REMARKS ON DESINGULARIZATION THEOREM

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1. Remarks on desingularization theorem

In this section, we discuss some remarks on desingularization theorem for the study of simple normal crossing varieties. First, we prove a slight generalization of Szabó's resolution lemma (cf. [Sz, Resolution Lemma], [BEV, Theorem 7.11], and [K07, Proposition 6]).

Theorem 1.1 (Resolution lemma for simple normal crossing varieties)**.** *Let X be a simple normal crossing variety and D a permissible Cartier divisor on X. Then there exists a proper birational morphism $f : Y \to X$ with the following properties*:

(1) *f is a composition of blow-ups of smooth subvarieties,*
(2) *Y is a simple normal crossing variety,*
(3) $\mathrm{Supp}(f_*^{-1}D \cup \mathrm{Exc}(f))$ *is a simple normal crossing divisor on Y, where $f_*^{-1}D$ is the strict transform of D on Y, and*
(4) *f is an isomorphism over U, where U is the largest open set of X such that* $\mathrm{Supp}D|_U$ *is a simple normal crossing divisor on U.*

Note that f is projective and the exceptional locus $\mathrm{Exc}(f)$ *is of pure codimension one in Y since f is a composition of blowing-ups.*

Proof. Let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition. By Szabó's resolution lemma (see, for example, [BEV, Theorem 7.11] and [K08, Proposition 6], we obtain a sequence of smooth blow-ups

$$\Pi : {}_rX \to {}_{r-1}X \to \cdots \to {}_1X \to {}_0X = X$$

whose centers are on the strict transforms of X_1 and have simple normal crossing with $\sum_{i \neq 1} X_i|_{X_1}$ such that

(A) $\mathrm{Exc}(\Pi)$ is a simple normal crossing divisor on a simple normal crossing variety ${}_rX$,
(B) Π is an isomorphism over U, and

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I will use these statements in some papers.

(C) $\mathrm{Supp}(\Pi_*^{-1}D+\sum_{i\neq 1} {}_rX_i+\mathrm{Exc}(\Pi))|_{{}_rX_1}$ is a simple normal crossing divisor on ${}_rX_1$, where ${}_rX_i$ is the strict transform of X_i on ${}_rX$ for every i.

We note that ${}_jX$ is a simple normal crossing variety for every j by the construction. Apply this argument to each irreducible component of X. Then we obtain a desired birational map $f : Y \to X$. □

Next, we discuss the elimination of indeterminacies for simple normal crossing varieties.

Theorem 1.2 (Elimination of indeterminacies for simple normal crossing varieties)**.** *Let X be a simple normal crossing variety and $g : X \dashrightarrow Y$ a rational map to a quasi-projective variety Y. Let $Z \subset X$ be the indeterminacy locus of g. Assume that Z contains no strata of X. Then there is a projective birational morphism $f : X' \to X$ from a simple normal crossing variety X' with the following properties*:

1. *f is a composition of blow-ups of smooth subvarieties,*
2. *the composition $g \circ f : X' \to Y$ is a morphism, and*
3. *f is an isomorphism over $X \setminus Z$.*

Proof. Since Y is quasi-projective, we can assume that $Y = \mathbb{P}^N$. By the assumption, the codimension of $Z \geq 2$. Let $X = \bigcup_{i\in I} X_i$ be the irreducible decomposition of X. We consider $\mathcal{O}_{\mathbb{P}^N}(1)$ and a basis $\{s_0, \cdots, s_N\}$ of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. We put $g_i = g|_{X_i}$ for every i. The line bundle $g_i^*\mathcal{O}_{\mathbb{P}^N}(1)$ on $X_i \setminus Z$ extends to a line bundle on X_i and $g_i^*s_0, \cdots g_i^*s_N$ are sections of $\mathcal{L}_i$ on X_i. Let $\mathcal{M}_i \subset \mathcal{L}_i$ be the subsheaf generated by $\{g_i^*s_0, \cdots, g_i^*s_N\}$. Then $I_i = \mathcal{M}_i \otimes \mathcal{L}_i^{-1}$ is an ideal sheaf on X_i. By the principalization theorem (see, for example, [K07, Theorem 3.26]), there is a sequence of smooth blow-ups

$$\Pi : \widetilde{X} = {}_rX \to {}_{r-1}X \to \cdots \to {}_1X \to {}_0X = X$$

whose centers are on the strict transforms of X_1 and have simple normal crossing with $\sum_{i\neq 1} X_i|_{X_1}$ (see [K07, Definitions 3.24 and 3.25]) such that

(A) $I_1\mathcal{O}_{\widetilde{X}_1} \subset \mathcal{O}_{\widetilde{X}_1}$ is a locally principal ideal sheaf, where $\widetilde{X}_1$ is the strict transform of X_1 on $\widetilde{X}$, and
(B) Π is an isomorphism over $X \setminus Z$.

We note that ${}_jX$ is a simple normal crossing variety for every j by the construction. We replace X with $\widetilde{X}$ and apply the same argument to the irreducible component X_2 of X. Repeat this argument. Then we obtain a projective birational morphism $f : X' \to X$ from a simple normal crossing variety X' such that $I_i\mathcal{O}_{X'_i}$ is a locally principal ideal sheaf for every i, where X'_i is the strict transform of X_i on X'. The global sections $(g_i \circ f)^*s_0, \cdots, (g_i \circ f)^*s_N \in H^0(X'_i, f^*\mathcal{L}_i)$ generate the locally free sheaf $I_i f^*\mathcal{L}_i$ on X'_i for every i. By the construction, we can glue the morphisms given by

$$X'_i \to \mathbb{P}^N : x \mapsto [(g_i \circ f)^*s_0(x) : \cdots : (g_i \circ f)^*s_N(x)]$$

for all i and we obtain a morphism $g \circ f : X' \to \mathbb{P}^N$. □

We think the following problem is one of the most important problems on simple normal crossing varieties.

Problem 1.3. Let X be a simple normal crossing variety. Are there any simple normal crossing varieties $\overline{X}$ such that

(i) $\overline{X}$ is complete,
(ii) $\overline{X}$ is projective if X is quasi-projective,
(iii) X is a Zariski open dense subset of $\overline{X}$,
(iv) $Z = \overline{X} \setminus X$ is a simple normal crossing divisor on $\overline{X}$, and
(v) Z contains no strata of $\overline{X}$.

We note that it was already solved by Szabó's resolution lemma when X is a simple normal crossing divisor on a smooth variety M. We recommend the reader to see fujino0 [F7].

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