NOTES ON RATIONAL CHAIN CONNECTEDNESS

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ABSTRACT. We extend Hacon–M^{c}Kernan's rational chain connectedness theorem to the complex analytic setting. As a consequence, we prove that the fibers of any resolution of singularities of complex analytic kawamata log terminal singularities are rationally chain connected. In contrast to the original approach, we avoid the use of extension theorems and instead rely on the minimal model program.

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1. INTRODUCTION

In this paper, we generalize the main results of [HM2] to projective morphisms between complex analytic spaces. The overall strategy of the proofs follows the same lines as in the algebraic case. A few remarks are in order. First, [HM2] was written before [BCHM], and the technically most demanding part of the proof relies on the extension theorem established in [HM1]. This extension theorem is so intricate that even experts may find it difficult to remember all of its statements. In this paper, we avoid using the extension theorem and instead rely on the minimal model program developed in [F5]. Of course, the minimal model program itself ultimately depends, at least in part, on extension theorems; thus, we are not entirely free from them. Nevertheless, our approach, which only uses standard arguments from the minimal model program, is arguably more accessible to the reader. While we work in the setting of complex analytic spaces, in the algebraic case, one can similarly obtain a proof of [HM2] that superficially avoids the use of extension theorems by applying [BCHM]. We emphasize once again that the existence of flips established in [HM3] still relies heavily on the extension theorem.

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In Theorem 5.1, we extend [HM2, Theorem 5.1] to the complex analytic setting. For convenience, we refer to [HM2, Theorem 5.1] as *Hacon–McKernan's rational chain connectedness theorem*; it constitutes the principal result of [HM2]. Since the precise formulation of Theorem 5.1 is rather technical and lengthy, we refrain from reproducing it here. As an immediate consequence, we obtain the following theorem.

Theorem 1.1 ([HM2, Theorem 1.2]). *Let X be a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f\colon X \to S$ be a projective morphism between complex analytic spaces. Assume that $-K_X$ is f-big and that $-(K_X + \Delta)$ is f-semiample. Let $g\colon Y \to X$ be any bimeromorphic morphism, and let $\pi\colon Y \to S$ be the induced morphism. Then the connected components of every fiber of π are rationally chain connected modulo the inverse image of the non-klt locus of (X, Δ).*

Theorem 1.1 admits several useful corollaries.

Corollary 1.2 ([HM2, Corollary 1.5]). *Let (X, Δ) be a divisorial log terminal pair. Then we have:*

(i) *The fibers of any bimeromorphic morphism $g\colon Y \to X$ are rationally chain connected.*
(ii) *Assume further that X is projective. Then X is rationally chain connected if and only if it is rationally connected.*

Corollary 1.3 ([HM2, Corollary 1.6]). *Let $f\colon X \dashrightarrow Y$ be a meromorphic map between normal complex varieties, and assume that (X, Δ) is a divisorial log terminal pair for some effective $\mathbb{R}$-divisor Δ. Let $\Gamma \subset X \times Y$ be the graph of f, and denote by $p\colon \Gamma \to X$ and $q\colon \Gamma \to Y$ the natural projections. Then for every point $x \in X$ at which f is not defined,*

$$q\big(p^{-1}(x)\big) \subset Y$$

is rationally chain connected.

Corollary 1.4. *Let $f\colon X \dashrightarrow Y$ be a meromorphic map between complex analytic spaces, and assume that (X, Δ) is divisorial log terminal. If Y contains no rational curves, then f is a morphism.*

Corollaries 1.2, 1.3, and 1.4 are stated for divisorial log terminal pairs. However, we note the following simple observation.

Remark 1.5. Let (X, Δ) be a kawamata log terminal pair. Then (X, Δ) is divisorial log terminal.

In [HM2], the following theorem is derived as an application of Hacon–McKernan's rational chain connectedness theorem (see [HM2, Theorem 5.1]). It is by now standard that Theorem 1.6 admits a more direct proof.

Theorem 1.6. *Let X be a normal projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Assume that $-(K_X + \Delta)$ is ample. Then X is rationally chain connected modulo* $\mathrm{Nklt}(X, \Delta)$. *Moreover, in the case where (X, Δ) is kawamata log terminal, X is rationally connected.*

Theorem 1.7 is an application of Theorem 1.6 and the theory of quasi-log structures.

Theorem 1.7 (Rational chain connectedness, see [F4, Theorem 1.14] and [F7, Theorem 9.8]). *Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$ and let $[X, \omega]$ be a quasi-log complex analytic space. Assume that $-\omega$ is π-ample. Then*

$\pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$ for every point $P \in S$. In particular, if $\pi^{-1}(P) \cap X_{-\infty} = \emptyset$ further holds, that is, $[X, \omega]$ is quasi-log canonical in a neighborhood of $\pi^{-1}(P)$, then $\pi^{-1}(P)$ is rationally chain connected.

The following corollary is a special case of Theorem 1.7. Consequently, Corollary 1.8 follows from Theorem 1.6 and is independent of Theorem 1.1 and Theorem 5.1.

Corollary 1.8 ([HM2, Corollary 1.3])**.** *Let X be a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f\colon X \to S$ be a projective morphism between normal complex varieties. Assume that:*

(1) *(X, Δ) is kawamata log terminal, $-(K_X + \Delta)$ is f-nef, and $-K_X$ is f-big; or*
(2) *(X, Δ) is log canonical and $-(K_X + \Delta)$ is f-ample.*

Then every connected component of every fiber of f is rationally chain connected.

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We follow [F1] for the basic definitions and notation of the minimal model program. For the complex analytic setting, we refer to [F5] and [F6]. For the basic properties of uniruled, rationally connected, and rationally chain connected varieties, we follow [D] and [K]. Throughout this paper, all algebraic varieties are defined over $\mathbb{C}$.

2. Preliminaries

We follow [F1, Chapter 2], [F5, Sections 2 and 3], and [F6, Chapter 2] for the standard definitions and notation in the minimal model program.

We begin by recalling several auxiliary definitions and basic properties that will be used throughout the paper.

Definition 2.1 (Boundary part)**.** Let $D = \sum_i d_i D_i$ be an effective $\mathbb{R}$-divisor, where $d_i \in \mathbb{R}$, each D_i is a prime divisor, and $D_i \neq D_j$ for $i \neq j$. Define

$$D^b := \sum_i \min\{d_i, 1\} D_i, \qquad D^{nb} := D - D^b.$$

Then $D^{nb} \geq 0$, and we call D^b the *boundary part* of D.

Definition 2.2 (Non-klt locus)**.** Let X be a normal complex variety, and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The *non-klt locus* of (X, Δ), denoted by $\mathrm{Nklt}(X, \Delta)$, is the smallest closed analytic subset $Z \subset X$ such that (X, Δ) is kawamata log terminal on $X \setminus Z$.

In this paper, we regard $\mathrm{Nklt}(X, \Delta)$ simply as a subset of X and do not consider its complex analytic structure.

Definition 2.3. Let $f\colon X \dashrightarrow Z$ be a dominant rational map between projective varieties, and let V be a Zariski closed subset of X. We say that V *dominates* Z if there exists an elimination of indeterminacy $g\colon Y \to X$ of f, that is, a birational morphism from a projective variety Y such that the induced map

$$h := f \circ g\colon Y \to Z$$

is a morphism and satisfies $h(g^{-1}(V)) = Z$. This notion is independent of the choice of the resolution g.

Definition 2.4 (Nefness)**.** Let $f\colon X \to Y$ be a projective morphism of complex analytic spaces, and let $W \subset Y$ be a subset. Let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X, or the sum of an $\mathbb{R}$-line bundle and an $\mathbb{R}$-Cartier divisor. We say that $\mathcal{L}$ is *f-nef over W* if $\mathcal{L} \cdot C \geq 0$ for every curve $C \subset X$ such that $f(C)$ is a point of W. If $\mathcal{L}$ is f-nef over Y, we simply say that it is f-nef.

There are several characterizations of relative bigness. For our purposes, the following property is sufficient, so we do not recall the definition here.

Lemma 2.5 (Kodaira's lemma)**.** *Let $f\colon X \to S$ be a projective surjective morphism of normal complex varieties, and let D be an f-big $\mathbb{R}$-divisor on X. Assume that S is a Stein space. Then there exists a decomposition*

$$D \sim_{\mathbb{Q},f} A + B,$$

where A is an f-ample $\mathbb{Q}$-divisor and B is an effective $\mathbb{R}$-divisor.

For the sake of completeness, we explicitly state the basepoint-free theorem in the complex analytic setting that will be used in this paper.

Lemma 2.6 (Basepoint-free theorem, see [HM2, Lemma 7.1], [F5, Theorem 8.1], and [F6, Theorem 5.3.1])**.** *Let (X, Δ) be a kawamata log terminal pair, and let $f\colon X \to S$ be a projective morphism of complex analytic spaces. Suppose that $-(K_X + \Delta)$ is f-nef over a point $s \in S$, and that $-K_X$ is f-big. Then $-(K_X + \Delta)$ is f-semiample over some open neighborhood of s in S.*

Proof of Lemma 2.6. We may shrink S around s without further mention. Since $-K_X$ is f-big, we can write

$$-K_X \sim_{\mathbb{Q},f} A + B,$$

where A is an f-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and B is an effective $\mathbb{Q}$-divisor. We set

$$\Theta := (1 - \varepsilon)\Delta + \varepsilon B.$$

For sufficiently small $\varepsilon > 0$, the pair (X, Θ) remains kawamata log terminal. Moreover, we have

$$-(K_X + \Theta) \sim_{\mathbb{Q},f} -(1 - \varepsilon)(K_X + \Delta) + \varepsilon A,$$

which is f-ample. Hence, by the basepoint-free theorem for $\mathbb{R}$-divisors in the complex analytic setting (see [F5, Theorem 8.1], and [F6, Theorem 5.3.1]), $-(K_X + \Delta)$ is f-semiample over some open neighborhood of $s \in S$, as desired. □

We next recall the notions of uniruledness, rational connectedness, and rational chain connectedness, which are central to the statements and arguments of this paper.

Definition 2.7 (Uniruledness, see [K, Chapter IV, 1.1 Definition])**.** Let X be an algebraic variety. We say that X is *uniruled* if there exist an algebraic variety Y of dimension $\dim X - 1$ and a dominant rational map

$$\mathbb{P}^1 \times Y \dashrightarrow X.$$

Definition 2.8 (Rational connectedness, see [K, Chapter IV, 3.6 Proposition])**.** Let X be a compact complex variety. We say that X is *rationally connected* if, for two general points $x_1, x_2 \in X$, there exists an irreducible rational curve $C \subset X$ containing both x_1 and x_2.

Definition 2.9 (Rational chain connectedness)**.** Let X be a compact complex analytic space. We say that X is *rationally chain connected* if, for any two points $x_1, x_2 \in X$, there exists a connected curve $C \subset X$ such that $x_1, x_2 \in C$ and every irreducible component of C is rational.

We record the following elementary lemma.

Lemma 2.10. *Let $g\colon Z \to Y$ and $f\colon Y \to X$ be proper surjective morphisms of complex analytic spaces. If $(f\circ g)^{-1}(P)$ is rationally chain connected, then $f^{-1}(P)$ is also rationally chain connected.*

Proof of Lemma 2.10. This follows since the natural morphism

$$(f \circ g)^{-1}(P) \to f^{-1}(P)$$

is surjective. □

Definition 2.11 ([HM2, Definition 1.1])**.** Let X be a compact complex analytic space, and let $V \subset X$ be a closed analytic subset. We say that X is *rationally chain connected modulo V* if

(1) $V = \emptyset$ and X is rationally chain connected, or
(2) $V \neq \emptyset$ and for every $x \in X$, there exist a connected pointed curve C with marked points $0, \infty \in C$, such that every irreducible component of C is rational, and a morphism $h_x\colon C \to X$ satisfying $h_x(0) = x$ and $h_x(\infty) \in V$.

The following fundamental results play a central role in this paper. Since they are well known, we state them without proof.

Theorem 2.12 ([BDPP, Corollary 0.3])**.** *Let Z be a smooth projective variety. Then Z is uniruled if and only if its canonical divisor K_Z is not pseudo-effective.*

Theorem 2.13 ([GHS, Corollary 1.4])**.** *Let X be a projective variety, and let $\phi\colon X \dashrightarrow Z$ be its maximal rationally connected fibration* (MRC fibration*, for short*)*. Then Z is not uniruled.*

We refer the reader to [K, Chapter IV.5, Maximal Rationally Connected Fibrations] for a comprehensive treatment of MRC fibrations.

3. Inequality for the Kodaira Dimension

In this section, we discuss an inequality for the Kodaira dimension that is less widely known and may at first appear technical. For this reason, we present it here in detail.

Theorem 3.1 ([HM1, Lemma 2.9 and Corollary 2.11])**.** *Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Let $\pi\colon X \to Y$ be a surjective morphism onto a smooth projective variety Y with connected fibers. Assume that*

$$\kappa\big(X_y, (K_X + \Delta)|_{X_y}\big) \geq 0,$$

where X_y is a sufficiently general fiber of π. Let H be any ample Cartier divisor on Y and let $\varepsilon > 0$ be any rational number. Then

$$\kappa\big(X, K_{X/Y} + \Delta + \varepsilon\pi^*H\big) \geq \dim Y. \tag{3.1}$$

In particular, if K_Y is pseudo-effective, then

$$\kappa\big(X, K_X + \Delta + \varepsilon\pi^*H\big) \geq \dim Y. \tag{3.2}$$

For the sake of completeness, we give a proof of Theorem 3.1.

Proof of Theorem 3.1. We divide the proof into three steps. In Step 1, we prove (3.1) under the additional assumption that π is equidimensional. In Step 2, we remove this assumption and establish (3.1) in full generality. Finally, in Step 3, we deduce (3.2).

Step 1. In this step, we prove (3.1) under the extra assumption that π is equidimensional.

Let a be a positive integer such that $a(K_X+\Delta)$ is Cartier and

$$\pi_*\mathcal{O}_X\big(a(K_{X/Y}+\Delta)\big)\neq 0.$$

Then, by [F2, Proposition 9.1], we have

$$\kappa\big(X, a(K_{X/Y}+\Delta)+\pi^*H\big)\geq \dim Y.$$

This implies

$$\kappa\big(X, K_{X/Y}+\Delta+\varepsilon\pi^*H\big)\geq \dim Y$$

for any rational number $\varepsilon>0$.

Step 2. In this step, we prove (3.1) in full generality.

By [AK, Theorem 2.1] (see also [ADK, Theorem 1.1]), we can construct a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\ h\ } & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ Y & \xleftarrow[\ g\ ]{} & Y' \end{array}$$

with the following properties:

(1) $\pi'\colon X'\to Y'$ is an equidimensional surjective morphism from a normal projective variety X' to a smooth projective variety Y',
(2) h and g are birational morphisms,
(3) X' has only quotient singularities, and there exists a nonempty Zariski open subset $U_{X'}\subset X'$ such that $(U_{X'}\subset X')$ is toroidal,
(4) $\operatorname{Exc}(h)\cup \operatorname{Supp} h^{-1}_*\Delta\subset X'\setminus U_{X'}$.

We write

$$K_{X'}+\Delta'=h^*(K_X+\Delta)+E,$$

where (X',Δ') is log canonical and E is an effective h-exceptional $\mathbb{Q}$-divisor. We also write

$$K_{Y'}=g^*K_Y+F,$$

where F is an effective g-exceptional divisor. Then

$$K_{X'/Y'}+\Delta'=h^*(K_{X/Y}+\Delta)+E-\pi'^*F.$$

Let H' be an ample Cartier divisor on Y', and let $\varepsilon'>0$ be a rational number such that

$$\kappa\big(Y', g^*H-\varepsilon'H'\big)\geq 0.$$

Then we obtain

$$\begin{aligned}\kappa\big(X, K_{X/Y}+\Delta+\varepsilon\pi^*H\big)&\geq \kappa\big(X', K_{X'/Y'}+\Delta'+\varepsilon\pi'^*g^*H\big)\\ &\geq \kappa\big(X', K_{X'/Y'}+\Delta'+\varepsilon\varepsilon'\pi'^*H'\big)\\ &\geq \dim Y.\end{aligned}$$

The last inequality follows from Step 1, since π' is equidimensional.

Step 3. In this step, we prove (3.2).

Assume that K_Y is pseudo-effective. Let ε' be a rational number such that $0 < \varepsilon' < \varepsilon$. Since

$$K_X + \Delta + \varepsilon\pi^* H = K_{X/Y} + \Delta + \varepsilon'\pi^* H + \pi^*\big(K_Y + (\varepsilon - \varepsilon')H\big),$$

we have

$$\kappa\big(X, K_X + \Delta + \varepsilon\pi^* H\big) \geq \kappa\big(X, K_{X/Y} + \Delta + \varepsilon'\pi^* H\big) \geq \dim Y,$$

where the last inequality follows from (3.1). This proves (3.2).

This completes the proof of Theorem 3.1. □

Corollary 3.2. *In Theorem 3.1, the assumption that (X, Δ) is log canonical can be replaced by the following slightly weaker condition, which is required in the proof of Proposition 4.1. Namely, Δ is an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is log canonical over the generic point of Y.*

Proof of Corollary 3.2. By taking a resolution of singularities, we may assume that X is smooth and that $\operatorname{Supp}\Delta$ is a simple normal crossing divisor on X. We can choose an effective $\mathbb{Q}$-divisor Δ' on X such that (X, Δ') is log canonical, $\Delta \geq \Delta'$, and $\Delta = \Delta'$ holds over the generic point of Y. Applying Theorem 3.1 to the pair (X, Δ') and using the inequality $\Delta \geq \Delta'$, we obtain the desired inequalities. □

We note that Theorem 3.1, and hence Corollary 3.2, ultimately rely on Campana's twisted weak positivity (see [F2, Theorem 1.1]). Alternatively, one may apply directly the theory of mixed ω-sheaves developed in [F3]. In fact, Theorem 3.1 and Corollary 3.2 follow immediately from Theorem 3.3 below. In any case, Theorem 3.1 and Corollary 3.2 are now well known to experts.

Theorem 3.3 ([F3, Corollary 9.5])**.** *Let $\pi\colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y. Let Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is log canonical over the generic point of Y. Let H be an ample Cartier divisor on Y.*

Then there exists a positive integer m with the following property: for every integer $a \geq 2$ such that $a(K_X + \Delta)$ is Cartier and

$$\pi_*\mathcal{O}_X\big(a(K_X + \Delta)\big) \neq 0,$$

the sheaf

$$\mathcal{O}_Y(mH) \otimes \pi_*\mathcal{O}_X\big(a(K_{X/Y} + \Delta)\big)$$

is generically globally generated. In particular, for every such a, we have

$$\kappa\big(X, a(K_{X/Y} + \Delta) + m\pi^* H\big) \geq 0.$$

4. A Criterion due to Hacon–McKernan

In this section, for completeness, we recall the criterion of Hacon and McKernan for rational chain connectedness modulo a subset. Our exposition follows [HM2, Section 4]. Lemma 4.4 is new and will play a crucial role from the perspective of minimal model theory.

Proposition 4.1 ([HM2, Proposition 4.1])**.** *Let X be a normal projective variety and let Δ be an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $h\colon X \to F$ be a dominant morphism and $t\colon F \dashrightarrow Z$ a dominant rational map between projective varieties. Assume that:*

(1) *The non-klt locus of* (X,Δ) *does not dominate* Z *via the induced map* $t\circ h\colon X \dashrightarrow Z$.
(2) $K_X+\Delta$ *has nonnegative Kodaira dimension on the general fiber of* $X \dashrightarrow Z$, *that is, if* $g\colon Y\to X$ *resolves the indeterminacy of* $X\dashrightarrow Z$ *and* Y *is smooth projective, then* $g^*(K_X+\Delta)$ *has Kodaira dimension at least zero on the general fiber of the induced morphism* $Y\to Z$.
(3) $\kappa(X,K_X+\Delta)\le 0$.
(4) *There exists an effective big* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor* A *on* F *satisfying* $h^*A\le\Delta$.

Then Z *is either a point or uniruled.*

Proof of Proposition 4.1. It suffices to show that Z is a point under the assumption that Z is not uniruled. By resolution of singularities, we may assume without loss of generality that Z is smooth.

Take a resolution of singularities $g\colon Y\to X$ such that the induced rational map $Y\dashrightarrow Z$ is a morphism $\psi\colon Y\to Z$, and such that $E':=\mathrm{Exc}(g)$ and $\mathrm{Exc}(g)\cup\mathrm{Supp}\, g_*^{-1}\Delta$ are simple normal crossing divisors on Y. We write

$$K_Y+\Theta=g^*(K_X+\Delta)+E,$$

where Θ and E are effective $\mathbb{Q}$-divisors with no common irreducible components, $g_*\Theta=\Delta$, and E is g-exceptional. Set

$$\Gamma:=\Theta+\varepsilon E',$$

where $\varepsilon>0$ is a sufficiently small rational number.

By assumption (1) and the above construction, the pair (Y,Γ) is kawamata log terminal over the generic point of Z. Moreover, by assumption (2), we have

$$\kappa\big(Y_z,(K_Y+\Gamma)|_{Y_z}\big)\ge 0,$$

where Y_z is a sufficiently general fiber of ψ.

By construction, the support of Γ contains the exceptional locus of g. Therefore, by assumption (4), Γ contains the pull-back of a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on F. Hence, possibly after replacing Γ by a $\mathbb{Q}$-linearly equivalent $\mathbb{Q}$-divisor, we can find an ample $\mathbb{Q}$-divisor G on Z such that

$$\psi^*G\le\Gamma.$$

We can arrange the above replacement so that (Y,Γ) remains kawamata log terminal over the generic point of Z.

Since Z is not uniruled by assumption, Theorem 2.12 implies that K_Z is pseudo-effective. Therefore, by Corollary 3.2, we obtain

$$\kappa\big(Y,K_Y+\Gamma\big)\ge\dim Z.$$

On the other hand,

$$\kappa\big(Y,K_Y+\Gamma\big)=\kappa\big(X,K_X+\Delta\big)\le 0$$

by assumption (3). It follows that $\dim Z=0$, that is, Z is a point. This completes the proof of Proposition 4.1. □

Lemma 4.2 ([HM2, Lemma 4.2])**.** *Let* F *be a projective variety.*

(i) F *is rationally chain connected if and only if every nonconstant dominant rational map* $t\colon F\dashrightarrow Z$ *of projective varieties has uniruled target.*
(ii) F *is rationally chain connected modulo* V *if and only if for every nonconstant dominant rational map* $t\colon F\dashrightarrow Z$ *of projective varieties, the target* Z *is either uniruled or dominated by* V.

Proof of Lemma 4.2. In Step 1, we prove (i). In Step 2, we prove (ii).

Step 1. In this step, we prove (i).

Assume first that F is rationally chain connected, and let $t\colon F \dashrightarrow Z$ be a nonconstant dominant rational map. Then Z is also rationally chain connected. In particular, Z is uniruled. This proves the "only if" direction.

Conversely, assume that every nonconstant dominant rational map $t\colon F \dashrightarrow Z$ has uniruled target. In particular, since the identity map $F \dashrightarrow F$ is a dominant rational map, it follows that F itself is uniruled.

Let $F \dashrightarrow Z$ be the maximal rationally connected fibration of F. By Theorem 2.13, the base Z is not uniruled. Hence Z must be a point. Therefore, F is rationally chain connected. This completes the proof of (i).

Step 2. In this step, we prove (ii). By (i), we may assume throughout this step that $V \neq \emptyset$.

Assume first that F is rationally chain connected modulo V, and let $t\colon F \dashrightarrow Z$ be a nonconstant dominant rational map. If V does not dominate Z, then Z is covered by images of rational curves. In particular, Z is uniruled. This proves the "only if" direction.

Conversely, assume that for every nonconstant dominant rational map $t\colon F \dashrightarrow Z$, the target Z is either uniruled or dominated by V. As in Step 1, by considering the identity map $F \dashrightarrow F$, we see that F is uniruled. Hence the maximal rationally connected fibration $F \dashrightarrow Z$ is nontrivial.

By Theorem 2.13, the base Z of the maximal rationally connected fibration is not uniruled. Therefore, by assumption, Z must be dominated by V. This implies that F is rationally chain connected modulo V.

This completes the proof of Lemma 4.2. □

Corollary 4.3 ([HM2, Corollary 4.3])**.** *Let X be a normal projective variety and let Δ be an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $h\colon X \to F$ be a dominant morphism of projective varieties. Assume that for every dominant rational map $t\colon F \dashrightarrow Z$ of projective varieties, one of the following holds:*

- *the non-klt locus of (X, Δ) dominates Z, or*
- *conditions (2)–(4) of Proposition 4.1 are satisfied.*

Then F is rationally chain connected modulo the image $R \subset F$ of the non-klt locus of (X, Δ).

Proof of Corollary 4.3. We apply the criterion given in Lemma 4.2. Let $t\colon F \dashrightarrow Z$ be an arbitrary dominant rational map of projective varieties.

If the image R of the non-klt locus of (X, Δ) dominates Z, there is nothing to prove. Otherwise, R does not dominate Z, and we claim that Z is either a point or uniruled.

Indeed, since R does not dominate Z, condition (1) of Proposition 4.1 is satisfied. By assumption, conditions (2)–(4) of Proposition 4.1 also hold. Therefore, Proposition 4.1 implies that Z is either uniruled or a point.

By Lemma 4.2, this shows that F is rationally chain connected modulo R. This completes the proof of Corollary 4.3. □

In this paper, we apply Corollary 4.3 via the following lemma.

Lemma 4.4. *Let W be a normal projective variety, and let Δ_W be an effective $\mathbb{Q}$-divisor on W such that $K_W + \Delta_W$ is $\mathbb{Q}$-Cartier. Let $\varphi\colon V \to W$ be a birational morphism from a* *smooth projective variety V such that both* $\mathrm{Exc}(\varphi)$ *and* $\mathrm{Exc}(\varphi)\cup \mathrm{Supp}\,\varphi_*^{-1}\Delta_W$ *are simple normal crossing divisors on V. Write*

$$K_V+\Delta_V=\varphi^*(K_W+\Delta_W)+E,$$

where Δ_V and E have no common irreducible components, $\varphi_*\Delta_V=\Delta_W$, and E is φ-exceptional.

Assume the following:

(a) $K_W+\Delta_W\sim_{\mathbb{Q}} P$ *for some effective $\mathbb{Q}$-divisor P, equivalently,* $\kappa(W,K_W+\Delta_W)\geq 0$.
(b) (W,Δ_W^b) *is divisorial log terminal.*
(c) $\kappa(W,K_W+\Delta_W^b)\leq 0$.
(d) *There exists an effective big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor Q on W such that $Q\leq \Delta_W^b$.*

Then V is rationally chain connected modulo $\mathrm{Nklt}(V,\Delta_V)$.

Proof of Lemma 4.4. Set

$$E^\dagger:=\mathrm{Exc}(\varphi)\quad\text{and}\quad \Theta:=\Delta_V+\varepsilon E^\dagger,$$

where $\varepsilon>0$ is a sufficiently small rational number. Then

$$\mathrm{Nklt}(V,\Theta^b)=\mathrm{Nklt}(V,\Theta)=\mathrm{Nklt}(V,\Delta_V).$$

We apply Corollary 4.3 to the pair (V,Θ^b) with respect to the identity morphism $V\to V$. Let $t\colon V\dashrightarrow Z$ be a dominant rational map of projective varieties. If $\mathrm{Nklt}(V,\Theta^b)$ dominates Z, there is nothing to prove. Hence we may assume that $\mathrm{Nklt}(V,\Theta^b)$ does not dominate Z. It suffices to verify conditions (2)–(4) in Proposition 4.1. Since $\mathrm{Exc}(\varphi)\subset \mathrm{Supp}\,\Theta^b$ by construction, condition (4) follows from assumption (d). Next, we compute

$$\begin{aligned}K_V+\Theta^b&=\varphi^*(K_W+\Delta_W)-\Theta^{nb}+E+\varepsilon E^\dagger\\&=\varphi^*(K_W+\Delta_W^b)+\varphi^*\Delta_W^{nb}-\Theta^{nb}+E+\varepsilon E^\dagger.\end{aligned}\tag{4.1}$$

Note that

$$\varphi^*\Delta_W^{nb}-\Theta^{nb}+E+\varepsilon E^\dagger$$

is φ-exceptional. Therefore, by (c) and (4.1),

$$\kappa(V,K_V+\Theta^b)\leq\kappa(W,K_W+\Delta_W^b)\leq 0.$$

Thus condition (3) holds. Finally, by (4.1), assumption (a), the effectivity of $E+\varepsilon E^\dagger$, and the inclusion

$$\mathrm{Supp}\,\Theta^{nb}\subset\mathrm{Nklt}(V,\Theta^b),$$

we see that condition (2) is satisfied.

Hence Corollary 4.3 applies, and we conclude that V is rationally chain connected modulo

$$\mathrm{Nklt}(V,\Theta^b)=\mathrm{Nklt}(V,\Delta_V).$$

This completes the proof. □

As a straightforward application of Lemma 4.4, we establish the rational connectedness of divisorial log terminal Fano pairs.

Corollary 4.5. *Let (X,Δ) be a projective divisorial log terminal pair such that $-(K_X+\Delta)$ is ample. Then X is rationally connected.*

Proof of Corollary 4.5. By slightly perturbing Δ, we may assume that (X, Δ) is kawamata log terminal and $-(K_X + \Delta)$ is ample. Since $-(K_X + \Delta)$ is ample, we take an effective $\mathbb{Q}$-divisor D such that

$$D \sim_{\mathbb{Q}} -(K_X + \Delta)$$

and $(X, \Delta + D)$ is kawamata log terminal. Then

$$K_X + \Delta + D \sim_{\mathbb{Q}} 0.$$

By Lemma 4.4, there exists a resolution $\varphi\colon V \to X$ such that V is rationally chain connected. Since V is smooth, it follows that V is rationally connected (see [K, Chapter IV. 3.10, Theorem]). Hence X is rationally connected. $\square$

We also need the following variant of Lemma 4.4 for the proof of Hacon–McKernan's rational chain connectedness theorem (see Theorem 5.1).

Lemma 4.6. *Let (X, Δ) be a projective divisorial log terminal pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\varphi\colon X \to W$ be a projective surjective morphism onto a normal projective variety W with connected fibers such that $-(K_X+\Delta)$ is φ-ample and $\varphi(\mathrm{Nklt}(X, \Delta)) = W$. Let $\psi\colon V \to X$ be a birational morphism from a smooth projective variety V such that both $\mathrm{Exc}(\psi)$ and $\mathrm{Exc}(\psi) \cup \mathrm{Supp}\, \psi_*^{-1}\Delta$ are simple normal crossing divisors on V. Write*

$$K_V + \Delta_V = \psi^*(K_X + \Delta) + E,$$

where Δ_V and E have no common irreducible components, $\psi_*\Delta_V = \Delta$, and E is ψ-exceptional. Then V is rationally chain connected modulo $\mathrm{Nklt}(V, \Delta_V)$.

Proof of Lemma 4.6. Let H be a sufficiently ample Cartier divisor on W such that $-(K_X+\Delta) + \varphi^*H$ is ample. Choose a general effective $\mathbb{Q}$-divisor A on X with sufficiently small coefficients such that

$$A \sim_{\mathbb{Q}} -(K_X + \Delta) + \varphi^*H.$$

Set $\Delta_X := \Delta + A$. Then

$$K_X + \Delta_X \sim_{\mathbb{Q}} \varphi^*H,$$

and hence $\kappa(X, K_X + \Delta_X) \geq 0$. Moreover, since the coefficients of A are sufficiently small and A is general, the pair (X, Δ_X) is divisorial log terminal and $A \leq \Delta_X$.

Pulling back to V, we obtain

$$K_V + \Delta_V + \psi_*^{-1}A = \psi^*(K_X + \Delta_X) + E.$$

Since A is general, we may further assume that

$$\mathrm{Exc}(\psi) \cup \mathrm{Supp}\, \psi_*^{-1}A \cup \mathrm{Supp}\, \psi_*^{-1}\Delta$$

is a simple normal crossing divisor on V.

We now apply Lemma 4.4 to the general fiber of $\varphi\colon X \to W$ with respect to the pair (X, Δ_X). The above construction ensures that the assumptions of Lemma 4.4 are satisfied on the general fiber. Therefore we conclude that V is rationally chain connected modulo

$$\mathrm{Nklt}(V, \Delta_V + \psi_*^{-1}A) = \mathrm{Nklt}(V, \Delta_V).$$

This completes the proof. $\square$

5. The Hacon–McKernan Rational Chain Connectedness Theorem

In this section, we establish the Hacon–McKernan rational chain connectedness theorem in the complex analytic setting: Theorem 5.1.

Theorem 5.1 ([HM2, Theorem 5.1]). *Let $f\colon X \to S$ be a projective morphism between normal complex varieties such that $f_*\mathcal{O}_X \simeq \mathcal{O}_S$. Let Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, Δ is f-big, and $K_X + \Delta \sim_{\mathbb{Q},f} 0$. Fix a point $s \in S$. After possibly shrinking S around s, we may construct a resolution $g\colon Y \to X$ and effective $\mathbb{Q}$-divisors Γ and G satisfying the following properties.*

(1) *Let $g\colon Y \to X$ be a resolution of singularities such that $\pi := f \circ g\colon Y \to S$ is projective, $\pi^{-1}(s)$ and $\mathrm{Exc}(g)$ are simple normal crossing divisors on Y, and*
$$\mathrm{Exc}(g) \cup \operatorname{Supp} g_*^{-1}\Delta \cup \pi^{-1}(s)$$
is a simple normal crossing divisor on Y. Given any bimeromorphic morphism $X' \to X$, we may assume further that $g\colon Y \to X$ factors through $X' \to X$.

(2) *We have*
$$K_Y + \Gamma \sim_{\mathbb{Q},\pi} E,$$
where Γ and E have no common irreducible components, $\mathrm{Supp}(\Gamma + E)$ is a simple normal crossing divisor on Y, E is g-exceptional, and Γ can be written as $\Gamma = A + B$ with A an effective general π-ample $\mathbb{Q}$-divisor with small coefficients and B an effective $\mathbb{Q}$-divisor.

(3) *We have $\mathrm{Nklt}(Y, \Gamma) \subset g^{-1}\,\mathrm{Nklt}(X, \Delta)$. In particular, if (X, Δ) is kawamata log terminal, then (Y, Γ) is kawamata log terminal.*

(4) *There exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor G on S such that*
$$\mathrm{Supp}(\Gamma + E) \cup \mathrm{Supp}(\pi^*G) \cup \pi^{-1}(s)$$
is a simple normal crossing divisor on Y. For $t \in [0, 1]$, set $\Delta_t := \Delta + tf^*G$. Let Γ_t and E_t be the effective $\mathbb{Q}$-divisors obtained from $\Gamma + t\pi^*G$ and E, respectively, by subtracting their common irreducible components, so that
$$K_Y + \Gamma_t \sim_{\mathbb{Q},\pi} E_t.$$
Let V_t be the closure of
$$\mathrm{Nklt}(Y, \Gamma_t) \setminus \mathrm{Nklt}(Y, \Gamma).$$
Let F be the union of the irreducible components of $\pi^{-1}(s)$ whose discrepancies with respect to $K_X + \Delta$ are greater than -1. Then $F = V_1$ holds.

(5) *Moreover, there exist rational numbers*
$$0 = t_0 < t_1 < \cdots < t_k \leq 1$$
such that
$$V_{t_i} = F_1 \cup \cdots \cup F_i,$$
where $F_1, \ldots, F_k$ are the irreducible components of F. For any $0 < \varepsilon \ll 1$, we have
$$V_{t_i - \varepsilon} = V_{t_{i-1}}.$$

(6) *Each F_i is rationally chain connected modulo*
$$W_i := F_i \cap \mathrm{Nklt}(Y, \Gamma_{t_{i-1}}).$$
Therefore, F is rationally chain connected modulo $\mathrm{Nklt}(Y, \Gamma)$, and hence modulo $g^{-1}\,\mathrm{Nklt}(X, \Delta)$.

(7) *In particular, if* (Y, Γ) *is kawamata log terminal, then* F_1 *is rationally connected.*

We give a detailed proof of Theorem 5.1.

Proof of Theorem 5.1. Throughout this proof, we freely shrink S around the point s without further mention. Let $g\colon Y \to X$ be a resolution of singularities such that $\pi := f \circ g\colon Y \to S$ is projective and both $\pi^{-1}(s)$ and $\mathrm{Exc}(g)$ are simple normal crossing divisors on Y (see [BM, Section 13]). Let $\mathfrak{m}_s$ denote the maximal ideal corresponding to $s \in S$, and let $\mu\colon S' \to S$ be the blow-up of S along $\mathfrak{m}_s$. We may further assume that $\pi\colon Y \to S$ factors through μ and that

$$\mathrm{Exc}(g) \cup \operatorname{Supp} g_*^{-1}\Delta \cup \pi^{-1}(s)$$

is a simple normal crossing divisor on Y. Finally, let $X' \to X$ be a bimeromorphic morphism. Then, by [H, Corollary 2] and [BM, Section 13], we can arrange for $g\colon Y \to X$ to factor through $X' \to X$.

Let F be the union of the irreducible components of $\pi^{-1}(s)$ whose discrepancies with respect to $K_X + \Delta$ are greater than -1. Write

$$K_Y + \Gamma' = g^*(K_X + \Delta) + E',$$

where Γ' and E' are effective $\mathbb{Q}$-divisors with no common irreducible components, $g_*\Gamma' = \Delta$, and E' is g-exceptional. Let $E^\dagger := \mathrm{Exc}(g)$ denote the reduced exceptional divisor. For a sufficiently small rational number $\delta > 0$, set

$$\Gamma'' := \Gamma' + \delta E^\dagger, \qquad E'' := E' + \delta E^\dagger.$$

Then $\mathrm{Nklt}(Y, \Gamma') = \mathrm{Nklt}(Y, \Gamma'')$. Since

$$\operatorname{Supp} \Gamma'' = \operatorname{Supp} g_*^{-1}\Delta \cup \mathrm{Exc}(g),$$

and Δ is f-big by assumption, we may write

$$\Gamma'' \sim_{\mathbb{Q},\pi} A' + B',$$

where A' is a π-ample $\mathbb{Q}$-divisor, B' is an effective $\mathbb{R}$-divisor, and A' and E'' have no common irreducible components. Moreover, by choosing the resolution $g\colon Y \to X$ suitably (see [BM, Section 13]), we may further assume that

$$\operatorname{Supp}(A' + B') \cup \operatorname{Supp} g_*^{-1}\Delta \cup \mathrm{Exc}(g) \cup \pi^{-1}(s)$$

is a simple normal crossing divisor on Y.

Consider

$$K_Y + \big((1-\varepsilon)\Gamma'' + \varepsilon B'\big) + \varepsilon A' \sim_{\mathbb{Q},\pi} g^*(K_X + \Delta) + E''$$

for $0 < \varepsilon \ll 1$. Canceling common components on both sides yields

$$K_Y + \Gamma \sim_{\mathbb{Q},\pi} g^*(K_X + \Delta) + E,$$

where Γ and E are effective $\mathbb{Q}$-divisors with no common irreducible components, E is g-exceptional, and $\varepsilon A' \leq \Gamma$. Since $g^*(K_X + \Delta) \sim_{\mathbb{Q},\pi} 0$, we obtain

$$K_Y + \Gamma \sim_{\mathbb{Q},\pi} E.$$

By construction, $\operatorname{Supp}(\Gamma + E)$ is a simple normal crossing divisor. Set $A := \varepsilon A'$ and $B := \Gamma - \varepsilon A'$ so that $\Gamma = A + B$. We note that we can freely replace A by a general effective $\mathbb{Q}$-divisor $\widetilde{A} \sim_{\mathbb{Q},\pi} A$. Moreover, we have

$$\mathrm{Nklt}(Y, \Gamma) \subset \mathrm{Nklt}(Y, \Gamma'') = \mathrm{Nklt}(Y, \Gamma') \subset g^{-1}\,\mathrm{Nklt}(X, \Delta),$$

where the first inclusion holds by $0 < \varepsilon \ll 1$. In particular, if (X, Δ) is kawamata log terminal, then so is (Y, Γ).

We write $F = \sum_{i=1}^{k} F_i$ for its decomposition into irreducible components. We note that we may slightly perturb the coefficients of those F_i that appear in B or E without changing $\mathrm{Nklt}(Y, \Gamma)$, as follows. For sufficiently small positive rational numbers ε_i, we replace B and A by $B + \sum_{i=1}^{k} \varepsilon_i F_i$ and $A - \sum_{i=1}^{k} \varepsilon_i F_i$, respectively. After subtracting the common irreducible components of $B + \sum_{i=1}^{k} \varepsilon_i F_i$ and E, and replacing $A - \sum_{i=1}^{k} \varepsilon_i F_i$ with a general effective $\mathbb{Q}$-divisor that is $\mathbb{Q}$-linearly equivalent to it over S, the non-klt locus $\mathrm{Nklt}(Y, \Gamma)$ remains unchanged.

Let $H_1, \dots, H_l$ be general Cartier divisors on S passing through s, where $l \gg 0$, and set

$$G := \frac{1}{2} \sum_{j=1}^{l} H_j.$$

Then

$$\mathrm{Supp}(\Gamma + E) \cup \mathrm{Supp}(\pi^* G) \cup \pi^{-1}(s)$$

is a simple normal crossing divisor on Y. We note that the morphism $\pi \colon Y \to S$ factors through the blow-up $\mu \colon S' \to S$.

For $t \in [0, 1]$, set $\Delta_t := \Delta + t f^* G$. Let Γ_t and E_t be the effective $\mathbb{Q}$-divisors obtained from $\Gamma + t\pi^* G$ and E, respectively, by subtracting their common irreducible components so that

$$K_Y + \Gamma_t \sim_{\mathbb{Q}, \pi} E_t.$$

Let V_t be the closure of

$$\mathrm{Nklt}(Y, \Gamma_t) \setminus \mathrm{Nklt}(Y, \Gamma).$$

By construction, we have $F = V_1$. Since we may slightly perturb the coefficients of those F_i that appear in B or E, there exist rational numbers

$$0 = t_0 < t_1 < \cdots < t_k \leq 1$$

such that, after renumbering the F_i if necessary,

$$V_{t_i} = F_1 \cup \cdots \cup F_i$$

for each i. Moreover, for any sufficiently small $0 < \varepsilon \ll 1$,

$$V_{t_i - \varepsilon} = V_{t_{i-1}}.$$

We have already verified (1)–(5). Assertion (6) follows from Lemma 5.2 below. Finally, since F_1 is rationally chain connected by (6) and is smooth, it is rationally connected (see [K, Chapter IV. 3.10 Theorem]). This proves (7) and completes the proof of Theorem 5.1. □

The nontrivial part of Theorem 5.1 is (6). This follows from the following lemma, whose proof is new and plays a central role in this paper.

Lemma 5.2. *Let F° be an irreducible component of F. Then there exists $\tau^\circ \in (0, 1]$ such that F° is a log canonical center of (Y, Γ_{τ°). For any sufficiently small positive rational number ε, the variety F° is rationally chain connected modulo*

$$W^\circ := F^\circ \cap \mathrm{Nklt}(Y, \Gamma_{\tau^\circ - \varepsilon}).$$

Proof of Lemma 5.2. We run a minimal model program over X and then apply Lemma 4.4 or Lemma 4.6.

Step 1 (Reduction to the non-klt locus on F°)**.** Observe that

$$K_Y + \Gamma_{\tau^\circ} \sim_{\mathbb{Q},\pi} E_{\tau^\circ},$$

and F° is an irreducible component of Γ_{τ°. By adjunction, we write

$$K_{F^\circ} + \Psi := (K_Y + \Gamma_{\tau^\circ})|_{F^\circ}, \quad \text{i.e., } \Psi = (\Gamma_{\tau^\circ} - F^\circ)|_{F^\circ}.$$

Then for any sufficiently small positive rational number ε, we have

$$\mathrm{Nklt}(F^\circ, \Psi) = \mathrm{Nklt}(F^\circ, (\Gamma_{\tau^\circ - \varepsilon})|_{F^\circ}) = F^\circ \cap \mathrm{Nklt}(Y, \Gamma_{\tau^\circ - \varepsilon}).$$

Hence it suffices to show that F° is rationally chain connected modulo $\mathrm{Nklt}(F^\circ, \Psi)$.

Step 2 (Running a $(K_Y + \Gamma^b_{\tau^\circ})$-MMP over X)**.** Recall that

$$K_Y + \Gamma^b_{\tau^\circ} \sim_{\mathbb{Q},\pi} E_{\tau^\circ} - \Gamma^{nb}_{\tau^\circ},$$

and that $A \le \Gamma^b_{\tau^\circ}$ by construction.

Take an open neighborhood U of s and a Stein compact subset $W \subset S$ containing U such that $\Gamma(W, \mathcal{O}_S)$ is noetherian. Then, by [F5, Lemma 9.4], we can run a $(K_Y + \Gamma^b_{\tau^\circ})$-minimal model program over X around $f^{-1}(W)$ with ample scaling, obtaining a finite sequence of flips and divisorial contractions:

$$Y =: Y_0 \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_m,$$

such that the strict transform $(F^\circ)_m$ of F° remains a divisor on Y_m. We note that, at each step of the minimal model program, it is necessary to shrink the spaces further around W. For any $\mathbb{R}$-divisor D on Y, we denote by $(D)_m$ its pushforward on Y_m.

We distinguish two cases:

(A) $K_{Y_m} + (\Gamma^b_{\tau^\circ})_m$ is nef over $f^{-1}(W)$, or

(B) Y_m admits a $(K_{Y_m} + (\Gamma^b_{\tau^\circ})_m)$-negative divisorial contraction that contracts $(F^\circ)_m$.

By adjunction, set

$$K_{(F^\circ)_m} + \Phi := (K_{Y_m} + (\Gamma_{\tau^\circ})_m)|_{(F^\circ)_m}.$$

Then one verifies that

$$K_{(F^\circ)_m} + \Phi^b = (K_{Y_m} + (\Gamma^b_{\tau^\circ})_m)|_{(F^\circ)_m}.$$

Moreover, $((F^\circ)_m, \Phi^b)$ is divisorial log terminal because $(Y_m, (\Gamma^b_{\tau^\circ})_m)$ is divisorial log terminal.

Furthermore, $(K_{Y_m} + (\Gamma_{\tau^\circ})_m)$ is $\mathbb{Q}$-linearly equivalent to $(E_{\tau^\circ})_m$ over S, and $(E_{\tau^\circ})_m$ shares no components with $(F^\circ)_m$, so

$$\kappa((F^\circ)_m, K_{(F^\circ)_m} + \Phi) \ge 0.$$

Since $\Gamma^b_{\tau^\circ} \ge A$, Φ^b contains an effective big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

Step 3 (Case (A))**.** In this step, we treat Case (A).

In Case (A), the negativity lemma implies $(E_{\tau^\circ})_m = 0$ over U, hence

$$K_{(F^\circ)_m} + \Phi \sim_{\mathbb{Q}} 0 \quad \text{and} \quad \kappa((F^\circ)_m, K_{(F^\circ)_m} + \Phi^b) \le \kappa((F^\circ)_m, K_{(F^\circ)_m} + \Phi) = 0.$$

Thus $((F^\circ)_m, \Phi)$ satisfies conditions (a)–(d) of Lemma 4.4.

Let $q\colon V \to (F^\circ)_m$ be a resolution such that

$$K_V + \Delta_V = q^*(K_{(F^\circ)_m} + \Phi) + E^\sharp, \tag{5.1}$$

as in Lemma 4.4. Then Lemma 4.4 implies that V is rationally chain connected modulo $\mathrm{Nklt}(V, \Delta_V)$.

We may assume that V is a common resolution of F° and $(F^\circ)_m$:

$$\begin{array}{ccc} & V & \\ {}^{p}\swarrow & & \searrow^{q} \\ F^\circ & & (F^\circ)_m. \end{array}$$

Let ν be any irreducible component of $\Delta_V^{\geq 1}$. Then

$$a(\nu, V, \Delta_V - E^\sharp) = a(\nu, V, \Delta_V) \leq -1.$$

Hence, by (5.1), we obtain

$$a(\nu, (F^\circ)_m, \Phi) \leq -1.$$

Since

$$K_{(F^\circ)_m} + \Phi = (K_{Y_m} + (\Gamma_{\tau^\circ})_m)|_{(F^\circ)_m} \sim_{\mathbb{Q}} 0,$$

it follows that

$$a(\nu, F^\circ, \Psi - (E_{\tau^\circ})|_{F^\circ}) = a(\nu, (F^\circ)_m, \Phi) \leq -1.$$

Therefore

$$a(\nu, F^\circ, \Psi) \leq -1.$$

This shows that

$$p(\mathrm{Nklt}(V, \Delta_V)) = p(\mathrm{Supp}\, \Delta_V^{\geq 1}) \subset \mathrm{Nklt}(F^\circ, \Psi).$$

Hence F° is rationally chain connected modulo $\mathrm{Nklt}(F^\circ, \Psi)$.

Step 4 (Case (B))**.** In this step, we treat Case (B).

In Case (B), $(F^\circ)_m$ is contracted by a $(K_{Y_m} + (\Gamma^b_{\tau^\circ})_m)$-negative divisorial contraction. By the $(K_{Y_m} + (\Gamma^b_{\tau^\circ})_m)$-negative divisorial contraction contracting $(F^\circ)_m$, we obtain a projective surjective morphism

$$(F^\circ)_m \to W$$

with connected fibers such that $-(K_{(F^\circ)_m} + \Phi^b)$ is ample over W and $\dim W < \dim (F^\circ)_m$. Since

$$K_{(F^\circ)_m} + \Phi^b \sim_{\mathbb{Q}} (E_{\tau^\circ})_m|_{(F^\circ)_m} - \Phi^{nb},$$

we obtain that $\mathrm{Supp}\, \Phi^{nb}$ dominates W. Hence, the non-klt locus $\mathrm{Nklt}((F^\circ)_m, \Phi^b)$ dominates W.

Let $q \colon V \to (F^\circ)_m$ be a resolution such that

$$K_V + \Delta_V = q^*(K_{(F^\circ)_m} + \Phi^b) + E^\flat,$$

as in Lemma 4.6. Then Lemma 4.6 implies that V is rationally chain connected modulo $\mathrm{Nklt}(V, \Delta_V)$.

We may assume that V is a common resolution of F° and $(F^\circ)_m$:

$$\begin{array}{ccc} & V & \\ {}^{p}\swarrow & & \searrow^{q} \\ F^\circ & & (F^\circ)_m. \end{array}$$

By the negativity lemma, for any divisor ν over F° we have

$$a(\nu, (F^\circ)_m, \Phi^b) \geq a(\nu, F^\circ, \Psi^b).$$

It follows that

$$p(\mathrm{Nklt}(V, \Delta_V)) \subset \mathrm{Nklt}(F^\circ, \Psi^b) = \mathrm{Nklt}(F^\circ, \Psi).$$

Thus F° is rationally chain connected modulo $\mathrm{Nklt}(F^\circ, \Psi)$.

This completes the proof of Lemma 5.2. □

Here, we make a remark on the treatment of our Theorem 5.1, which is essentially [HM2, Theorem 5.1], and point out the differences from the way Hacon and M^cKernan handle the corresponding result.

Remark 5.3. Although we work with complex analytic spaces rather than algebraic varieties, Theorem 5.1 is essentially the same as [HM2, Theorem 5.1]. However, we do not address the statement concerning the Kodaira dimension in [HM2, Theorem 5.1 (4)]. The proof of [HM2, Theorem 5.1 (4)] relies on a highly nontrivial extension theorem; see [HM2, Theorem 5.2] and [HM1, Corollary 3.17] for details. Since this extension result is technically demanding even for specialists, we avoid using it in the present paper.

Instead, we prove Theorem 5.1 by running the minimal model program (see [F5]). Of course, the existence of flips itself ultimately depends on extension theorems, and therefore our approach does not yield a substantial simplification of the overall theory. Nevertheless, we find the minimal model program more accessible and conceptually transparent than working directly with the highly technical extension results.

Remark 5.4. Roughly speaking, Hacon and McKernan directly prove that $\kappa_\sigma(F^\circ, K_{F^\circ} + \Psi^b) \leq 0$ by using their difficult extension theorem (see [HM1, Corollary 3.17]). They say that it is the tricky part of their proof of Theorem 1.1 (see [HM2, Theorem 1.2]). In contrast, in our proof of Lemma 5.2, it may happen that

$$\kappa(F^\circ, K_{F^\circ} + \Psi^b) > \kappa\big((F^\circ)_m, K_{(F^\circ)_m} + \Phi^b\big).$$

Thus our method does not allow us to directly deduce $\kappa(F^\circ, K_{F^\circ} + \Psi^b) \leq 0$. Instead, we prove that a resolution of $((F^\circ)_m, \Phi)$ (resp. $((F^\circ)_m, \Phi^b)$) is rationally chain connected modulo the non-klt locus. It then follows that the same property holds for (F°, Ψ) (resp. (F°, Ψ^b)).

6. Proofs of the Results

In this section, we prove the results stated in Section 1.

Proof of Theorem 1.1. By taking the Stein factorization, we may assume that $f_*\mathcal{O}_X \simeq \mathcal{O}_S$. Fix a point $P \in S$ and shrink S around P without further comment. Since $-(K_X + \Delta)$ is f-semiample by assumption, we take an effective $\mathbb{R}$-divisor D_1 such that

$$-(K_X + \Delta) \sim_{\mathbb{R},f} D_1$$

and

$$\mathrm{Nklt}(X, \Delta) = \mathrm{Nklt}(X, \Delta_1),$$

where $\Delta_1 := \Delta + D_1$. Since $\Delta_1 \sim_{\mathbb{R},f} -K_X$ and is f-big, we write

$$\Delta_1 \sim_{\mathbb{Q},f} A + B,$$

where A is an effective f-ample $\mathbb{Q}$-divisor and B is an effective $\mathbb{R}$-divisor. Set

$$\Delta_2 := (1 - \varepsilon)\Delta_1 + \varepsilon B$$

for some $0 < \varepsilon \ll 1$. Then

$$-(K_X + \Delta_2) \sim_{\mathbb{R},f} \varepsilon A,$$

which is f-ample, and

$$\mathrm{Nklt}(X, \Delta_2) \subset \mathrm{Nklt}(X, \Delta_1) = \mathrm{Nklt}(X, \Delta).$$

By slightly perturbing Δ_2, we obtain an effective $\mathbb{Q}$-divisor Δ_3 such that $-(K_X+\Delta_3)$ is f-ample and

$$\operatorname{Nklt}(X,\Delta_3)=\operatorname{Nklt}(X,\Delta_2)\subset\operatorname{Nklt}(X,\Delta).$$

Take an effective $\mathbb{Q}$-divisor D_2 such that

$$-(K_X+\Delta_3)\sim_{\mathbb{Q},f} D_2$$

and

$$\operatorname{Nklt}(X,\Delta_4)=\operatorname{Nklt}(X,\Delta_3)\subset\operatorname{Nklt}(X,\Delta),$$

where $\Delta_4:=\Delta_3+D_2$. Set $\Delta^\dagger:=\Delta_4$. Then

$$K_X+\Delta^\dagger\sim_{\mathbb{Q},f}0,$$

$\Delta^\dagger$ is f-big, and

$$\operatorname{Nklt}(X,\Delta^\dagger)\subset\operatorname{Nklt}(X,\Delta).$$

Therefore, we can apply Theorem 5.1. By Lemma 2.10, we may replace Y by a higher model if necessary. By Theorem 5.1 (6), $\pi^{-1}(P)$ is rationally chain connected modulo $g^{-1}\operatorname{Nklt}(X,\Delta^\dagger)$. Consequently, it is rationally chain connected modulo $g^{-1}\operatorname{Nklt}(X,\Delta)$. □

Proof of Corollary 1.2. Throughout this proof, we shrink X around a point P without further comment. By perturbing the coefficients of Δ, we may assume that (X,Δ) is kawamata log terminal and that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. We apply Theorem 1.1 to the identity map $f\colon X\to X$. Then any fiber of $g\colon Y\to X$ is rationally chain connected. This proves (i).

For (ii), if X is rationally connected, then it is clearly rationally chain connected. Thus, it suffices to prove the "only if" part. Let $g\colon Y\to X$ be a resolution of singularities. By (i), g has rationally chain connected fibers, and hence Y is rationally chain connected. Since Y is smooth, it follows that Y is rationally connected (see [K, Chapter IV. 3.10 Theorem]). Therefore, X is rationally connected. This completes the proof. □

Proof of Corollary 1.3. We apply Corollary 1.2 (i) to $p\colon\Gamma\to X$. Then $p^{-1}(x)$ is rationally chain connected. Thus, by Lemma 2.10, $q(p^{-1}(x))$ is rationally chain connected. □

Proof of Corollary 1.4. Let Γ be the graph of $f\colon X\dashrightarrow Y$, and denote by $p\colon\Gamma\to X$ the natural projection. By Corollary 1.2 (i), every positive-dimensional fiber of p is rationally chain connected. Since Y contains no rational curves, every fiber of p must be zero-dimensional. Hence p is an isomorphism, and therefore f is a morphism. □

Proof of Theorem 1.6. First, by perturbing Δ slightly, we may assume that $-(K_X+\Delta)$ is an ample $\mathbb{Q}$-divisor without changing $\operatorname{Nklt}(X,\Delta)$. Take a dlt blow-up $f\colon X'\to X$ such that

$$K_{X'}+\Delta'=f^*(K_X+\Delta).$$

We may assume that $(X',(\Delta')^b)$ is $\mathbb{Q}$-factorial and divisorial log terminal (see [F1, Theorem 4.4.21]). By construction, $-(K_{X'}+\Delta')$ is semiample and big. Let D' be a general effective $\mathbb{Q}$-divisor with $D'\sim_{\mathbb{Q}}-(K_{X'}+\Delta')$, and set $\Delta^\dagger:=\Delta'+D'$. Then

$$K_{X'}+\Delta^\dagger\sim_{\mathbb{Q}}0,$$

$(\Delta^\dagger)^b\geq D'$, $(X',(\Delta^\dagger)^b)$ is divisorial log terminal, and

$$\operatorname{Nklt}(X',(\Delta^\dagger)^b)=\operatorname{Nklt}(X',(\Delta')^b).$$

Applying Lemma 4.4 to a resolution of X', it follows that X is rationally chain connected modulo $\mathrm{Nklt}(X, \Delta)$. If (X, Δ) is kawamata log terminal, then X is rationally connected by Corollary 4.5. □

Proof of Theorem 1.7. This theorem is a corollary of Theorem 1.6. In the algebraic setting, the proof is given in [F4, Section 13]. Note that Theorem 1.1, and therefore Theorem 5.1, are not needed in this proof. In the complex analytic setting, the same argument applies, since the theory of quasi-log structures is established in [F7]. □

Proof of Corollary 1.8. By taking the Stein factorization, we may assume that $f_*\mathcal{O}_X \simeq \mathcal{O}_S$. Throughout this proof, we fix a point $s \in S$ and shrink S around s without further comment. If (X, Δ) is log canonical, then $[X, K_X + \Delta]$ naturally becomes a quasi-log canonical pair. Hence, (2) is a special case of Theorem 1.7. For (1), as in the proof of Lemma 2.6, we construct an effective $\mathbb{R}$-divisor Θ such that (X, Θ) is kawamata log terminal and $-(K_X + \Theta)$ is π-ample. Thus, (1) follows from (2). This completes the proof. □

For the reader's convenience, we present an alternative proof of Corollary 1.8 (1) using Theorem 1.1. This argument follows the strategy of Hacon and M^{c}Kernan in [HM2].

Proof of Corollary 1.8 (1) as an application of Theorem 1.1. As usual, we fix a point $P \in S$ and shrink S around P without further comment. By the basepoint-free theorem (see Lemma 2.6), $-(K_X + \Delta)$ is f-semiample. Since $\mathrm{Nklt}(X, \Delta) = \emptyset$, Theorem 1.1 together with Lemma 2.10 implies that $f^{-1}(P)$ is rationally chain connected. □

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