1. Remarks on desingularization theorem

In this section, we discuss some remarks on desingularization theorem for the study of simple normal crossing varieties. First, we prove a slight generalization of Szabó’s resolution lemma (cf. [Sz, Resolution Lemma], [BEV, Theorem 7.11], and [K07, Proposition 6]).

**Theorem 1.1** (Resolution lemma for simple normal crossing varieties). Let $X$ be a simple normal crossing variety and $D$ a permissible Cartier divisor on $X$. Then there exists a proper birational morphism $f : Y \to X$ with the following properties:

1. $f$ is a composition of blow-ups of smooth subvarieties,
2. $Y$ is a simple normal crossing variety,
3. $\text{Supp}(f^{-1}_1 D \cup \text{Exc}(f))$ is a simple normal crossing divisor on $Y$, where $f^{-1}_1 D$ is the strict transform of $D$ on $Y$, and
4. $f$ is an isomorphism over $U$, where $U$ is the largest open set of $X$ such that $\text{Supp}D|_U$ is a simple normal crossing divisor on $U$.

Note that $f$ is projective and the exceptional locus $\text{Exc}(f)$ is of pure codimension one in $Y$ since $f$ is a composition of blow-ups.

**Proof.** Let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition. By Szabó’s resolution lemma (see, for example, [BEV, Theorem 7.11] and [K08, Proposition 6]), we obtain a sequence of smooth blow-ups

$$\Pi : \pi X \to \pi^{-1} X \to \cdots \to \pi X \to \pi X = X$$

whose centers are on the strict transforms of $X_1$ and have simple normal crossing with $\sum_{i \neq 1} X_i|_{X_1}$ such that

(A) $\text{Exc}(\Pi)$ is a simple normal crossing divisor on a simple normal crossing variety $\pi X$,

(B) $\Pi$ is an isomorphism over $U$, and

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*Date: 2010/8/16, Version 1.07.*

*I will use these statements in some papers.*
Supp(\(\Pi^{-1}D + \sum_{i \neq 1} X_i + \text{Exc}(\Pi)\)) |_{\tilde{X}_1} is a simple normal crossing divisor on \(\tilde{X}_1\), where \(\tilde{X}_i\) is the strict transform of \(X_i\) on \(\tilde{X}\) for every \(i\).

We note that \(\tilde{X}\) is a simple normal crossing variety for every \(j\) by the construction. Apply this argument to each irreducible component of \(X\). Then we obtain a desired birational map \(f : Y \to X\). □

Next, we discuss the elimination of indeterminacies for simple normal crossing varieties.

**Theorem 1.2** (Elimination of indeterminacies for simple normal crossing varieties). Let \(X\) be a simple normal crossing variety and \(g : X \dasharrow Y\) a rational map to a quasi-projective variety \(Y\). Let \(Z \subset X\) be the indeterminacy locus of \(g\). Assume that \(Z\) contains no strata of \(X\). Then there is a projective birational morphism \(f : X_0 \to X\) from a simple normal crossing variety \(X_0\) with the following properties:

1. \(f\) is a composition of blow-ups of smooth subvarieties,
2. the composition \(g \circ f : X_0 \to Y\) is a morphism, and
3. \(f\) is an isomorphism over \(X \setminus Z\).

**Proof.** Since \(Y\) is quasi-projective, we can assume that \(Y = \mathbb{P}^N\). By the assumption, the codimension of \(Z \geq 2\). Let \(X = \bigcup_{i} X_i\) be the irreducible decomposition of \(X\). We consider \(\mathcal{O}_{\mathbb{P}^N}(1)\) and a basis \(\{s_0, \ldots, s_N\}\) of \(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))\). We put \(g_i = g|_{X_i}\) for every \(i\). The line bundle \(g_i^*\mathcal{O}_{\mathbb{P}^N}(1)\) on \(X_i \setminus Z\) extends to a line bundle on \(X_i\) and \(g_i^*s_0, \ldots, g_i^*s_N\) are sections of \(\mathcal{L_i}\) on \(X_i\). Let \(\mathcal{M}_i \subset \mathcal{L}_i\) be the subsheaf generated by \(\{g_i^*s_0, \ldots, g_i^*s_N\}\). Then \(I_i = \mathcal{M}_i \otimes \mathcal{L}_i^{-1}\) is an ideal sheaf on \(X_i\). By the principalization theorem (see, for example, [K07, Theorem 3.26]), there is a sequence of smooth blow-ups

\[
\Pi : \tilde{X} = \pi X \to r_{-1}X \to \cdots \to r_1X \to rX = X
\]

whose centers are on the strict transforms of \(X_1\) and have simple normal crossing with \(\sum_{i \neq 1} X_i|_{X_1}\) (see [K07, Definitions 3.24 and 3.25]) such that

(A) \(I_i\mathcal{O}_{\tilde{X}_1} \subset \mathcal{O}_{\tilde{X}_1}\) is a locally principal ideal sheaf, where \(\tilde{X}_1\) is the strict transform of \(X_1\) on \(\tilde{X}\), and

(B) \(\Pi\) is an isomorphism over \(X \setminus Z\).

We note that \(\pi X\) is a simple normal crossing variety for every \(j\) by the construction. We replace \(X\) with \(\tilde{X}\) and apply the same argument to the irreducible component \(X_2\) of \(X\). Repeat this argument. Then we obtain a projective birational morphism \(f : X' \to X\) from a simple normal crossing variety \(X'\) such that \(I_i\mathcal{O}_{X'}\) is a locally principal ideal
sheaf for every $i$, where $X'_i$ is the strict transform of $X_i$ on $X'$. The global sections $(g_i \circ f)^*s_0, \ldots, (g_i \circ f)^*s_N \in H^0(X'_i, f^*\mathcal{L}_i)$ generate the locally free sheaf $I_{i,f^*\mathcal{L}_i}$ on $X'_i$ for every $i$. By the construction, we can glue the morphisms given by

$$X'_i \to \mathbb{P}^N : x \mapsto [(g_i \circ f)^*s_0(x) : \cdots : (g_i \circ f)^*s_N(x)]$$

for all $i$ and we obtain a morphism $g \circ f : X' \to \mathbb{P}^N$. \qed

We think the following problem is one of the most important problems on simple normal crossing varieties.

**Problem 1.3.** Let $X$ be a simple normal crossing variety. Are there any simple normal crossing varieties $\overline{X}$ such that

(i) $\overline{X}$ is complete,

(ii) $\overline{X}$ is projective if $X$ is quasi-projective,

(iii) $X$ is a Zariski open dense subset of $\overline{X}$,

(iv) $Z = \overline{X} \setminus X$ is a simple normal crossing divisor on $\overline{X}$, and

(v) $Z$ contains no strata of $\overline{X}$.

We note that it was already solved by Szabó’s resolution lemma when $X$ is a simple normal crossing divisor on a smooth variety $M$. We recommend the reader to see [Sz].

**Acknowledgments.** I thank Professors Takeshi Abe, Hiraku Kawanoue, Kenji Matsuki, and Shigefumi Mori.

**References**


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