

is surjective. Therefore, we see that

$$H^0(\mathrm{Nklt}(\bar{W}, B_{\bar{W}}), \mathcal{O}_{\mathrm{Nklt}(\bar{W}, B_{\bar{W}})}) \cong \mathbb{C}$$

and $\mathrm{Nklt}(\bar{W}, B_{\bar{W}})$ is connected. □

Corollary 2.4. *Let (X, Δ) be a Fano log pair with semi-log canonical singularities and W_0 its minimal slc stratum. Then there is an effective $\mathbb{Q}$-divisor B_{W_0} on W_0 such that (W_0, B_{W_0}) is a Fano log pair with klt singularities and that*

$$K_{W_0} + B_{W_0} \sim_{\mathbb{Q}} (K_X + \Delta)|_{W_0}$$

Hence W_0 is rationally connected ([Zha06]) *and $\pi_1(W_0) = 1$* ([Tak00, Theorem 1.1])*.*

Corollary 2.5. *Let (X, Δ) be a Fano log pair with slc singularities. Then any union of slc strata of (X, Δ) is rationally chain connected.*

Proof. Since any union of slc strata of (X, Δ) is connected, it suffices to prove that any single slc stratum are rationally chain connected.

Let W be an slc stratum of (X, Δ) and E the union of all slc strata that are strictly contained in W. Let $\nu\colon \bar{W} \to W$ be the normalization. By Lemma 2.3 there is an effective $\mathbb{Q}$-divisor $B_{\bar{W}}$ on $\bar{W}$ such that $K_{\bar{W}} + B_{\bar{W}} \sim_{\mathbb{Q}} (K_X + \Delta)|_{\bar{W}}$, which is anti-ample, and $\mathrm{Nklt}(\bar{W}, B_{\bar{W}}) \subset \nu^{-1}(E)$. By [BP11, Corollary 1.4], $\bar{W}$ is rationally connected modulo $\mathrm{Nklt}(\bar{W}, B_{\bar{W}})$, that is, for any general point w of $\bar{W}$ there exists a rational curve C_w passing through w and intersecting $\mathrm{Nklt}(\bar{W}, B_{\bar{W}})$. In particular, $\bar{W}$ is rationally connected modulo $\nu^{-1}(E)$. It follows that W is rationally connected modulo E which is the union of lower dimensional slc strata. Thus we can run induction on dimension of the slc strata, noting that the minimal slc stratum of (X, Δ) is rationally connected by Corollary 2.4. □

We prepare an important lemma for the proof of Theorem 2.7.

Lemma 2.6. *Let W be a non-minimal slc stratum of a Fano log pair (X, Δ) with semi-log canonical singularities, and let E be the union of all slc strata that are strictly contained in W. Let $\nu\colon \bar{W} \to W$ be the normalization. Then $\nu^{-1}(E)$ is connected.*

Proof. By [Fuj14a, Theorem 1.2] (see also Theorem 1.5) and adjunction (see, for example, [Fuj17a, Theorem 6.3.5 (i)]), $[W, \omega]$ is a qlc pair, where $\omega = (K_X + \Delta)|_W$, such that $\mathrm{Nqklt}(W, \omega) = E$. By [FL17, Theorem 1.1], we see that $[\bar{W}, \nu^*\omega]$ is also qlc with $\mathrm{Nqklt}(\bar{W}, \nu^*\omega) = \nu^{-1}(E)$. Since $-\nu^*\omega$ is ample, $H^i(\bar{W}, \mathcal{I}_{\mathrm{Nqklt}(\bar{W}, \nu^*\omega)}) = 0$ for every $i > 0$ by the vanishing theorem (see, for example, [Fuj17a, Theorem 6.3.5 (ii)]). Note that $\mathcal{I}_{\mathrm{Nqklt}(\bar{W}, \nu^*\omega)}$ is the defining ideal sheaf of $\mathrm{Nqklt}(\bar{W}, \nu^*\omega)$ on $\bar{W}$. It follows from the long exact sequence of cohomology that the natural restriction map

$$H^0(\bar{W}, \mathcal{O}_{\bar{W}}) \to H^0(\mathrm{Nqklt}(\bar{W}, \nu^*\omega), \mathcal{O}_{\mathrm{Nqklt}(\bar{W}, \nu^*\omega)})$$

is surjective. Therefore, we see that $H^0(\mathrm{Nqklt}(\bar{W}, \nu^*\omega), \mathcal{O}_{\mathrm{Nqklt}(\bar{W}, \nu^*\omega)}) \cong \mathbb{C}$ and $\nu^{-1}(E) = \mathrm{Nqklt}(\bar{W}, \nu^*\omega)$ is connected. □

Theorem 2.7. *Let (X, Δ) be a Fano log pair with slc singularities and W the union of some slc strata of (X, Δ). Then $\pi_1(W) = 1$.*

Proof. Note that W is connected and contains the minimal slc stratum W_0 of (X, Δ). Let $W^{(0)} := W$. Suppose that $W^{(i)}$ is defined. We define $W^{(i+1)}$ to be the union of slc strata that are strictly contained in an irreducible component of $W^{(i)}$. Thus we obtain a filtration of reduced subschemes of W:

$$W = W^{(0)} \supset W^{(1)} \supset \cdots \supset W^{(k)} = W_0$$

We want to show by inverse induction on i that $\pi_1(W^{(i)}) = 1$ for any $i \geq 0$. In particular, it will follow that $\pi_1(W) = \pi_1(W^{(0)}) = 1$.

By Corollary 2.4 we know that $\pi_1(W^{(k)}) = \pi_1(W_0) = 1$.

Now assuming $\pi_1(W^{(i)}) = 1$, we need to show that $\pi_1(W^{(i-1)}) = 1$. For simplicity of notation, let us denote $Z := W^{(i-1)}$and $E := W^{(i)}$. We construct a covering family of open subsets of Z in the Euclidean topology: Let Z_j be the irreducible components of Z and U_j an open neighborhood of $E_j := Z_j \cap E$ in Z_j such that E_j is a deformation retract of U_j (cf. [BHPV04, Chapter 1, Theorem 8.8]). Let $U = \cup_j U_j$ and $V_j := Z_j \cup U$. Then $\{V_j\}_j$ is an open covering of Z such that $V_{j_1} \cap V_{j_2} = U$ for any $j_1 \neq j_2$, which is connected.

Note that the U_j's are closed subsets of U and $U_{j_1} \cap U_{j_2} = E_{j_1} \cap E_{j_2}$ for $j_1 \neq j_2$, so the deformation retractions $U_j \times I \to U_j$ from U_j onto E_j coincide on $(U_{j_1} \cap U_{j_2}) \times I$ for any $j_1 \neq j_2$, and thus glue to a continuous map $U \times I \to U$ which is a deformation retraction of U onto E. Here I denotes the unit interval $[0, 1]$. Similarly, there is a deformation retraction from V_j onto $Z_j \cup E$. It follows that $\pi_1(U) = \pi_1(E) = 1$ and $\pi_1(V_j) = \pi_1(Z_j \cup E)$. Applying the van Kampen theorem, we obtain an isomorphism

$$*_j\, \pi_1(V_j) \xrightarrow{\sim} \pi_1(Z). \tag{2.1}$$

where $*_j \pi_1(V_j)$ denotes the free product of the $\pi_1(V_j)$'s. Therefore, it suffices to show that $\pi_1(V_j) = \pi_1(Z_j \cup E)$ is trivial for each j.

Let $\nu\colon \bar{Z}_j \to Z_j$ be the normalization. Then by Lemma 2.3 we can find an effective $\mathbb{Q}$-divisor $B_{\bar{Z}_j}$ on $\bar{Z}_j$ such that $K_{\bar{Z}_j} + B_{\bar{Z}_j} \sim_{\mathbb{Q}} (K_X + \Delta)|_{\bar{Z}_j}$, which is anti-ample. We note that $\nu^{-1}(E_j)$ is connected by Lemma 2.6. By [FPR15, Proposition 3.1], there is a homotopy equivalence between the double mapping cylinder $E \cup_\nu \nu^{-1}(E_j) \times I \cup_\iota \bar{Z}_j$ and $Z_j \cup E$ where $\iota\colon \nu^{-1}(E_j) \hookrightarrow \bar{Z}_j$ is the inclusion map, and it follows that (cf. [FPR15, Corollary 3.2, (ii)])

$$\pi_1(Z_j \cup E) \cong \pi_1(E \cup_\nu \nu^{-1}(E_j) \times I \cup_\iota \bar{Z}_j) \cong \pi_1(E) *_{\pi_1(\nu^{-1}(E_j))} \pi_1(\bar{Z}_j). \tag{2.2}$$

By [HM07, Corollary 1.4] the induced homomorphism $\pi_1(\mathrm{Nklt}(\bar{Z}_j, B_{\bar{Z}_j})) \to \pi_1(\bar{Z}_j)$ is surjective. Therefore, $\pi_1(\nu^{-1}(E_j)) \to \pi_1(\bar{Z}_j)$ is surjective since $\mathrm{Nklt}(\bar{Z}_j, B_{\bar{Z}_j}) \subset \nu^{-1}(E_j) \subset \bar{Z}_j$, and so is the induced homomorphism $\pi_1(E) \to \pi_1(Z_j \cup E)$ by (2.2). Since $\pi_1(E) = 1$ by the induction hypothesis, we have the desired triviality of $\pi_1(Z_j \cup E)$. ◻

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