$\overline{\mathbf{C}}_{\mathbf{n,n-1}}$ REVISITED

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Abstract. The main purpose of this paper is to make $\overline{C}_{n,n-1}$, which is the main theorem of [Ka1], more accessible.

1. Introduction

In spite of its importance, the proof of $\overline{C}_{n,n-1}$ is not so easy to access for the younger generation, including myself. After [Ka1] was published, the birational geometry has drastically developed. When Kawamata wrote [Ka1], the following techniques and results are not known nor fully matured.

- Kawamata's covering trick,
- moduli theory of curves, especially, the notion of level structures and the existence of tautological families,
- various notions of singularities such as rational singularities, canonical singularities, and so on.

See [Ka2, §2], [AK, Section 5], [AO, Part II], [vGO], [V2], and [KM]. In the mid 1990s, de Jong gave us fantastic results: [dJ1] and [dJ2]. The alteration paradigm generated the weak semistable reduction theorem [AK]. This paper shows how to recover the main theorem of [Ka1] by using the weak semistable reduction. The proof may look much simpler than Kawamata's original proof (note that we have to read [V1] to understand [Ka1]). However, the alteration theorem grew out from the deep investigation of the moduli space of stable pointed curves (see [dJ1] and [dJ2]). So, don't misunderstand the real value of this paper. We note that we do not enforce Kawamata's arguments. We only recover his main result. Of course, this paper is not self-contained.

The following result is the main theorem of [Ka1]. We call this $\overline{C}_{n,n-1}$ in this paper. Here, n means the dimension of X.

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Theorem 1.1 ([Ka1, Theorem 1]). *Let $f : X \longrightarrow Y$ be a dominant morphism of algebraic varieties defined over the complex number field $\mathbb{C}$. Assume that the general fibre $X_y = f^{-1}(y)$ is an irreducible curve. Then we have the following inequality for logarithmic Kodaira dimensions*:

$$\overline{\kappa}(X) \geq \overline{\kappa}(Y) + \overline{\kappa}(X_y).$$

In Section 2, we will give a proof to [Ka1, Theorem 2], which is stronger than $\overline{C}_{n,n-1}$. See the inequality $(\overline{C}'_{n,n-1})$ in the first paragraph of the proof below.

Note that our reference list does not cover all the papers treating the related topics. We apologize in advance to the colleagues whose works were not appropriately mentioned in this paper.

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Notation. We will work over $\mathbb{C}$ throughout this paper. For the basic properties of the logarithmic Kodaira dimension, see [I1], [I2], [I3], and [Ka1, §1].

(i) Let X be a (not necessarily complete) variety. Then $\overline{\kappa}(X)$ denotes the *logarithmic Kodaira dimension* of X.

(ii) Let $f : X \longrightarrow Y$ be a dominant morphism between varieties and D a $\mathbb{Q}$-divisor on X. We can write $D = D_{\mathrm{hor}} + D_{\mathrm{ver}}$ such that every irreducible component of D_{hor} (resp. D_{ver}) is mapped (resp. not mapped) onto Y. If $D = D_{\mathrm{hor}}$ (resp.$D = D_{\mathrm{ver}}$), D is said to be *horizontal* (resp. *vertical*).

(iii) Let $f : X \longrightarrow Y$ be a birational morphism. Then $\mathrm{Exc}(f)$ denotes the exceptional locus of f.

2. $\overline{C}_{n,n-1}$

Here, we prove the following theorem. It is easy to see that this statement is equivalent to Theorem 1.1 by the basic properties of the logarithmic Kodaira dimension.

Theorem 2.1 ($\overline{C}_{n,n-1}$). *Let $f : X \longrightarrow Y$ be a surjective morphism with connected fibers between non-singular projective varieties X and Y. Let C and D be simple normal crossing divisors on X and Y. We put $X_0 := X \setminus C$ and $Y_0 := Y \setminus D$. Assume that $f(X_0) \subset Y_0$. Then*

$$\overline{\kappa}(X_0) \geq \overline{\kappa}(Y_0) + \overline{\kappa}(F_0),$$

where F_0 is a sufficiently general fiber of $f_0 := f|_{X_0} : X_0 \longrightarrow Y_0$.

Before we start the proof, let us recall the following trivial lemma. We will frequently use it without mentioning it.

Lemma 2.2. *Let X be a complete normal variety. Let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that $D_1 \geq D_2$. Then $\kappa(D_1) \geq \kappa(D_2)$.*

Proof of Theorem 2.1. By Theorem 2 in [Ka1], it is sufficient to prove

$$(\overline{C}'_{n.n-1}) \qquad \kappa(K_X + C - f^*(K_Y + D)) \geq \overline{\kappa}(F_0).$$

Step 1. By Theorem 2.1 in [AK] (see also [Kr, Chapter 2, Remark 4.5 and Section 9]), we have the following commutative diagram:

$$\begin{array}{ccccc} X & \longleftarrow & X' & \supset & U_{X'} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Y' & \supset & U_{Y'} \end{array}$$

such that $p : X' \longrightarrow X$ and $q : Y' \longrightarrow Y$ are projective birational morphisms, X' is quasi-smooth (in particular, $\mathbb{Q}$-factorial) and Y' is non-singular, the inclusion on the right are toroidal embeddings, and such that

1. $f' : (U_{X'} \subset X') \longrightarrow (U_{Y'} \subset Y')$ is toroidal and equi-dimensional,
2. Let $C' := (p^*C)_{\text{red}}$ and $D' := (q^*D)_{\text{red}}$. Then $C' \subset X' \setminus U_{X'}$ and $D' \subset Y' \setminus U_{Y'}$.

Since

$$\overline{\kappa}(X_0) = \kappa(K_X + C) = \kappa(K_{X'} + C')$$

and

$$\overline{\kappa}(Y_0) = \kappa(K_Y + D) = \kappa(K_{Y'} + D'),$$

we can replace $f : X \longrightarrow Y$ with $f' : X' \longrightarrow Y'$. For the simplicity of the notation, we omit the superscript $'$. So, we can assume that $f : X \longrightarrow Y$ is toroidal with the above extra assumptions.

Step 2. By taking a Kawamata's Kummer cover $q : Y' \longrightarrow Y$, we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{p} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[q]{} & Y' \end{array}$$

such that $f' : X' \longrightarrow Y'$ is weakly semistable, where X' is the normalization of $X \times_Y Y'$ (see [AK, Section 5]). We put $G := X \setminus U_X$ and $H := Y \setminus U_Y$. Then we have

$$K_X + C - f^*(K_Y + D) \geq K_X + C_{\text{hor}} + G_{\text{ver}} - f^*(K_Y + H).$$

Therefore, we can check that

$$p^*(K_X + C - f^*(K_Y + D)) \geq K_{X'/Y'} + (p^*C)_{\mathrm{hor}}.$$

We note that $(p^*C)_{\mathrm{hor}} = p^*(C_{\mathrm{hor}})$. So, it is sufficient to prove that $\kappa(K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \overline{\kappa}(F_0)$.

Step 3. Let F be a general fiber of $f : X \longrightarrow Y$. We put $g := g(F)$: the genus of F.

Case $(g \geq 2)$**.** In this case,

$$\kappa(K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \kappa(K_{X'/Y'}) \geq 1 = \overline{\kappa}(F_0).$$

The last inequality is well-known. So, we stop the proof in this case.

Case $(g = 1)$**.** It is well-known that

$$\kappa(K_{X'/Y'}) \geq \mathrm{Var}(f') = \mathrm{Var}(f) \geq 0.$$

For the definition of the *variation* $\mathrm{Var}(f)$, see, for instance, [V3, p.329]. So, if C is vertical or $\mathrm{Var}(f) \geq 1$, then we obtain

$$\kappa(K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) \geq \overline{\kappa}(F_0).$$

Therefore, we can assume that $\mathrm{Var}(f) = 0$ and C is not vertical. By Kawamata's covering trick, we obtain the following commutative diagram:

$$\begin{tikzcd} X' \arrow[d, "f'"'] & X'' \arrow[l, "\pi"'] \arrow[d, "f''"] \\ Y' & Y'', \arrow[l, "\eta"] \end{tikzcd}$$

where $\eta : Y'' \longrightarrow Y'$ is a Kawamata's Kummer cover from a non-singular projective variety Y'', $f'' : X'' := X' \times_{Y'} Y'' \longrightarrow Y''$ is weakly semistable, and f'' is birationally equivalent to $Y'' \times E \longrightarrow Y''$. Here, E is an elliptic curve. Note that, if we need, we blow-up Y' and replace X' with its base change before taking the cover. For details, see [AK, Lemma 6.2] and the proof of [Ka2, Corollary 19]. Since

$$\pi^*(K_{X'/Y'} + (p^*C)_{\mathrm{hor}}) = K_{X''/Y''} + \pi^*((p^*C)_{\mathrm{hor}}),$$

it is sufficient to prove $\kappa(K_{X''/Y''} + \pi^*((p^*C)_{\mathrm{hor}})) \geq 1$. Let $\alpha : \widetilde{X} \longrightarrow Y'' \times E$, $\beta : \widetilde{X} \longrightarrow X''$ be a common resolution. Since X'' has only rational Gorenstein singularities, X'' has at worst canonical singularities. Thus, we obtain

$$\kappa(K_{X''/Y''} + \pi^*((p^*C)_{\mathrm{hor}})) = \kappa(K_{\widetilde{X}/Y''} + \beta^*\pi^*((p^*C)_{\mathrm{hor}})).$$

On the other hand,

$$K_{\widetilde{X}/Y''} = K_{\widetilde{X}/Y''\times E} + K_{Y''\times E/Y''} =: A$$

is an effective α-exceptional divisor such that $\mathrm{Supp}A = \mathrm{Exc}(\alpha)$. Let B be an irreducible component of $\beta^*\pi^*((p^*C)_{\mathrm{hor}})$ such that B is dominant onto Y''. Then

$$m(A + \beta^*\pi^*((p^*C)_{\mathrm{hor}})) \geq \alpha^*\alpha_*B,$$

for a sufficiently large integer m. Therefore, if is sufficient to prove $\kappa(Y''\times E, \alpha_*B) \geq 1$. It is true by [F2, Corollary 5.4]. Thus, we finish the proof when $g = 1$.

Case $(g = 0)$**.** As in the above case, we can take a Kawamata's Kummer cover and obtain the following commutative diagram:

$$\begin{tikzcd} X' \arrow[d, "f'"'] & X'' \arrow[l, "\pi"'] \arrow[d, "f''"] \\ Y' & Y'', \arrow[l, "\eta"] \end{tikzcd}$$

where f'' is birationally equivalent to $Y''\times\mathbb{P}^1 \longrightarrow Y''$. We can further assume that all the horizontal components of $\pi^*((p^*C)_{\mathrm{hor}})$ are mapped onto Y'' birationally.

Lemma 2.3 (cf. [F1, Section 7])**.** *Let $f : V \longrightarrow W$ be a surjective morphism between non-singular projective varieties with connected fibers. Assume that f is birationally equivalent to $W\times\mathbb{P}^1 \longrightarrow \mathbb{P}^1$. Let $\{C_k\}$ be a set of distinct irreducible divisors such that $f : C_k \longrightarrow W$ is birational for every k $(k \leq 3)$. Then*

$$\kappa(K_{V/W} + C_1 + C_2) \geq 0$$

and

$$\kappa(K_{V/W} + C_1 + C_2 + C_3) \geq 1.$$

Proof. By modifying V and W birationally (see also [F1, Lemma 7.8]) and replacing C_k with its strict transform, we can assume that there exists a simple normal crossing divisor Σ on W such that

$$\varphi_{ij} : V_0 := f^{-1}(W_0) \simeq W_0 \times \mathbb{P}^1$$

with $\varphi_{ij}(C_i|_{V_0}) = W_0 \times \{0\}$ and $\varphi_{ij}(C_j|_{V_0}) = W_0 \times \{\infty\}$ for $i \neq j$, where $W_0 := W \setminus \Sigma$. We can further assume that there exists $\psi_{ij} : V \longrightarrow \mathbb{P}^1$ such that $\psi_{ij}|_{V_0} = p_2 \circ \varphi_{ij}$, where p_2 is the second projection

$W_0 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. We also assume that $\cup_k C_k \cup (f^*\Sigma)_{\mathrm{red}}$ is a simple normal crossing divisor. we obtain

$$\begin{aligned} \wedge{\psi_{ij}}^* \left(\frac{dz}{z}\right) &\in \mathrm{Hom}_{\mathcal{O}_V}(f^*(K_W+\Sigma), K_V + C_i + C_j + (f^*\Sigma)_{\mathrm{red}}) \\ &\simeq H^0(V, K_{V/W} + C_i + C_j + (f^*\Sigma)_{\mathrm{red}} - f^*\Sigma) \\ &\subset H^0(V, K_{V/W} + C_i + C_j) \end{aligned}$$

for $i \neq j$, where z denotes a suitable inhomogeneous coordinate of $\mathbb{P}^1$ (see [F1, Lemma 7.12]). Therefore,

$$\dim_{\mathbb{C}} H^0(V, K_{V/W} + C_1 + C_2) \geq 1$$

and

$$\dim_{\mathbb{C}} H^0(V, K_{V/W} + C_1 + C_2 + C_3) \geq 2.$$

Thus, we obtain the required result. □

Apply Lemma 2.3 to $\widetilde{X} \longrightarrow Y''$, where $\beta : \widetilde{X} \longrightarrow X''$ is a resolution of X''. Then we obtain

$$\kappa(K_{\widetilde{X}/Y''} + \beta^*\pi^*((p^*C)_{\mathrm{hor}})) \geq \overline{\kappa}(F_0).$$

Thus, we complete the proof. □

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