REMARKS ON LOG PLURICANONICAL REPRESENTATIONS

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Abstract. We show the finiteness of log pluricanonical representations under the assumption of the existence of a good minimal model.

Contents

1. Introduction

This short paper is a supplement to [FG] (see also [HX]). One of the main purposes of this paper is to establish:

Theorem 1.1. *Let (X, Δ) be a projective log canonical pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume that (X, Δ) has a good minimal model. Then there exists a positive integer k such that the image of*

$$\rho_{km} \colon \operatorname{Bir}(X, \Delta) \to \operatorname{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(km(K_X + \Delta)))$$

is a finite group for every positive integer m, where

$$\operatorname{Bir}(X, \Delta) := \{\sigma \mid \sigma \colon (X, \Delta) \dashrightarrow (X, \Delta) \text{ is } B\text{-birational}\}.$$

We make five important remarks on Theorem 1.1.

Remark 1.2. In Theorem 1.1, k is the smallest positive integer such that $k(K_X + \Delta)$ and $k(K_{X'} + \Delta')$ are both Cartier, where (X', Δ') is a good minimal model of (X, Δ).

Remark 1.3. If $K_X + \Delta$ is semiample, that is, (X, Δ) itself is a good minimal model of (X, Δ), then Theorem 1.1 is nothing but [FG, Theorem 1.1] (see also Remark 1.2). In this paper, we will show that we can reduce Theorem 1.1 to [FG, Theorem 1.1]. We note that [FG, Theorem 1.1] plays a crucial role in the study of the abundance conjecture for semi-log canonical pairs.

Remark 1.4 (Log canonical pairs of log general type)**.** If (X, Δ) is a projective log canonical pair such that $K_X + \Delta$ is big, then $\operatorname{Bir}(X, \Delta)$ is already a finite group (see [FG, Corollary 3.13]). Therefore, Theorem 1.1 is obvious when $K_X + \Delta$ is big.

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Remark 1.5 (Kawamata log terminal pairs)**.** We have already established some better results for kawamata log terminal pairs. For the details, see [FG, §3.1], where we do not need the minimal model program. Hence, in this paper, we are mainly interested in a log canonical pair (X, Δ) which is not kawamata log terminal. For smooth varieties, see also [U, §14].

Remark 1.6. In a recent preprint [F5], the first author established the finiteness of relative log pluricanonical representations in a suitable complex analytic setting. For the details, see [F5].

Similarly to Theorem 1.1, we can prove the following theorem.

Theorem 1.7. *Let V be a smooth quasi-projective variety and let X be a smooth projective completion of V such that $\Delta := X \setminus V$ is a simple normal crossing divisor on X. Assume that (X, Δ) has a good minimal model. Then there exists a positive integer k such that the image of*

$$\rho_{km} \colon \mathrm{PBir}(V) \to \mathrm{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(km(K_X + \Delta)))$$

is a finite group for every positive integer m, where

$$\mathrm{PBir}(V) := \{\sigma \mid \sigma \colon V \dashrightarrow V \text{ is proper birational}\}.$$

Note that Theorem 1.7 implies Theorem 1.8.

Theorem 1.8 (Affine varieties)**.** *Let V be a smooth affine variety and let X be a smooth projective completion of V such that $\Delta := X \setminus V$ is a simple normal crossing divisor on X. Then there exists a positive integer k such that the image of*

$$\rho_{km} \colon \mathrm{Aut}(V) \to \mathrm{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(km(K_X + \Delta)))$$

is a finite group for every positive integer m, where

$$\mathrm{Aut}(V) := \{\sigma \mid \sigma \colon V \to V \text{ is an isomorphism}\}.$$

To the best knowledge of the authors, Theorem 1.8 is nontrivial and new.

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Throughout this paper, we will work over $\mathbb{C}$, the field of complex numbers. We will freely use the standard notation as in [F2] and [F3].

2. Preliminaries

In this section, we recall some basic definitions and properties necessary for this paper. For the standard notation of the theory of minimal models, see, for example, [F2], [F3], and so on. Let us start with the definition of B-birational maps, which was first introduced in [F1].

Definition 2.1 (B-birational maps)**.** Let (X, Δ) be a projective log canonical pair. We say that a birational map $\sigma \colon X \dashrightarrow X$ is *B-birational* if there exists a commutative diagram

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ X & \overset{}{\underset{\sigma}{\dashrightarrow}} & X \end{array}$$

such that Z is a normal projective variety, α and β are projective birational morphisms, and

$$\alpha^*(K_X + \Delta) = \beta^*(K_X + \Delta)$$

holds. We put

$$\mathrm{Bir}(X, \Delta) := \{\sigma \mid \sigma\colon (X, \Delta) \dashrightarrow (X, \Delta) \text{ is } B\text{-birational}\}.$$

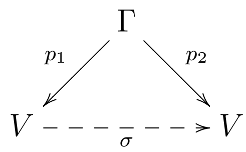
Then $\mathrm{Bir}(X, \Delta)$ has a natural group structure. We take a positive integer m such that $m(K_X + \Delta)$ is Cartier. Then it is easy to see that we have a group homomorphism

$$\rho_m\colon \mathrm{Bir}(X, \Delta) \to \mathrm{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(m(K_X + \Delta))).$$

We call it the *B-pluricanonical representation* or *log pluricanonical representation* for the pair (X, Δ).

For the details of $\mathrm{Bir}(X, \Delta)$, see [FG]. Let us recall the notion of proper birational maps (see, for example, [I]).

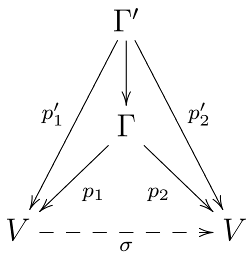
Definition 2.2 (Proper birational maps)**.** Let V be a quasi-projective variety. We say that a birational map $\sigma\colon V \dashrightarrow V$ is *proper birational* if p_1 and p_2 are projective, where Γ is the graph of $\sigma\colon V \dashrightarrow V$ and p_1 and p_2 are projections from Γ to V as in the following commutative diagram (see [I, Proposition 2.17 (i)]).



We put

$$\mathrm{PBir}(V) := \{\sigma \mid \sigma\colon V \dashrightarrow V \text{ is proper birational}\}.$$

Then it is easy to see that $\mathrm{PBir}(V)$ has a natural group structure (see [I, Proposition 2.17 (iii)]). We note that if V is affine and normal then σ is an isomorphism for every $\sigma \in \mathrm{PBir}(V)$ by Zariski's main theorem (see [I, Theorem 2.19 and Corollary]). We further assume that V is smooth and X is a smooth projective completion of V such that $\Delta := X \setminus V$ is a simple normal crossing divisor on X. Let $\Gamma' \to \Gamma$ be a projective resolution of singularities and let Y be a projective completion of Γ' such that $\Delta_Y := Y \setminus \Gamma'$ is a simple normal crossing divisor on Y. Then we have the following commutative diagram.



Let m be any positive integer. In this case, for every $\sigma \in \mathrm{PBir}(V)$, we can define $\sigma^* \in \mathrm{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ as follows:

$$\begin{aligned}\sigma^* := (\overline{p}_1'^*)^{-1} \circ \overline{p}_2'^*\colon H^0(X, \mathcal{O}_X(m(K_X + \Delta))) &\to H^0(Y, \mathcal{O}_Y(m(K_Y + \Delta_Y))) \\ &\to H^0(X, \mathcal{O}_X(m(K_X + \Delta))),\end{aligned}$$

where $\overline{p}'_i\colon Y \to X$ is the extension of $p'_i\colon \Gamma' \to V$ for $i = 1, 2$. We can easily see that it is independent of the choice of (Y, Δ_Y) (see [I, Theorem 11.1]). Then we can consider the following group homomorphism

$$\rho_m\colon \mathrm{PBir}(V) \to \mathrm{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$$

by setting $\rho_m(\sigma) := \sigma^*$ for every positive integer m.

For the sake of completeness, we recall the definition of good minimal models in the sense of Birkar–Shokurov.

Definition 2.3 (Good minimal models)**.** Let (X, Δ) be a projective log canonical pair. Let $\psi\colon X \dashrightarrow X'$ be a birational map and let E be the reduced ψ^{-1}-exceptional divisor on X', that is, $E = \sum_j E_j$ where E_j are the ψ^{-1}-exceptional prime divisors on X'. Then (X', Δ') is called a *good minimal model* of (X, Δ) (in the sense of Birkar–Shokurov) if

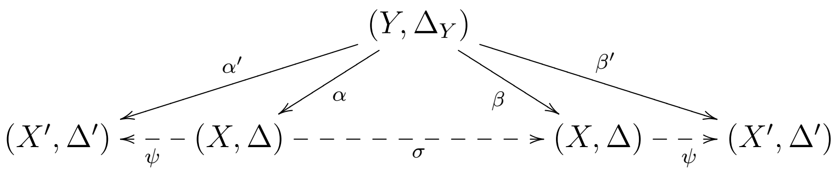
(1) (X', Δ') is a projective $\mathbb{Q}$-factorial divisorial log terminal pair, where $\Delta' := \psi_*\Delta + E$,
(2) $K_{X'} + \Delta'$ is semiample, and
(3) $a(D, X, \Delta) < a(D, X', \Delta')$ holds for every ψ-exceptional prime divisor D on X.

In Definition 2.3, we note that we can prove $a(P, X, \Delta) \le a(P, X', \Delta')$ for every prime divisor P over X by the negativity lemma.

3. Proofs

In this section, we prove Theorems 1.1, 1.7, and 1.8.

Proof of Theorem 1.1. Let $\psi\colon (X, \Delta) \dashrightarrow (X', \Delta')$ be a good minimal model. Then we can construct the following commutative diagram



where Y is a smooth projective variety with

$$\alpha^*(K_X + \Delta) =: K_Y + \Delta_Y := \beta^*(K_X + \Delta)$$

such that the support of Δ_Y is a simple normal crossing divisor on Y. Then we can write

$$K_Y + \Delta_Y^{>0} = \alpha'^*(K_{X'} + \Delta') + E$$

and

$$K_Y + \Delta_Y^{>0} = \beta'^*(K_{X'} + \Delta') + F$$

such that E and F are both effective with $\alpha'_*E = 0$ and $\beta'_*F = 0$. We note that $E - F$ is α'-nef and $\alpha'_*(F - E) \ge 0$. This implies $F \ge E$ by the well-known negativity lemma. Similarly, we can prove that $E \ge F$ holds. Thus we have $E = F$. This means that $\psi \circ \sigma \circ \psi^{-1} \in \mathrm{Bir}(X', \Delta')$. We take the smallest positive integer k such that $k(K_X + \Delta)$ and $k(K_{X'} + \Delta')$ are both Cartier. By [FG, Theorem 1.1], we know that the image of

$$\rho'_{km}\colon \mathrm{Bir}(X', \Delta') \to \mathrm{Aut}_{\mathbb{C}} H^0(X', \mathcal{O}_{X'}(km(K_{X'} + \Delta')))$$

is a finite group for every positive integer m. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Bir}(X,\Delta) & \xrightarrow{\rho_{km}} & \mathrm{Aut}_{\mathbb{C}}\, H^0(X, \mathcal{O}_X(km(K_X+\Delta))) \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \Pi}\ \simeq \\ \mathrm{Bir}(X',\Delta') & \xrightarrow[\rho'_{km}]{} & \mathrm{Aut}_{\mathbb{C}}\, H^0(X', \mathcal{O}_{X'}(km(K_{X'}+\Delta'))), \end{array}$$

where $\pi(\sigma) = \psi \circ \sigma \circ \psi^{-1}$ for $\sigma \in \mathrm{Bir}(X,\Delta)$ and $\Pi(g) = (\psi^{-1})^* \circ g \circ \psi^*$ for $g \in \mathrm{Aut}_{\mathbb{C}}\, H^0(X, \mathcal{O}_X(km(K_X+\Delta)))$. Hence we obtain the desired finiteness. We finish the proof of Theorem 1.1. □

The proof of Theorem 1.7 is essentially the same as that of Theorem 1.1.

Proof of Theorem 1.7. Let $\psi\colon (X,\Delta) \to (X',\Delta')$ be a good minimal model. As in Definition 2.2, we construct (Y,Δ_Y) and obtain the following commutative diagram:

$$\begin{array}{ccccccc} & & & (Y,\Delta_Y) & & & \\ & {\scriptstyle \alpha'}\swarrow & {\scriptstyle \alpha}\swarrow & & \searrow{\scriptstyle \beta} & \searrow{\scriptstyle \beta'} & \\ (X',\Delta') & \overset{\psi}{\dashleftarrow} & (X,\Delta) & \overset{\sigma}{\dashrightarrow} & (X,\Delta) & \overset{\psi}{\dashrightarrow} & (X',\Delta'), \end{array}$$

where α (resp. β) is the extension of $p'_1\colon \Gamma' \to V$ (resp. $p'_2\colon \Gamma' \to V$). By construction, we have $\Delta_Y = \alpha^{-1}(\Delta) = \beta^{-1}(\Delta)$ since $\sigma\colon V \dashrightarrow V$ is proper birational (see [I, Lemma 11.2]). As in the proof of Theorem 1.1, we write

$$K_Y + \Delta_Y = \alpha'^*(K_{X'} + \Delta') + E$$

and

$$K_Y + \Delta_Y = \beta'^*(K_{X'} + \Delta') + F$$

with $E \geq 0$, $F \geq 0$, $\alpha'_* E = 0$, and $\beta'_* F = 0$. Then, by the negativity lemma, we can prove that $E = F$ holds, that is, $\psi \circ \sigma \circ \psi^{-1} \in \mathrm{Bir}(X',\Delta')$. We take the smallest positive integer k such that $k(K_{X'} + \Delta')$ is Cartier. Then, by [FG, Theorem 1.1]. the image of

$$\rho'_{km}\colon \mathrm{Bir}(X',\Delta') \to \mathrm{Aut}_{\mathbb{C}}\, H^0(X', \mathcal{O}_{X'}(km(K_{X'}+\Delta')))$$

is a finite group for every positive integer m. Hence, by the same argument as in the proof of Theorem 1.1, we obtain the desired finiteness. We finish the proof of Theorem 1.7. □

Finally, we prove Theorem 1.8, which is an easy application of Theorem 1.7.

Proof of Theorem 1.8. As we mentioned before, $\mathrm{PBir}(V) = \mathrm{Aut}(V)$ holds by Zariski's main theorem since V is a smooth affine variety. Moreover, it is well known that (X,Δ) has a good minimal model when $\kappa(X, K_X + \Delta) \geq 0$ since V is affine (see the proof of [F4, Proposition 4.1]). Thus, the desired statement follows from Theorem 1.7. We finish the proof of Theorem 1.8. □

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