ON REDUCTION THEOREM (PRIVATE NOTE)

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Abstract. In this note, we will explain reduction theorem.

The following is the definition of an (*elementary*) *pre limiting contraction*.

Definition 1 (Pre limiting contractions)**.** We call $f : (X, D) \to Z$ an (*elementary*) *pre limiting contraction* or *pl contraction* if

(1) (X, D) is a $\mathbb{Q}$-factorial dlt pair;
(2) f is small and $-(K_X + D)$ is f-ample;
(3) there exists an irreducible component $S \subset \lfloor D \rfloor$ such that S is f-negative;
(4) $\rho(X/Z) = 1$.

Caution 2. I don't know what is the best definition of *pre limiting contractions* ! See [S, 1.1] and [FA, 18.6 Definition]. All definitions in this note may be changed.

Definition 3 (PL-flips)**.** A *pl-flip* is the S-flip of a pl contraction f in Definition 1.

The following is the definition of *log flips* in this note.

Definition 4 (Log flips)**.** By a *log flip* we mean the $(K_X + D)$-flip of a contraction $f : (X, D) \to Z$ assuming that

(a) (X, D) is klt,
(b) f is small,
(c) $-(K_X + D)$ is f-nef, and
(d) D is a $\mathbb{Q}$-divisor.

Definition 5. Let $f : X \dashrightarrow Y$ be a birational map. Let $\{E_i\}$ be the exceptional divisors of f^{-1} and D an $\mathbb{R}$-divisor on X. The *birational transform* of D is defined as

$$(D)_Y := f_*D + \sum E_i.$$

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Remark 6. For the definitions of *minimal models* and *canonical models*, see [KM, Definition 3.50].

The following is the main theorem of this note. This is essentially the same as [FA, Chapter 18].

Theorem 7 (Reduction Theorem). *Log flips exist in dimension n provided that:*

$(PLF)^{el}_n$ *(elementary) pl-flips exist in dimension n; and*
$(ST)_n$ *special termination holds in dimension n.*

Proof. Let (X, D) be a klt pair and let $f : X \to Z$ be a contraction as in Definition 4. We define $T := f(\mathrm{Exc}(f)) \subset Z$. We can assume that Z is affine without loss of generality.

Step 1. *Let H' be a Cartier divisor on Z such that*

(i) *$H := f^*H' = f_*^{-1}H'$ contains $\mathrm{Exc}(f)$.*
(ii) *H' is reduced and contains $\mathrm{Sing}(Z)$ and the singular locus of $\mathrm{Supp} f(D)$.*
(iii) *Fix a resolution $\pi : Z' \to Z$. Let $F_j \subset Z'$ be divisors that generate $N^1(Z'/Z)$. We assume that H' contains $\pi(F_j)$ for every j. (This usually implies that H' is reducible.) We note that we can assume that $\mathrm{Supp}\pi(F_j)$ contains no irreducible components of $\mathrm{Supp} f(D)$ for every j without loss of generality. Therefore, we can assume that H and D have no common irreducible components.*

The main consequence of the last assumption is the following:

(iv) *Let $h : Y \to Z$ be any proper birational morphism such that Y is $\mathbb{Q}$-factorial. Then the irreducible components of the proper transform of H' and the h-exceptional divisors generate $N^1(Y/Z)$.*

Step 2. *By Hironaka's desingularization theorem, there is a log resolution $h : Y \to X \to Z$ for $(X, D + H)$ that is an isomorphism over $Z \setminus H'$.*

Then $K_Y + (D + H)_Y$ is a $\mathbb{Q}$-factorial dlt pair, where $(D + H)_Y$ is a birational transform of $D + H$. Observe that $h^*(H')$ contains $h^{-1}(T)$ and $h^*(H')$ contains all h-exceptional divisors.

Step 3. *Apply the log MMP with respect to $K_Y + (D + H)_Y$ over Z. We successively construct the objects $(h_i : Y_i \to Z, (D+H)_{Y_i})$ such that $\lfloor (D + H)_{Y_i} \rfloor$ contains the support of h_i^*H', and every flipping curve for h_i is contained in $\mathrm{Supp} h_i^*H'$. If C_i is a flipping curve, then $C_i \subset h_i^*H'$ and $C_i \cdot h_i^*H' = 0$. By Step 1 (iv) and Step 2, there is an irreducible component $F_i \subset h_i^*H'$ such that $C_i \cdot F_i \neq 0$. Thus a suitable irreducible component of h_i^*H' intersects C_i negatively. This means that all flips*

that we need are (elementary) pl-flips. By the special termination, we end up with a $\mathbb{Q}$-factorial dlt pair $\overline{h} : (\overline{Y}, (D+H)_{\overline{Y}}) \to Z$ such that $K_{\overline{Y}} + (D+H)_{\overline{Y}}$ is $\overline{h}$-nef.

Step 4 (c.f. [KM, Theorem 7.44]). *This step is called "subtracting H". It is independent of other steps. So we use the different notation in Step 4. Of course, we assume $(PLF)_n^{el}$ and $(ST)_n$ throughout this step.*

Theorem 8 (Subtracting Theorem). *Let $(X, S+B+H)$ be an n-dimensional $\mathbb{Q}$-factorial dlt pair with effective $\mathbb{Q}$-divisors S, B, and H such that $\lfloor S \rfloor = S$, $\lfloor B \rfloor = 0$. Let $f : X \to Y$ be a projective birational morphism. Assume the following:*

(i) *$H \equiv_f -\sum b_j S_j$, where $b_j \in \mathbb{Q}_{\geq 0}$, where S_j is an irreducible component of S for every j.*
(ii) *$K_X + S + B + H$ is f-nef.*

Then $(X, S+B)$ has a minimal model over Y.

Proof. We give a proof in the form of several lemmas by running the log MMP over Y guided by H. The notation and the assumptions of Theorem 8 are assumed in these lemmas. □

Lemma 9. *There exists a rational number $\lambda \in [0,1]$ such that*

(1) *$K_X + S + B + \lambda H$ is f-nef, and*
(2) *if $\lambda > 0$, then there exists a $(K_X + S + B)$-negative extremal ray R over Y such that $R \cdot (K_X + S + B + \lambda H) = 0$.*

Proof. This follows from Cone Theorem. See, for example, [KM, Complement 3.6]. We note that [KM, §3.1] assumes that the pair has only klt singularities. However, Rationality Theorem holds for dlt pairs. Therefore, [KM, Complement 3.6] is true for dlt pairs. See [KM, Theorem 3.15, Remark 3.16]. □

If $\lambda = 0$, then the theorem is proved. Therefore, we assume that $\lambda > 0$ and let $\phi : X \to V$ be the contraction of R.

Lemma 10. *If ϕ contracts a divisor E, then the above conditions (i) and (ii) still hold if we replace $f : X \to Y$ with $V \to Y$ and B, S, H with $\phi_* B$, $\phi_* S$, $\lambda \phi_* H$.*

Proof. It is obvious. We note that ϕ is a divisorial contraction. □

Lemma 11. *If ϕ is a flipping contraction, then ϕ is an (elementary) pl contraction. If $p : X \dashrightarrow X^+$ is the flip of ϕ, then the above conditions (i) and (ii) still hold if we replace $f : X \to Y$ with $f^+ : X^+ \to Y$ and B, S, H with $p_* B$, $p_* S$, $\lambda p_* H$.*

Proof. One has to prove that ϕ is an (elementary) pl contraction. By hypothesis $R \cdot (K_X + S + B + \lambda H) = 0$ and $R \cdot (K_X + S + B) < 0$, one sees $R \cdot H > 0$. Hence by condition (i), there exists j_0 such that $R \cdot S_{j_0} < 0$. The latter part is obvious. □

Lemma 12. *We can apply the above procedure to the new set up in the cases Lemma 10 and Lemma 11 if $H \neq 0$. After repeating this finitely many times, H becomes 0, and one obtain a minimal model of $(X, S + B)$ over Y. In particular, Theorem 8 holds.*

Proof. It is obvious that Lemma 10 does not occur infinitely many times. The flip in Lemma 11 is a $(K_X + S + B)$-flip with the flipping curve on S. Hence it cannot repeat infinitely many times by the special termination. Therefore, we obtain a minimal model. □

Step 5. *We go back to the original setting. Apply Theorem 8 to $\overline{h} : (\overline{Y}, (D + H)_{\overline{Y}}) \to Z$, which was obtained in Step 3. More precisely, we put $f = \overline{h}$, $X = \overline{Y}$, $Y = Z$, $S + B + H = (D + H)_{\overline{Y}}$, $B = \{(D + H)_{\overline{Y}}\}$, and $H =$ the strict transform of H', and apply Theorem 8. Then we obtain*

$$\widetilde{h} : (\widetilde{Y}, D_{\widetilde{Y}}) \to Z$$

such that $\widetilde{Y}$ is $\mathbb{Q}$-factorial, $K_{\widetilde{Y}} + D_{\widetilde{Y}}$ is dlt and $\widetilde{h}$-nef. By the negativity lemma ([KM, Lemma 3.38])*, we can easily check that $\widetilde{h}$ is small and $(\widetilde{Y}, D_{\widetilde{Y}})$ is klt. This is a minimal model of (X, D) over Z.*

Step 6. *By the base point free theorem, we obtain the canonical model of the pair (X, D) over Z, which is the required flip.*

Therefore, we finish the proof of the reduction theorem. □

Corollary 13. *In dimension $n \leq 4$, $(PLF)_n^{el}$ implies the existence of all log flips.*

Proof. The special termination $(ST)_n$ holds if $n \leq 4$. Thus, this corollary is obvious by Theorem 7. □

References

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