On relative good minimal models
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CHAPTER 1

1.

1.7.6say 1.1 (Sufficiently general fibers). Let $f : X \to Y$ be a morphism between algebraic varieties. Then a sufficiently general fiber F of $f : X \to Y$ means that $F = f^{-1}(y)$ where y is any point contained in a countable intersection of Zariski dense open subsets of Y.

Note that a sufficiently general fiber is sometimes called a very general fiber in the literature.

2.

To make Theorem ?? more useful, we note the following result, which is buried in [Bir4], [HaX1], and [Lai]. Although Theorem 2.1 holds for klt pairs (X, Δ), we state it for smooth varieties for simplicity.

Tthm1.1 THEOREM 2.1. *Let $f : X \to Y$ be a projective surjective morphism from a smooth quasi-projective variety X to a normal quasi-projective variety Y with connected fibers. Then the following conditions are equivalent.*

(i) *X has a relative good minimal model over Y.*
(ii) *$X_{\overline{\eta}}$ has a good minimal model, where $X_{\overline{\eta}}$ is the geometric generic fiber of $f : X \to Y$.*
(iii) *the geometric generic fiber of the Iitaka fibration of $X_{\overline{\eta}}$ has a good minimal model.*
(iv) *F has a good minimal model, where F is a sufficiently general fiber of $f : X \to Y$.*
(v) *the geometric generic fiber of the Iitaka fibration of F has a good minimal model.*
(vi) *G has a good minimal model, where G is a general fiber of $f : X \to Y$.*
(vii) *the geometric generic fiber of the Iitaka fibration of G has a good minimal model.*

In order to understand Theorem 2.1, we give some supplementary results.

Tthm1.2 **Theorem 2.2.** *Let (X,Δ) be a projective klt pair such that Δ is a $\mathbb{Q}$-divisor. Then (X,Δ) has a good minimal model if and only if*

$$\kappa(X, K_X+\Delta)=\kappa_\sigma(X, K_X+\Delta).$$

Proof. For the proof, see [GL, Theorem 4.3] or [DHP, Remark 2.6]. □

Tlem1.3 **Lemma 2.3.** *Let $f:X\to Y$ be a projective surjective morphism between normal varieties with connected fibers and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X,Δ) is klt. Let $X_{\overline{\eta}}$ be the geometric generic fiber of $f:X\to Y$. We put $\Delta_{\overline{\eta}}=\Delta|_{X_{\overline{\eta}}}$. Then we have*

$$\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}}+\Delta_{\overline{\eta}})=\kappa(F, K_F+\Delta|_F)$$

and

$$\kappa_\sigma(X_{\overline{\eta}}, K_{X_{\overline{\eta}}}+\Delta_{\overline{\eta}})=\kappa_\sigma(F, K_F+\Delta|_F)$$

where F is a sufficiently general fiber of $f:X\to Y$.

Proof. This is obvious by the definitions of Iitaka's D-dimension κ and Nakayama's numerical Kodaira dimension κ_σ. For the details, see Definition ??, [Nak2] and [Le]. □

By combining Lemma 2.2 with Lemma 2.3, we have:

Tcor1.4 **Corollary 2.4.** *Let $f:X\to Y$ be a projective surjective morphism between normal varieties and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X,Δ) is klt. Then $(X_{\overline{\eta}},\Delta_{\overline{\eta}})$ has a good minimal model if and only if $(F,\Delta|_F)$ has a good minimal model where F is a sufficiently general fiber of $f:X\to Y$.*

Proof. This statement is obvious by Lemma 2.2 and Lemma 2.3. □

The following theorem follows from [Lai, Theorem 4.4] (see also [Bir4, Theorem 1.5] and [HaX1, Theorem 2.12]).

Tthm1.5 **Theorem 2.5.** *Let X be a smooth projective variety with non-negative Kodaira dimension. Then X has a good minimal model if and only if the geometric generic fiber of the Iitaka fibration of X has a good minimal model.*

Proof. See [Lai, Theorem 4.4], [Bir4, Theorem 5.1], and [HaX1, Theorem 2.12]. □

Let us give a sketch of the proof of Theorem 2.1 for the reader's convenience.

Sketch of the proof of Theorem 2.1. We divide the proof into several steps.

Tstep1 STEP 1 ((ii)$\Longleftrightarrow$(iii), (iv)$\Longleftrightarrow$(v), and (vi)$\Longleftrightarrow$(vii)). This step is nothing but Theorem Tthm1.5 2.5.

Tstep2 STEP 2 ((ii)$\Longleftrightarrow$(iv)). This step is a special case of Corollary Tcor1.4 2.4.

Tstep3 STEP 3 ((vi)$\Longrightarrow$(iv)). This is obvious since a sufficiently general fiber of $f : X \to Y$ is a general fiber of $f : X \to Y$.

Tstep4 STEP 4 ((i)$\Longrightarrow$(vi)). Let $f' : X' \to Y$ be a relative good minimal model of X over Y. Let $y \in Y$ be a general point. We consider the fibers $G = f^{-1}(y)$ and $G' = f'^{-1}(y)$. If G' is not $\mathbb{Q}$-factorial, then we take a small projective $\mathbb{Q}$-factorization of G'. Then we see that G' is a good minimal model of G.

Tstep5 STEP 5 ((iv)$\Longrightarrow$(i)). This is a special case of hacon-xu1 [HaX1, Theorem 2.12] (see also the proof of birkar4 [Bir4, Theorem 5.1]).

We complete the proof of Theorem Tthm1.1 2.1. $\square$

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[1] Replace [La1] and [La2] with [Laz1] and [Laz2] respectively.