On relative good minimal models
2014/9/2

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1.1 (Sufficiently general fibers). Let $f : X \to Y$ be a morphism between algebraic varieties. Then a sufficiently general fiber $F$ of $f : X \to Y$ means that $F = f^{-1}(y)$ where $y$ is any point contained in a countable intersection of Zariski dense open subsets of $Y$.

Note that a sufficiently general fiber is sometimes called a very general fiber in the literature.

2. To make Theorem 1.1 more useful, we note the following result, which is buried in [Bir4], [HaX1], and [Lai]. Although Theorem 1.1 holds for klt pairs $(X, \Delta)$, we state it for smooth varieties for simplicity.

**Theorem 2.1.** Let $f : X \to Y$ be a projective surjective morphism from a smooth quasi-projective variety $X$ to a normal quasi-projective variety $Y$ with connected fibers. Then the following conditions are equivalent.

- (i) $X$ has a relative good minimal model over $Y$.
- (ii) $X_\eta$ has a good minimal model, where $X_\eta$ is the geometric generic fiber of $f : X \to Y$.
- (iii) the geometric generic fiber of the Iitaka fibration of $X_\eta$ has a good minimal model.
- (iv) $F$ has a good minimal model, where $F$ is a sufficiently general fiber of $f : X \to Y$.
- (v) the geometric generic fiber of the Iitaka fibration of $F$ has a good minimal model.
- (vi) $G$ has a good minimal model, where $G$ is a general fiber of $f : X \to Y$.
- (vii) the geometric generic fiber of the Iitaka fibration of $G$ has a good minimal model.

In order to understand Theorem 2.1, we give some supplementary results.
**Theorem 2.2.** Let \((X, \Delta)\) be a projective klt pair such that \(\Delta\) is a \(\mathbb{Q}\)-divisor. Then \((X, \Delta)\) has a good minimal model if and only if
\[
\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta).
\]

**Proof.** For the proof, see [GL, Theorem 4.3] or [DHP, Remark 2.6].

**Lemma 2.3.** Let \(f : X \to Y\) be a projective surjective morphism between normal varieties with connected fibers and let \(\Delta\) be an effective \(\mathbb{Q}\)-divisor on \(X\) such that \((X, \Delta)\) is klt. Let \(X_\pi\) be the geometric generic fiber of \(f : X \to Y\). We put \(\Delta_\pi = \Delta_{|X_\pi}\). Then we have
\[
\kappa(X_\pi, K_{X_\pi} + \Delta_\pi) = \kappa(F, K_F + \Delta_{|F})
\]
and
\[
\kappa_\sigma(X_\pi, K_{X_\pi} + \Delta_\pi) = \kappa_\sigma(F, K_F + \Delta_{|F})
\]
where \(F\) is a sufficiently general fiber of \(f : X \to Y\).

**Proof.** This is obvious by the definitions of Iitaka’s \(D\)-dimension \(\kappa\) and Nakayama’s numerical Kodaira dimension \(\kappa_\sigma\). For the details, see Definition 2.4.8def [Nak2] and Definition 2.4.8def [Le].

By combining Lemma 2.2 with Lemma 2.3, we have:

**Corollary 2.4.** Let \(f : X \to Y\) be a projective surjective morphism between normal varieties and let \(\Delta\) be an effective \(\mathbb{Q}\)-divisor on \(X\) such that \((X, \Delta)\) is klt. Then \((X_\pi, \Delta_\pi)\) has a good minimal model if and only if \((F, \Delta_{|F})\) has a good minimal model where \(F\) is a sufficiently general fiber of \(f : X \to Y\).

**Proof.** This statement is obvious by Lemma 2.2 and Lemma 2.3.

The following theorem follows from [Lai, Theorem 4.4] (see also [Bir4, Theorem 1.5] and [HaX1, Theorem 2.12]).

**Theorem 2.5.** Let \(X\) be a smooth projective variety with non-negative Kodaira dimension. Then \(X\) has a good minimal model if and only if the geometric generic fiber of the Iitaka fibration of \(X\) has a good minimal model.

**Proof.** See [Lai, Theorem 4.4], [Bir4, Theorem 5.1], and [HaX1, Theorem 2.12].

Let us give a sketch of the proof of Theorem 2.1 for the reader’s convenience.

**Sketch of the proof of Theorem 2.1.** We divide the proof into several steps.
STEP 1 ((ii)\leftrightarrow(iii), (iv)\leftrightarrow(v), and (vi)\leftrightarrow(vi)). This step is nothing but Theorem 2.5.

STEP 2 ((ii)\leftrightarrow(iv)). This step is a special case of Corollary 2.4.

STEP 3 ((vi)\implies(iv)). This is obvious since a sufficiently general fiber of $f : X \to Y$ is a general fiber of $f : X \to Y$.

STEP 4 ((i)\implies(vi)). Let $f' : X' \to Y$ be a relative good minimal model of $X$ over $Y$. Let $y \in Y$ be a general point. We consider the fibers $G = f^{-1}(y)$ and $G' = f'^{-1}(y)$. If $G'$ is not $\mathbb{Q}$-factorial, then we take a small projective $\mathbb{Q}$-factorization of $G'$. Then we see that $G'$ is a good minimal model of $G$.

STEP 5 ((iv)\implies(i)). This is a special case of [HaX1, Theorem 2.12] (see also the proof of [Bir4, Theorem 5.1]).

We complete the proof of Theorem 2.1. \hfill \Box
## Bibliography

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<th>Reference</th>
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\[1\] Replace [La1] and [La2] with [Laz1] and [Laz2] respectively.