ON QUASI-ALBANESE MAPS

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Abstract. We discuss Iitaka's theory of quasi-Albanese maps in details. We also give a detailed proof of Kawamata's theorem on the quasi-Albanese maps for varieties of the logarithmic Kodaira dimension zero. Note that Iitaka's theory is an application of Deligne's mixed Hodge theory for smooth algebraic varieties.

Contents

1. Introduction

In this paper, we discuss Iitaka's theory of quasi-Albanese maps. We give a detailed proof of:

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Theorem 1.1 (see [I1] and Theorem 3.16)**.** *Let X be a smooth algebraic variety defined over $\mathbb{C}$. Then there exists a morphism $\alpha\colon X \to A$ to a quasi-abelian variety A such that*

(i) *for any other morphism $\beta\colon X \to B$ to a quasi-abelian variety B, there is a morphism $f\colon A \to B$ such that $\beta = f \circ \alpha$*

$$\begin{array}{ccc} X & \xrightarrow{\beta} & B \\ {\scriptstyle\alpha}\downarrow & \nearrow_{f} & \\ A & & \end{array}$$

and

(ii) *f is uniquely determined.*

A quasi-abelian variety in Theorem 1.1 is sometimes called a semi-abelian variety in the literature, which is an extension of an abelian variety by an algebraic torus as an algebraic group. Note that if X is complete in Theorem 1.1 then A is nothing but the Albanese variety of X. Theorem 1.1 depends on Deligne's theory of mixed Hodge structures for smooth complex algebraic varieties.

We also give a detailed proof of Kawamata's theorem on the quasi-Albanese maps for varieties of the logarithmic Kodaira dimension zero.

Theorem 1.2 (see [Ka2] and Theorem 10.1)**.** *Let X be a smooth variety such that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then the quasi-Albanese map $\alpha\colon X \to A$ is dominant and has irreducible general fibers.*

The original proof of Theorem 1.2 in [Ka2] needs some deep results on the theory of variations of (mixed) Hodge structure. They are the hardest parts of [Ka2] to follow. In Section 7, we give many supplementary comments on various semipositivity theorems, which clarify Kawamata's original approach to Theorem 1.2 in [Ka2]. In Section 8, we explain how to avoid using the theory of variations of (mixed) Hodge structure for the proof of Theorem 1.2. A vanishing theorem in [F1] related to the theory of mixed Hodge structures is sufficient for the proof of Theorem 1.2.

When the logarithmic irregularity $\overline{q}(X)$ of X is $\dim X$ in Theorem 1.2, we have the following theorem, which is a slight refinement of [MPT, Theorem A].

Theorem 1.3 (see [MPT, Theorem A] and [FMPT])**.** *Let X be a smooth variety with $\overline{\kappa}(X) = 0$ and the logarithmic irregularity $\overline{q}(X) = \dim X$. Then the quasi-Albanese map $\alpha\colon X \to A$ is birational and there exists a closed subset Z of A with $\operatorname{codim}_A Z \geq 2$ such that $\alpha\colon X \setminus$*

$\alpha^{-1}(Z) \to A \setminus Z$ is an isomorphism and $\alpha^{-1}(Z)$ is of pure codimension one.

As an easy consequence of Theorem 1.3, we have:

Corollary 1.4 (see [MPT, Corollary B])**.** *Let X be a smooth affine variety with $\dim X = n$. Then X is isomorphic to $\mathbb{G}_m^n$ if and only if $\overline{\kappa}(X) = 0$ and $\overline{q}(X) = n$.*

Corollary 1.5 is also an easy consequence of Theorem 1.3.

Corollary 1.5. *Let X be a nonempty Zariski open set of a quasi-abelian variety A. Then $\overline{\kappa}(X) = 0$ if and only if $\operatorname{codim}_A(A \setminus X) \geq 2$.*

One of the main motivations of this paper is to understand Theorem 1.2 in detail. The original proof of Theorem 1.2 in [Ka2] looks inaccessible because the theory of variations of (mixed) Hodge structure was not fully matured when [Ka2] was written around 1980. Moreover, some details are omitted in [Ka2]. In [Ka2], Kawamata could and did use only [D1], [Gri], and [Sc] for the Hodge theory. Although the semipositivity theorem in [F1] (see also [FF1] and [FFS]) does not recover Kawamata's statement on semipositivity (see [Ka2, Theorem 32]), it is natural and is sufficient for us to carry out Kawamata's proof of Theorem 1.2 in [Ka2] with some suitable modifications. The author has been unable to follow [Ka2, Theorem 32]. Moreover, the vanishing theorem in [F1] gives a more elementary approach to Theorem 1.2 and makes Theorem 1.2 independent of the theory of variations of (mixed) Hodge structure. The author hopes that this paper will make Iitaka's theory of quasi-Albanese maps and Kawamata's result on the quasi-Albanese maps of varieties of the logarithmic Kodaira dimension zero accessible.

We look at the organization of this paper. Section 2 is a preliminary section. In Subsections 2.1 and 2.2, we collect some basic definitions and results of the logarithmic Kodaira dimensions and the quasi-abelian varieties in the sense of Iitaka, respectively. Section 3 is devoted to the theory of quasi-Albanese maps and varieties due to Shigeru Iitaka. We explain it in details following Iitaka's paper [I1] with many supplementary arguments. Theorem 3.16, which is Theorem 1.1, is the main result of this section. In Section 4, we prove some basic properties of quasi-abelian varieties for the reader's convenience. In Section 5, we quickly explain a birational characterization of abelian varieties and a bimeromorphic characterization of complex tori without proof. In Section 6, we recall the subadditivity of the logarithmic Kodaira dimensions in some special cases. We use them for the proof of Theorem 1.2. Section 7 is devoted to the explanation of some semipositivity theorems related to the theory of variations of (mixed) Hodge structure. We hope that this section will help the reader to understand [Ka2]. In Section 8, we discuss some weak positivity theorems. Our approach in Section 8 does not use the theory of variations of (mixed) Hodge structure. We use a generalization of the Kollár vanishing theorem. This section makes Theorem 1.2 independent of the theory of variations of (mixed) Hodge structure. In Section 9, we discuss finite covers of quasi-abelian varieties. We need them for the proof of Theorem 1.2. In Section 10, we prove Theorem 1.2 in details. In Section 11, which is the final section, we prove Theorem 1.3 and Corollaries 1.4 and 1.5.

1.6 (Historical note)**.** If I remember correctly, I wrote the following two preprints:

- Osamu Fujino, Subadditivity of the logarithmic Kodaira dimension for morphisms of relative dimension one revisited

and

- Osamu Fujino, On quasi-Albanese maps

in 2014 and circulated them as Kyoto Math 2015-02 and 2015-03 in the preprint series in Department of Mathematics, Kyoto University, respectively. In 2018, I gave a series of lectures on the Iitaka conjecture in Osaka. Then I published [F8], which is a completely revised and expanded version of the above first preprint. Although I did not put the above preprints on the arXiv, some people have cited the above second preprint as a reference on the theory of quasi-Albanese maps. Hence, I put it on the arXiv now and will plan to publish it somewhere. Note that I added Section 11 when I revised this paper in 2024.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. A variety means a reduced and irreducible separated scheme of finite type over $\mathbb{C}$. We will use the standard notation as in [F3] and [F4]. The theory of algebraic groups which are not affine nor projective is not so easy to access. Hence we make efforts to minimize the use of the general theory of algebraic groups for the reader's convenience. In this paper, we do not even use [D2, Lemme (10.1.3.3)]. We do not use the theory of minimal models.

2. Preliminaries

In this section, we collect some basic definitions and results on the logarithmic Kodaira dimensions and the quasi-abelian varieties (see, for example, [I1], [I2], [I4], [I5], and so on). For the basic properties of the Kodaira dimensions and some related topics, see, for example, [U] and [Mo] (see also [F8]).

2.1. **Logarithmic Kodaira dimensions and irregularities.** First, we recall the logarithmic Kodaira dimensions and the logarithmic irregularities following Iitaka. For the details, see [I1], [I2], [I4], and [I5].

Definition 2.1 (Logarithmic Kodaira dimension)**.** Let X be an algebraic variety. By Nagata (see [Nag]), we have a complete algebraic variety $\overline{X}$ which contains X as a dense Zariski open subset. By Hironaka (see [Hi]), we have a smooth projective variety $\overline{W}$ and a projective birational morphism $\mu\colon \overline{W} \to \overline{X}$ such that if $W = \mu^{-1}(X)$, then $\overline{D} = \overline{W} - W = \mu^{-1}(\overline{X} - X)$ is a simple normal crossing divisor on $\overline{W}$. The *logarithmic Kodaira dimension* $\overline{\kappa}(X)$ of X is defined as

$$\overline{\kappa}(X) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D})$$

where κ denotes Iitaka's D-dimension.

Definition 2.2 (Logarithmic irregularity)**.** Let X be an algebraic variety. We take $(\overline{W}, \overline{D})$ as in Definition 2.1. Then we put

$$\overline{q}(X) = \dim_{\mathbb{C}} H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))$$

and call it the *logarithmic irregularity* of X. We put

$$T_1(X) = H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))$$

following Iitaka [I1].

It is easy to see:

Lemma 2.3. *$\overline{\kappa}(X)$, $\overline{q}(X)$, and $T_1(X)$ are well-defined, that is, they are independent of the choice of the pair $(\overline{W}, \overline{D})$.*

This lemma is well known. We give a proof for the reader's convenience.

Proof. By Hironaka's resolution (see [Hi]), it is sufficient to prove that

$$\kappa(\overline{W}, K_{\overline{W}}+\overline{D})=\kappa(\overline{W}_1, K_{\overline{W}_1}+\overline{D}_1)$$

and

$$H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))=H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1))$$

where $f\colon \overline{W}_1\to\overline{W}$ is a projective birational morphism from a smooth projective variety $\overline{W}_1$ and $\overline{D}_1=\mathrm{Supp} f^*\overline{D}$. By the local calculation, we see that

$$f^*\Omega^1_{\overline{W}}(\log \overline{D})\subset \Omega^1_{\overline{W}_1}(\log \overline{D}_1). \tag{2.1}$$

Therefore, we obtain

$$H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))\subset H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1)).$$

On the other hand, it is obvious that

$$H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1))\subset H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))$$

since $\Omega^1_{\overline{W}}(\log \overline{D})$ is locally free. Thus, we have

$$H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1))=H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D})).$$

By (2.1), we have

$$K_{\overline{W}_1}+\overline{D}_1=f^*(K_{\overline{W}}+\overline{D})+E$$

where E is an effective f-exceptional divisor on $\overline{W}_1$. Therefore, it is obvious that

$$\kappa(\overline{W}, K_{\overline{W}}+\overline{D})=\kappa(\overline{W}_1, K_{\overline{W}_1}+\overline{D}_1)$$

holds. □

Lemma 2.4. *Let X be a variety and let U be a nonempty Zariski open set of X. Then we have*

$$\overline{\kappa}(X)\le\overline{\kappa}(U)$$

and

$$\overline{q}(X)\le\overline{q}(U).$$

Proof. It is obvious by the definitions of $\overline{\kappa}$ and $\overline{q}$. □

We may sometimes use Lemma 2.4 implicitly. We need Lemma 2.5 for the proof of Corollary 1.5.

Lemma 2.5. *Let X be a smooth variety. Assume that F is a closed subset of X with $\mathrm{codim}_X F\ge 2$. Then we have*

$$\overline{\kappa}(X)=\overline{\kappa}(X-F).$$

Proof. We take a smooth complete algebraic variety $\overline{X}$ such that $\overline{D} = \overline{X} - X$ is a simple normal crossing divisor on $\overline{X}$. Then we have

$$\overline{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + \overline{D})$$

by definition. Let $\overline{F}$ be the closure of F in $\overline{X}$. Note that $\mathrm{codim}_{\overline{X}}\overline{F} \geq 2$. We take a resolution

$$f\colon Y \to \overline{X}$$

such that f is an isomorphism over $\overline{X} - \overline{F}$ and that $\mathrm{Supp} f^{-1}(\overline{F})$ and $\mathrm{Supp}\left(f^{-1}(\overline{F}) \cup f^*\overline{D}\right)$ are simple normal crossing divisors on Y. We put $\Delta_1 = \mathrm{Supp} f^*\overline{D}$ and $\Delta_2 = \mathrm{Supp}\left(f^{-1}(\overline{F}) \cup f^*\overline{D}\right)$. Then we have

$$K_Y + \Delta_1 = f^*(K_{\overline{X}} + \overline{D}) + E$$

where E is an effective f-exceptional divisor on Y. Therefore, we obtain

$$\overline{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + \overline{D}) = \kappa(Y, K_Y + \Delta_1).$$

By definition,

$$\overline{\kappa}(X - F) = \kappa(Y, K_Y + \Delta_2).$$

Since $\Delta_2 - \Delta_1$ is an effective f-exceptional divisor on Y by $\mathrm{codim}_{\overline{X}}\overline{F} \geq 2$, we have

$$\overline{\kappa}(X - F) = \kappa(Y, K_Y + \Delta_2) = \kappa(\overline{X}, K_{\overline{X}} + \overline{D}) = \overline{\kappa}(X).$$

This is the desired equality. □

We will freely use the following lemmas throughout this paper.

Lemma 2.6. *Let $f\colon X \to Y$ be a dominant morphism of algebraic varieties. Then $\overline{\kappa}(X) \geq \overline{\kappa}(Y)$ holds.*

Proof. By using Hironaka's resolution of singularities, we may assume that X and Y are both smooth. Let $\overline{f}\colon \overline{X} \to \overline{Y}$ be a compactification of $f\colon X \to Y$, that is, $\overline{X}$ and $\overline{Y}$ are smooth complete varieties and $\Delta_{\overline{X}} := \overline{X} - X$ and $\Delta_{\overline{Y}} := \overline{Y} - Y$ are simple normal crossing divisors on $\overline{X}$ and $\overline{Y}$, respectively. Since

$$\overline{f}^*\Omega^1_{\overline{Y}}(\log \Delta_{\overline{Y}}) \subset \Omega^1_{\overline{X}}(\log \Delta_{\overline{X}}),$$

we have

$$K_{\overline{X}} + \Delta_{\overline{X}} = \overline{f}^*(K_{\overline{Y}} + \Delta_{\overline{Y}}) + E,$$

where E is effective. This implies that

$$\overline{\kappa}(Y) = \kappa(\overline{Y}, K_{\overline{Y}} + \Delta_{\overline{Y}}) = \kappa(\overline{X}, \overline{f}^*(K_{\overline{Y}} + \Delta_{\overline{Y}})) \leq \kappa(\overline{X}, K_{\overline{X}} + \Delta_{\overline{X}}) = \overline{\kappa}(X).$$

We finish the proof. □

Lemma 2.7 ([I2, Theorem 3])**.** *Let $f\colon X \to Y$ be a finite étale morphism of algebraic varieties. Then $\overline{\kappa}(X) = \overline{\kappa}(Y)$ holds.*

Proof. Let $\widetilde{Y} \to Y$ be a resolution of singularities of Y. We replace Y and X with $\widetilde{Y}$ and $X \times_Y \widetilde{Y}$, respectively. Then we may assume that X and Y are smooth. Let $\overline{f} \colon \overline{X} \to \overline{Y}$ be a compactification of $f \colon X \to Y$, that is, $\overline{X}$ and $\overline{Y}$ are smooth complete varieties and $\Delta_{\overline{X}} := \overline{X} - X$ and $\Delta_{\overline{Y}} := \overline{Y} - Y$ are simple normal crossing divisors on $\overline{X}$ and $\overline{Y}$, respectively. Let

$$\overline{f} \colon \overline{X} \xrightarrow{\overline{g}} \overline{Z} \xrightarrow{\overline{h}} \overline{Y}$$

be the Stein factorization of $\overline{f}$. We note that $\overline{h}$ is finite and $\overline{g}$ is birational. We put $\overline{Z} := \overline{h}^{-1}(Y)$ and $\Delta_{\overline{Z}} := \overline{Z} - Z$. Then we can easily check that

$$K_{\overline{Z}} + \Delta_{\overline{Z}} = \overline{h}^*(K_{\overline{Y}} + \Delta_{\overline{Y}})$$

and

$$K_{\overline{X}} + \Delta_{\overline{X}} = \overline{g}^*(K_{\overline{Z}} + \Delta_{\overline{Z}}) + E,$$

where E is effective and $\overline{g}$-exceptional. Thus, we have

$$\overline{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + \Delta_{\overline{X}}) = \kappa(\overline{Z}, K_{\overline{Z}} + \Delta_{\overline{Z}}) = \kappa(\overline{Y}, K_{\overline{Y}} + \Delta_{\overline{Y}}) = \overline{\kappa}(Y).$$

This is what we wanted. □

2.2. **Quasi-abelian varieties in the sense of Iitaka.** From now on, we quickly recall the basic properties of quasi-abelian varieties in the sense of Iitaka (see [I1] and [I2]). This paper shows that Iitaka's definition of quasi-abelian varieties is reasonable and natural from the viewpoint of the birational geometry.

Definition 2.8 (Quasi-abelian varieties in the sense of Iitaka)**.** Let G be a connected algebraic group. Then we have the Chevalley decomposition (see, for example, [C, Theorem 1.1] and [BSU, Theorem 1.1.1]):

$$1 \to \mathcal{G} \to G \to \mathcal{A} \to 1$$

in which $\mathcal{G}$ is the maximal affine algebraic subgroup of G and $\mathcal{A}$ is an abelian variety. If $\mathcal{G}$ is an algebraic torus $\mathbb{G}_m^d$ of dimension d, then G is called a *quasi-abelian* variety (in the sense of Iitaka).

We make some important remarks.

Remark 2.9. A quasi-abelian variety in Definition 2.8 is sometimes called a *semi-abelian variety* in the literature (see, for example, [BSU], [NW, Definition 5.1.20], and so on). We also note that the Chevalley decomposition in Definition 2.8 is called *Chevalley's structure theorem* in [BSU].

Remark 2.10. Let G be a quasi-abelian variety, that is, $\mathcal{G}$ is an algebraic torus $\mathbb{G}_m^d$ in Definition 2.8. Then, it is known that G is a principal $\mathbb{G}_m^d$-bundle over $\mathcal{A}$ in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]).

Note that the definition of quasi-abelian varieties in the sense of Iitaka (see Definition 2.8) is different from the definition in [AK, 3. Quasi-Abelian Varieties]. We also note that if G is a quasi-abelian variety in the sense of Iitaka then G is a quasi-abelian variety in the sense of [AK] (see, for example, [AK, 3.2.21 Main Theorem]).

Remark 2.11. It is well known that every algebraic group is quasi-projective (see, for example, [C, Corollary 1.2]).

Although we do not need the following fact, we can easily check:

Remark 2.12. Let G be a connected algebraic group. Then G is a quasi-abelian variety if and only if G contains no $\mathbb{G}_a$ as an algebraic subgroup (see [I2, Lemma 3]).

We note the following important property.

Lemma 2.13 (see [I2, Lemma 4])**.** *A quasi-abelian variety is a commutative algebraic group.*

Proof. We take $\tau \in G$ and consider the group homomorphism:

$$\Psi_\tau(\sigma) = \tau\sigma\tau^{-1} \colon \mathcal{G} \to G.$$

Since $\mathcal{G}$ is rational and $\mathcal{A}$ is an abelian variety, we see that $\Psi_\tau \colon \mathcal{G} \to \mathcal{G}$. Therefore, we obtain

$$G \ni \tau \mapsto \Psi_\tau \in \mathrm{Hom}(\mathcal{G}, \mathcal{G}).$$

Note that $\mathrm{Hom}(\mathcal{G}, \mathcal{G})$ is discrete because $\mathcal{G}$ is an algebraic torus. Thus, we obtain $\Psi_1 = \Psi_\tau$. Therefore, $\mathcal{G}$ is contained in the center of G. Moreover, if $\sigma, \tau \in G$, then we have

$$[\tau, \sigma] = \tau\sigma\tau^{-1}\sigma^{-1} \in \mathcal{G}$$

since $\mathcal{A}$ is commutative. Let ρ be any element of $\mathcal{G}$. Then it is easy to see that

$$[\tau\rho, \sigma] = [\tau, \sigma]$$

since $\mathcal{G}$ is contained in the center of G. Note that G is a principal $\mathcal{G}$-bundle over $\mathcal{A}$ as a complex manifold. Therefore, the morphism

$$G \ni \tau \mapsto [\tau, \sigma] \in \mathcal{G}$$

factors through a holomorphic map

$$\mathcal{A} \to \mathcal{G},$$

which is obviously trivial since $\mathcal{A}$ is complete. Hence, we obtain

$$[\tau, \sigma] = 1$$

for every $\sigma, \tau \in G$. This implies that G is commutative. $\square$

Remark 2.14. Let G be a quasi-abelian variety. By Lemma 2.13, G is a commutative group. Therefore, from now on, we write the group law in G additively if there is no danger of confusion. The unit element of G is denoted by 0. Note that an algebraic torus $\mathbb{G}_m^d$ is a quasi-abelian variety in the sense of Iitaka.

Lemma 2.15. *Let A be a quasi-abelian variety and let B be an algebraic subgroup of A. Then B and A/B are quasi-abelian varieties.*

Proof. We can construct the following big commutative diagram of Chevalley decompositions.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_B & \longrightarrow & B & \longrightarrow & \mathcal{A}_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_A & \longrightarrow & A & \longrightarrow & \mathcal{A}_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_{A/B} & \longrightarrow & A/B & \longrightarrow & \mathcal{A}_{A/B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that subgroups and homomorphic images of an algebraic torus are again algebraic tori. Hence we see that B and A/B are quasi-abelian varieties. We finish the proof. $\square$

We sometimes treat quasi-abelian varieties as commutative complex Lie groups.

Lemma 2.16. *Let G be a quasi-abelian variety. Then the universal cover of G is $\mathbb{C}^{\dim G}$ and G is $\mathbb{C}^{\dim G}/L$ for some lattice L as a complex Lie group. Of course, L is nothing but the topological fundamental group $\pi_1(G)$ of G. Note that the group law of G is induced by the usual addition of $\mathbb{C}^{\dim G}$.*

Proof. By Lemma 2.13, G is a commutative complex Lie group. Therefore, the universal cover is $\mathbb{C}^{\dim G}$ and there is a discrete subgroup L of

$\mathbb{C}^{\dim G}$ such that $G = \mathbb{C}^{\dim G}/L$ as a complex Lie group. By construction, the group law in G is induced by the usual addition of $\mathbb{C}^{\dim G}$ (see also the proof of Lemma 3.8). $\square$

In this paper, we mainly treat non-projective algebraic groups as complex Lie groups. We note the following famous example. It says that two different algebraic groups may be analytically isomorphic. Of course, we can not directly use Serre's GAGA principle for non-projective varieties.

Example 2.17 (Vector extensions of elliptic curves)**.** Let E be an elliptic curve. We take $0 \neq \xi \in H^1(E, \mathcal{O}_E) \simeq \mathbb{C}$. Then we have a non-trivial $\mathbb{G}_a$-bundle G over E associated to ξ in the Zariski topology. It is well known that there exists a short exact sequence of commutative algebraic groups

$$0 \to \mathbb{G}_a \to G \to E \to 0, \tag{2.2}$$

that is, G is a commutative algebraic group which is an extension of E by $\mathbb{G}_a$. Since $\xi \neq 0$, we can check that G is analytically isomorphic to $(\mathbb{C}^\times)^2$. Hence, $\mathbb{G}_m^2$ and G are analytically isomorphic but are two different algebraic groups. Note that (2.2) is the Chevalley decomposition of G. For some related topics, see [BSU], [Har, Chapter VI, Example 3.2], [Mu, Footnote in page 33], and so on.

In Section 3, we will discuss Iitaka's quasi-Albanese maps and prove the existence of quasi-Albanese maps and varieties in details.

Definition 2.18 (Quasi-Albanese maps)**.** Let X be a smooth variety. The *quasi-Albanese map* $\alpha\colon X \to A$ is a morphism to a quasi-abelian variety A such that

(i) for any other morphism $\beta\colon X \to B$ to a quasi-abelian variety B, there is a morphism $f\colon A \to B$ such that $\beta = f \circ \alpha$

$$\begin{array}{ccc} X & \xrightarrow{\ \beta\ } & B \\ {\scriptstyle \alpha}\big\downarrow & \nearrow_{f} & \\ A & & \end{array}$$

and

(ii) f is uniquely determined.

Note that A is usually called the *quasi-Albanese variety* of X.

If X is complete in Definition 2.18, then A is nothing but the Albanese variety of X.

3. Quasi-Albanese maps due to Iitaka

In this section, we discuss Iitaka's quasi-Albanese maps and varieties following [I1] and [I2]. We recommend the reader to study the basic results on the Albanese maps and varieties before reading this section (see, for example, [B, V.11–14], [U, §9], [GH, Chapter 2, Section 6], and so on).

Let us start with the following easy lemma on singular homology groups. In this section, X is a smooth complete algebraic variety and D is a simple normal crossing divisor on X. The Zariski open set $X \setminus D$ of X is denoted by V.

Lemma 3.1. *Let X be a smooth complete algebraic variety and let D be a simple normal crossing divisor on X. We put $V = X \setminus D$. Then the map*

$$\iota_*\colon H_1(V, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$

is surjective, where $\iota\colon V \hookrightarrow X$ is the natural open immersion.

Proof. We put $n = \dim X$. We have the following long exact sequence:

$$\cdots \longrightarrow H^{2n-1}(X, D; \mathbb{Z}) \xrightarrow{p} H^{2n-1}(X, \mathbb{Z}) \longrightarrow H^{2n-1}(D, \mathbb{Z}) \longrightarrow \cdots .$$

Note that $H^{2n-1}(D, \mathbb{Z}) = 0$ since D is an $(n-1)$-dimensional simple normal crossing variety. Therefore, p is surjective. We also note that

$$H^{2n-1}(X, D; \mathbb{Z}) \simeq H_c^{2n-1}(V, \mathbb{Z}).$$

We have the following commutative diagram:

$$\begin{array}{ccc} H_c^{2n-1}(V, \mathbb{Z}) & \xrightarrow{p} & H^{2n-1}(X, \mathbb{Z}) \\ {\scriptstyle D_V}\downarrow\simeq & & \simeq\downarrow{\scriptstyle D_X} \\ H_1(V, \mathbb{Z}) & \xrightarrow[\iota_*]{} & H_1(X, \mathbb{Z}). \end{array}$$

Note that the duality maps D_V and D_X are both isomorphisms by Poincaré duality. Since p is surjective, we see that ι_* is also surjective. For the details of Poincaré duality, see, for example, [Hat, Section 3.3]. □

Lemma 3.2. *The natural injection*

$$\iota^*\colon H^1(X, \mathbb{C}) \to H^1(V, \mathbb{C})$$

is nothing but

$$a_1 \oplus a_2\colon H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega_X^1) \to H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega_X^1(\log D))$$

where a_1 is the identity on $H^1(X, \mathcal{O}_X)$ and a_2 is the natural inclusion

$$H^0(X, \Omega_X^1) \hookrightarrow H^0(X, \Omega_X^1(\log D))$$

by Deligne's theory of mixed Hodge structures. Note that we have

$$b_1(V) - b_1(X) = \overline{q}(V) - q(X)$$

where

$$\overline{q}(V) = \dim H^0(X, \Omega^1_X(\log D)) \quad \textit{and} \quad q(X) = \dim H^0(X, \Omega^1_X).$$

Of course,

$$b_1(V) = \dim_{\mathbb{C}} H^1(V, \mathbb{C}) \quad \textit{and} \quad b_1(X) = \dim_{\mathbb{C}} H^1(X, \mathbb{C}).$$

Proof. By Lemma 3.1, ι^* is injective. By Deligne's mixed Hodge theory (see [D1]), we have

$$H^1(V, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X(\log D)).$$

By the Hodge decomposition, we have

$$H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X).$$

Since $\iota^* \colon H^1(X, \mathbb{C}) \to H^1(V, \mathbb{C})$ is a morphism of mixed Hodge structures (see [D1]), we obtain the desired description of ι^*. □

Let us describe the theory of quasi-Albanese maps and varieties due to Shigeru Iitaka (see [I1]).

3.3 (Quasi-Albanese maps and varieties)**.** We take a basis

$$\{\omega_1, \cdots, \omega_q\}$$

of $H^0(X, \Omega^1_X)$, where $q = q(X) = \dim H^0(X, \Omega^1_X)$. Note that $b_1(X) = 2q$ by the Hodge theory. We take

$$\varphi_1, \cdots, \varphi_d \in H^0(X, \Omega^1_X(\log D))$$

with $d = \overline{q}(V) - q(X)$ such that

$$\{\omega_1, \cdots, \omega_q, \varphi_1, \cdots, \varphi_d\}$$

is a basis of $H^0(X, \Omega^1_X(\log D))$. Let

$$\{\xi_1, \cdots, \xi_{2q}\}$$

be a basis of the free part of $H_1(X, \mathbb{Z})$. We take

$$\eta_1, \cdots, \eta_d \in \mathrm{Ker}\iota_* \subset H_1(V, \mathbb{Z})$$

such that

$$\{\xi_1, \cdots, \xi_{2q}, \eta_1, \cdots, \eta_d\}$$

is a basis of the free part of $H_1(V, \mathbb{Z})$ (see Lemma 3.1). We put $\overline{q} = \overline{q}(V)$,

$$A_i = \left(\int_{\xi_i} \omega_1, \cdots \int_{\xi_i} \omega_q, \int_{\xi_i} \varphi_1, \cdots, \int_{\xi_i} \varphi_d\right) \in \mathbb{C}^{\overline{q}}$$

for $1 \leq i \leq 2q$, and

$$B_j = \left(\int_{\eta_j} \omega_1, \cdots \int_{\eta_j} \omega_q, \int_{\eta_j} \varphi_1, \cdots, \int_{\eta_j} \varphi_d \right) \in \mathbb{C}^{\overline{q}}$$

for $1 \leq j \leq d$.

Lemma 3.4. *Let γ be a torsion element of $H_1(V, \mathbb{Z})$. Then we have*

$$\int_{\gamma} \omega = 0$$

for every $\omega \in H^0(X, \Omega^1_X(\log D))$.

Proof. It is obvious since

$$m \int_{\gamma} \omega = \int_{m\gamma} \omega = 0$$

if $m\gamma = 0$ in $H_1(V, \mathbb{Z})$. □

Lemma 3.5. *We have*

$$\int_{\eta_j} \omega_k = 0$$

for every j and k.

Proof. We see that

$$\int_{\eta_j} \omega_k = \int_{\eta_j} \iota^* \omega_k = \int_{\iota_* \eta_j} \omega_k = 0$$

since $\iota_* \eta_j = 0$. □

Lemma 3.6 (see [I1, Lemma 2])**.** *Let φ be an arbitrary element of $H^0(X, \Omega^1_X(\log D))$. Assume*

$$\int_{\eta} \varphi = 0$$

for every $\eta \in \mathrm{Ker}\iota_* \subset H_1(V, \mathbb{Z})$. Then we have $\varphi \in H^0(X, \Omega^1_X)$.

Proof. Assume that $\varphi \in H^0(X, \Omega^1_X(\log D)) \setminus H^0(X, \Omega^1_X)$. Then φ has a pole along some D_a, where D_a is an irreducible component of D. Let p be a general point of D_a. We take a local holomorphic coordinate system $(z_1, \cdots, z_n)$ around p such that D_a is defined by $z_1 = 0$. In this case, we can write

$$\varphi = \alpha(z) \frac{dz_1}{z_1} + \beta(z)$$

around p, where $\beta(z)$ is a holomorphic 1-form. We may assume that $\alpha(z) = \alpha(z_2, \cdots, z_n)$ by Weierstrass division theorem (see, for example, [GH]). Since $d\varphi = 0$, we obtain

$$d\varphi = d\alpha \wedge \frac{dz_1}{z_1} + d\beta = 0.$$

Thus we have $d\alpha = 0$. This means that α is a constant. Let us consider a circle γ_a around D_a at p. Then we obtain $\iota_*\gamma_a = 0$ in $H_1(X, \mathbb{Z})$ and

$$0 = \int_{\gamma_a} \varphi = \alpha \int_{\gamma_a} \frac{dz_1}{z_1} = \alpha 2\pi\sqrt{-1}.$$

This implies that $\alpha = 0$. Thus, φ is holomorphic at p. This is a contradiction. Therefore, we have $\varphi \in H^0(X, \Omega^1_X)$. □

Lemma 3.7 (see [I2, Proposition 2])**.** *The above vectors* $A_1, \cdots, A_{2q}$, $B_1, \cdots, B_d$ *are* $\mathbb{R}$-$\mathbb{C}$ *linearly independent. This means that*

$$\sum_{i=1}^{2q} a_i A_i + \sum_{j=1}^{d} b_j B_j = 0$$

for $a_i \in \mathbb{R}$ *and* $b_j \in \mathbb{C}$ *then* $a_i = 0$ *for every* i *and* $b_j = 0$ *for every* j.

Proof. We put

$$\widehat{A}_i = \left(\int_{\xi_i} \omega_1, \cdots, \int_{\xi_i} \omega_q \right)$$

for $1 \leq i \leq 2q$. Then $\widehat{A}_1, \cdots, \widehat{A}_{2q}$ are $\mathbb{R}$-linearly independent, which is well known by the Hodge theory. By Lemma 3.5, we have $a_i = 0$ for every i. We put

$$\widehat{B}_j = \left(\int_{\eta_j} \varphi_1, \cdots, \int_{\eta_j} \varphi_d \right)$$

for $1 \leq j \leq d$. It is sufficient to prove that $\widehat{B}_1, \cdots, \widehat{B}_d$ are $\mathbb{C}$-linearly independent. If $\widehat{B}_1, \cdots, \widehat{B}_d$ are $\mathbb{C}$-linearly dependent, then the rank of the $d \times d$ matrix

$$\left(\int_{\eta_j} \varphi_i \right)_{i,j}$$

is less than d. This means that there is $(c_1, \cdots, c_d) \neq 0$ such that

$$\int_{\eta_j} \sum_{i=1}^{d} c_i \varphi_i = 0$$

for every j. Therefore, we see that

$$\sum_{i=1}^{d} c_i\varphi_i \in H^0(X, \Omega^1_X)$$

by Lemma 3.6. This contradicts the choice of $\{\varphi_1, \cdots, \varphi_d\}$. Thus, $\widehat{B}_1, \cdots, \widehat{B}_d$ are $\mathbb{C}$-linearly independent. □

By the proof of Lemma 3.7, we can choose $\varphi_1, \cdots, \varphi_d$ such that

$$\int_{\eta_j} \varphi_k = \delta_{jk}.$$

We put

$$L = \sum_i \mathbb{Z}A_i + \sum_j \mathbb{Z}B_j,$$
$$L_1 = \sum_i \mathbb{Z}\widehat{A}_i,$$

and

$$L_0 = \sum_j \mathbb{Z}\widehat{B}_j.$$

Then we get the following short exact sequence of complex Lie groups:

$$0 \longrightarrow \mathbb{C}^d/L_0 \longrightarrow \mathbb{C}^{\overline{q}}/L \longrightarrow \mathbb{C}^q/L_1 \longrightarrow 0. \tag{3.1}$$

Note that $T = \mathbb{C}^d/L_0$ is an algebraic torus $\mathbb{G}_m^d$ and that $\mathcal{A}_X = \mathbb{C}^q/L_1$ is the Albanese variety of X. More explicitly, if $(z_1, \cdots, z_d)$ is the standard coordinate system of $\mathbb{C}^d$, then the isomorphism

$$\mathbb{C}^d/L_0 \xrightarrow{\sim} \mathbb{G}_m^d$$

is given by

$$(z_1, \cdots, z_d) \mapsto (\exp 2\pi\sqrt{-1}z_1, \cdots, \exp 2\pi\sqrt{-1}z_d).$$

We call

$$\widetilde{\mathcal{A}}_V = \mathbb{C}^{\overline{q}}/L$$

the quasi-Albanese variety of V. By the above description, we see that $\widetilde{\mathcal{A}}_V$ is a principal $\mathbb{G}_m^d$-bundle over an abelian variety $\mathcal{A}_X$ as a complex manifold. We have to check:

Lemma 3.8. *The quasi-Albanese variety $\widetilde{\mathcal{A}}_V$ is a quasi-abelian variety.*

Proof. We put $A = \widetilde{\mathcal{A}}_V$ and $B = \mathcal{A}_X$ for simplicity. Note that A is a principal $\mathbb{G}_m^d$-bundle over B as a complex manifold. We consider the following group homomorphism:

$$\rho\colon \mathbb{G}_m^d \to \mathrm{PGL}(d, \mathbb{C})$$

given by

$$\rho(\lambda_1, \cdots, \lambda_d) = \begin{pmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_d \end{pmatrix}.$$

By ρ, we obtain $\mathbb{P}^d$-bundle $Z = A \times_\rho \mathbb{P}^d$ over $B = A/\mathbb{G}_m^d$ which is a compactification of A. It is easy to see that the divisor $\Delta = Z \setminus A$ is a simple normal crossing divisor on Z and is ample over B. Moreover, we can easily see that $Z \to B$ and $A \to B$ are locally trivial in the Zariski topology. From now, we will see that the multiplication

$$\psi \colon A \times A \longrightarrow A$$

of A as a complex Lie group is algebraic. By construction, the map ψ can be extended to holomorphic maps

$$g_1 \colon Z \times A \longrightarrow Z \quad \text{and} \quad g_2 \colon A \times Z \longrightarrow Z$$

since Z is a $\mathbb{G}_m^d$-equivariant embedding of A. Therefore, we obtain a holomorphic map

$$g \colon Z \times Z \setminus \Sigma \longrightarrow Z \hookrightarrow \mathbb{P}^N,$$

where $\Sigma = (\Delta \times Z) \cap (Z \times \Delta)$. Of course, g is an extension of $\psi \colon A \times A \to A$. Note that $\operatorname{codim}_{Z\times Z}\Sigma \geq 2$. We consider $g^*\mathcal{O}_{\mathbb{P}^N}(1)$. This line bundle can be extended to a line bundle $\mathcal{L}$ on $Z \times Z$. Moreover, we can see

$$l_i := g^* X_i \in H^0(Z \times Z, \mathcal{L})$$

for $0 \leq i \leq N$, where $[X_0 : \cdots : X_N]$ are homogeneous coordinates of $\mathbb{P}^N$. Therefore, we obtain a rational map $h \colon Z \times Z \dashrightarrow Z$, which is given by the linear system spanned by $\{l_0, \cdots, l_N\}$ and is an extension of g. Thus, the multiplication

$$\psi \colon A \times A \to A$$

is algebraic since $\psi = h|_{A\times A}$. Let

$$\iota \colon A \to A$$

be the inverse. We can easily see that ι extends to

$$Z \setminus \Delta_{\mathrm{sing}} \to Z \hookrightarrow \mathbb{P}^N,$$

where Δ_{sing} is the singular locus of Δ. Hence, as in the case of g, we obtain a birational map $Z \dashrightarrow Z$, which is an extension of $\iota \colon A \to A$. Thus, ι is algebraic. This means that $A = \widetilde{\mathcal{A}}_V$ is an algebraic group. So, $\widetilde{\mathcal{A}}_V$ is a quasi-abelian variety. Note that the short exact sequence (3.1) is nothing but the Chevalley decomposition. $\square$

Lemma 3.9. *Let ω be an element of $H^0(X, \Omega^1_X(\log D))$. We fix a point $0 \in V$. Then we have a multivalued holomorphic function*

$$\int_0^p \omega$$

on V. For a point $p \in V$, we can define $\alpha_V \colon V \to \widetilde{\mathcal{A}}_V$ by

$$\alpha_V(p) = \left(\int_0^p \omega_1, \cdots, \int_0^p \omega_q, \int_0^p \varphi_1, \cdots, \int_0^p \varphi_d \right) \in \widetilde{\mathcal{A}}_V.$$

This map is independent of the choice of the path from 0 to p in V. Thus we get a quasi-Albanese map:

$$\alpha_V \colon V \to \widetilde{\mathcal{A}}_V.$$

It is a holomorphic map.

Proof. Let γ be a 2-cycle on V. Then

$$\int_{\partial\gamma} \omega = \int_\gamma d\omega = 0$$

for every $\omega \in H^0(X, \Omega^1_X(\log D))$. This is because ω is d-closed by Deligne (see [D1]). Therefore, α_V is well-defined. □

Lemma 3.10 (see [I1, Proposition 3])**.** *The map α_V in Lemma 3.9 is algebraic.*

Proof. Note that $A = \widetilde{\mathcal{A}}_V$ is a principal $\mathbb{G}_m^d$-bundle over $B = \mathcal{A}_X$ as a complex manifold. We consider the group homomorphism

$$\rho' \colon \mathbb{G}_m^d \to \mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C}) \times \cdots \times \mathrm{PGL}(2, \mathbb{C})$$

given by

$$\rho'(\lambda_1, \cdots, \lambda_d) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & 0 \\ 0 & \lambda_d \end{pmatrix}.$$

Then we obtain a $\mathbb{G}_m^d$-equivariant embedding $Z' = A \times_{\rho'} (\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)$ of A over B.

Claim. *The holomorphic map*

$$\alpha_V \colon V \to \widetilde{\mathcal{A}}_V$$

given in Lemma 3.9 can be extended to a rational map

$$\beta_X \colon X \dashrightarrow Z'.$$

Proof of Claim. We note that it is sufficient to prove that there exists a meromorphic extension β_X of α_V since X and Z' are smooth complete algebraic varieties. Let p be a point of $D \subset X$. Let $(z_1, \cdots, z_n)$ be a local holomorphic coordinate system of X at p such that D is defined by $z_1 \cdots z_r = 0$. In this case, we can write

$$\varphi_i = \sum_{b=1}^{r} \alpha_{ib} \frac{dz_b}{z_b} + \widetilde{\varphi}_i$$

where $\alpha_{ib} \in \mathbb{C}$ and $\widetilde{\varphi}_i$ is a holomorphic 1-form for every i around p (see the proof of Lemma 3.6). Let δ_a be a circle around $D_a = (z_a = 0)$ near p. Then $\iota_* \delta_a = 0$. Therefore, we have

$$\delta_a = \sum_j m_{ja} \eta_j + \widetilde{\delta}_a,$$

where $m_{ja} \in \mathbb{Z}$ and $\widetilde{\delta}_a$ is a torsion element. Thus we have

$$\alpha_{ia} = \frac{1}{2\pi\sqrt{-1}} \int_{\delta_a} \varphi_i = \frac{1}{2\pi\sqrt{-1}} \sum_j m_{ja} \int_{\eta_j} \varphi_i = \frac{m_{ia}}{2\pi\sqrt{-1}}.$$

Without loss of generality, we may assume that $0 \in V$ is near p. For a point $p' \in V$ near p, we have

$$\begin{aligned} &\exp\left(2\pi\sqrt{-1}\int_0^{p'} \varphi_i\right) \\ &= c\exp\left(\sum_b m_{ib} \log z_b(p')\right) \cdot \exp\left(2\pi\sqrt{-1}\int_0^{p'} \widetilde{\varphi}_i\right) \qquad (3.2) \\ &= c\prod_b z_b(p')^{m_{ib}} \cdot \exp\left(2\pi\sqrt{-1}\int_0^{p'} \widetilde{\varphi}_i\right) \end{aligned}$$

for some constant c. We consider the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\alpha_V} & \widetilde{\mathcal{A}}_V \\ \downarrow & & \downarrow \\ X & \dashrightarrow & Z' \\ \downarrow & & \downarrow{\scriptstyle \pi_{Z'}} \\ B & = & B \end{array}$$

Note that $X \to B$ is nothing but the Albanese map of X. Let U be a small open set of B in the classical topology. Then

$$\pi^{-1}(U) \simeq U \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times,$$

where $\pi\colon \widetilde{\mathcal{A}}_V \to B = \mathcal{A}_X$, and

$$\pi_{Z'}^{-1}(U) \simeq U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

over U. Over U, it is easy to see that α_V can be extended to a meromorphic map $X \dashrightarrow Z'$ in the sense of Remmert by (3.2) (see [GR, Chapter 10, §6, 3. Graph of a Finite System of Meromorphic Functions]). For the definition of meromorphic mappings in the sense of Remmert, see, for example, [U, Definition 2.2]. Therefore, α_V can be extended to a meromorphic map β_X from X to Z' in the sense of Remmert. By Serre's GAGA principle (see, for example, [SGA1, Exposé XII]), a meromorphic map $\beta_X\colon X \dashrightarrow Z'$ is a rational map between smooth complete algebraic varieties. □

Thus we obtain that α_V in Lemma 3.9 is algebraic. □

Lemma 3.11. *We have that*

$$(\alpha_V)_*\colon H_1(V, \mathbb{Z}) \to H_1(\widetilde{\mathcal{A}}_V, \mathbb{Z})$$

is surjective. Moreover, we have

$$\mathrm{Ker}(\alpha_V)_* = H_1(V, \mathbb{Z})_{\mathrm{tor}},$$

where $H_1(V, \mathbb{Z})_{\mathrm{tor}}$ is the torsion part of $H_1(V, \mathbb{Z})$.

Proof. Let $H_1(V, \mathbb{Z})_{\mathrm{free}}$ be the free part of $H_1(V, \mathbb{Z})$. Note that $\widetilde{\mathcal{A}}_V = \mathbb{C}^{\overline{q}}/L$ by construction, where

$$\mathbb{C}^{\overline{q}} = (H^0(X, \Omega^1_X(\log D)))^*$$

and L is an embedding of $H_1(V, \mathbb{Z})_{\mathrm{free}}$ into $(H^0(X, \Omega^1_X(\log D)))^*$. On the other hand,

$$H_1(\widetilde{\mathcal{A}}_V, \mathbb{Z}) = \pi_1(\widetilde{\mathcal{A}}_V) = L$$

by construction. By the construction of the lattice L and the quasi-Albanese map $\alpha_V\colon V \to \widetilde{\mathcal{A}}_V$, it is obvious that

$$(\alpha_V)_*\colon H_1(V, \mathbb{Z}) \to H_1(\widetilde{\mathcal{A}}_V, \mathbb{Z})$$

is surjective and that

$$\mathrm{Ker}(\alpha_V)_* = H_1(V, \mathbb{Z})_{\mathrm{tor}}.$$

This is the desired property. □

Lemma 3.12. *We have that*

$$\alpha_V^*\colon T_1(\widetilde{\mathcal{A}}_V) \to T_1(V)$$

is an isomorphism.

Proof. By Lemma 3.11,

$$(\alpha_V)_*\colon H_1(V,\mathbb{Q}) \to H_1(\widetilde{\mathcal{A}}_V,\mathbb{Q})$$

is an isomorphism. Therefore, we obtain

$$(\alpha_V)^*\colon H^1(\widetilde{\mathcal{A}}_V,\mathbb{Q}) \to H^1(V,\mathbb{Q})$$

is also an isomorphism. Moreover, it is an isomorphism of mixed Hodge structures (see [D1]). Therefore, we have an isomorphism

$$\alpha_V^*\colon T_1(\widetilde{\mathcal{A}}_V) \to T_1(V)$$

by Deligne (see [D1]). □

The following lemma is useful and important.

Lemma 3.13. *Let W be a quasi-abelian variety. Then the quasi-Albanese map*

$$\alpha_W\colon W \to \widetilde{\mathcal{A}}_W$$

is an isomorphism.

Proof. By translation, we may assume that $\alpha_W(0) = 0$. By Lemma 3.11,

$$(\alpha_W)_*\colon H_1(W,\mathbb{Z}) \to H_1(\widetilde{\mathcal{A}}_W,\mathbb{Z}) \tag{3.3}$$

is an isomorphism. By Lemma 3.12,

$$\alpha_W^*\colon T_1(\widetilde{\mathcal{A}}_W) \to T_1(W) \tag{3.4}$$

is an isomorphism. Then α_W induces an isomorphism of complex vector spaces

$$(\alpha_W)_*\colon T_{W,0} \to T_{\widetilde{\mathcal{A}}_W,0},$$

where $T_{W,0}$ is the tangent space of W at 0 and $T_{\widetilde{\mathcal{A}}_W,0}$ is the tangent space of $\widetilde{\mathcal{A}}_W$ at 0. By considering the exponential maps, we can recover α_W by (3.3) and (3.4). By the isomorphisms in (3.3) and (3.4), α_W is an isomorphism of complex Lie groups. Note that α_W is algebraic. Therefore, α_W is an isomorphism between smooth algebraic varieties. □

Lemma 3.14. *Let $f\colon V \to T$ be a morphism to a quasi-abelian variety T. Then there exists a unique algebraic morphism $\widetilde{f}\colon \widetilde{\mathcal{A}}_V \to T$ such that $f = \widetilde{f} \circ \alpha_V$*

$$\begin{array}{ccc} V & \xrightarrow{f} & T \\ {\scriptstyle \alpha_V}\downarrow & \nearrow_{\widetilde{f}} & \\ \widetilde{\mathcal{A}}_V & & \end{array}$$

where $\alpha_V \colon V \to \widetilde{\mathcal{A}}_V$ *is a quasi-Albanese map of* V*.*

Proof. We take a point $0 \in V$. By translations, we may assume that $\alpha_V(0) = 0$ and $f(0) = 0$. Let $\{u_1, \cdots, u_k\}$ be a basis of $T_1(T)$. We may assume that

$$f^*u_1, \cdots, f^*u_l$$

are linearly independent, where

$$l = \dim_{\mathbb{C}}\langle f^*u_1, \cdots, f^*u_k\rangle.$$

We take $v_1, \cdots, v_m \in T_1(V)$ such that

$$\{v_1, \cdots, v_m, f^*u_1, \cdots, f^*u_l\}$$

is a basis of $T_1(V)$. Since $f_* \colon H_1(V, \mathbb{Z}) \to H_1(T, \mathbb{Z})$, by using the basis $\{v_1, \cdots, v_m, f^*u_1, \cdots, f^*u_l\}$ of $T_1(V)$, we can easily construct a holomorphic map

$$\widetilde{f} \colon \widetilde{\mathcal{A}}_V \longrightarrow \widetilde{\mathcal{A}}_T \xrightarrow[\alpha_T^{-1}]{\sim} T$$

(see Lemma 3.13) satisfying $f = \widetilde{f} \circ \alpha_V$. Therefore, there is a commutative diagram:

$$\begin{array}{ccc} T_1(V) & \xleftarrow{f^*} & T_1(T) \\ {\scriptstyle \alpha_V^*}\uparrow & \swarrow {\scriptstyle \widetilde{f}^*} & \\ T_1(\widetilde{\mathcal{A}}_V) & & \end{array}$$

which determines $\widetilde{f}^*$ uniquely. This is because α_V^* is an isomorphism (see Lemma 3.12). As in the proof of Lemma 3.13, by considering the exponential maps, we see that $\widetilde{f}$ can be uniquely recovered by $\widetilde{f}^*$. Thus, $\widetilde{f}$ is unique. Therefore, all we have to do is to prove that $\widetilde{f}$ is algebraic. It is sufficient to prove that the graph

$$\Gamma = \{(x, \widetilde{f}(x)) \,|\, x \in \widetilde{\mathcal{A}}_V\} \subset \widetilde{\mathcal{A}}_V \times T$$

is an algebraic variety. We consider the map

$$\alpha_n \colon V^{2n} = V \times \cdots \times V \to \widetilde{\mathcal{A}}_V$$

given by

$$\alpha_n(z_1, \cdots, z_{2n}) = \alpha_V(z_1) + \cdots + \alpha_V(z_n) - \alpha_V(z_{n+1}) - \cdots - \alpha_V(z_{2n}).$$

We put

$$F_n = \overline{\mathrm{Im}\alpha_n},$$

that is, the Zariski closure of $\mathrm{Im}\alpha_n$. Then F_n is an irreducible algebraic subvariety of $\widetilde{\mathcal{A}}_V$ for every n such that

$$F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots.$$

Therefore, there is a positive integer n_0 such that

$$F_{n_0} = F_{n_0+1} = \cdots .$$

Note that F_{n_0} is a quasi-abelian subvariety of $\widetilde{\mathcal{A}}_V$ because it is closed under the group law of $\widetilde{\mathcal{A}}_V$. Moreover, by the universality of $\widetilde{\mathcal{A}}_V$ proved above, F_{n_0} is not contained in a quasi-abelian proper subvariety of $\widetilde{\mathcal{A}}_V$. This implies that $F_{n_0} = \widetilde{\mathcal{A}}_V$. Note that $\widetilde{f}$ is a homomorphism of complex Lie groups. We consider the following commutative diagram:

$$\begin{array}{ccc} V^{2n_0} & \xrightarrow{f_{n_0}} & T \\ {\scriptstyle \alpha_{n_0}}\downarrow & \nearrow_{\widetilde{f}} & \\ \widetilde{\mathcal{A}}_V & & \end{array}$$

where

$$f_{n_0}(z_1, \cdots, z_{2n_0}) = f(z_1) + \cdots + f(z_{n_0}) - f(z_{n_0+1}) - \cdots - f(z_{2n_0}).$$

We consider the Zariski closure of

$$\{(\alpha_{n_0}(x), f_{n_0}(x)) \mid x \in V^{2n_0}\} \subset \Gamma \subset \widetilde{\mathcal{A}}_V \times T.$$

Then it is an algebraic subvariety of $\widetilde{\mathcal{A}}_V \times T$ and coincides with the graph Γ. This implies that $\widetilde{f}$ is algebraic. □

Lemma 3.15. *Let $f\colon V_1 \to V_2$ be a morphism between smooth algebraic varieties. Then f induces an algebraic morphism $f_*\colon \widetilde{\mathcal{A}}_{V_1} \to \widetilde{\mathcal{A}}_{V_2}$ which satisfies the following commutative diagram.*

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \alpha_{V_1}}\downarrow & & \downarrow{\scriptstyle \alpha_{V_2}} \\ \widetilde{\mathcal{A}}_{V_1} & \xrightarrow[f_*]{} & \widetilde{\mathcal{A}}_{V_2} \end{array}$$

Moreover, f_* is unique.

Proof. It is almost obvious by Lemma 3.14. We apply Lemma 3.14 to the map $\alpha_{V_2} \circ f\colon V_1 \to \widetilde{\mathcal{A}}_{V_2}$. Then we obtain the desired map $f_*\colon \widetilde{\mathcal{A}}_{V_1} \to \widetilde{\mathcal{A}}_{V_2}$ uniquely. □

We summarize:

Theorem 3.16 (Iitaka's quasi-Albanese varieties and maps)**.** *Let V be a smooth algebraic variety. Then there exists a quasi-abelian variety $\widetilde{\mathcal{A}}_V$ and a morphism $\alpha_V\colon V \to \widetilde{\mathcal{A}}_V$ with the following property:*

for any quasi-abelian variety T and any morphism $f\colon V \to T$, there exists a unique morphism $\widetilde{f}\colon \widetilde{\mathcal{A}}_V \to T$ such that $\widetilde{f} \circ \alpha_V = f$.

$$\begin{array}{ccc} V & \xrightarrow{f} & T \\ {\scriptstyle \alpha_V}\downarrow & \nearrow_{\widetilde{f}} & \\ \widetilde{\mathcal{A}}_V & & \end{array}$$

The quasi-abelian variety $\widetilde{\mathcal{A}}_V$, determined up to isomorphism by this condition, is called the quasi-Albanese variety *of V. The map $\alpha_V\colon V \to \widetilde{\mathcal{A}}_V$ is called the* quasi-Albanese map *of V. By the construction of $\widetilde{\mathcal{A}}_V$, $\widetilde{\mathcal{A}}_V$ is nothing but the Albanese variety of V when V is complete.*

Anyway, Theorem 3.16 is a generalization of the theory of Albanese maps and varieties for non-compact smooth complex algebraic varieties. We close this section with an easy corollary of Theorem 3.16.

Corollary 3.17 (cf. Remark 2.12)**.** *Let $f\colon \mathbb{A}^1 \to G$ be an algebraic morphism from $\mathbb{A}^1$ to a quasi-abelian variety G. Then $f(\mathbb{A}^1)$ is a point.*

Proof. Note that $T_1(\mathbb{A}^1) = 0$. Thus the quasi-Albanese variety $\widetilde{\mathcal{A}}_{\mathbb{A}^1}$ is a point. Since f factors through $\widetilde{\mathcal{A}}_{\mathbb{A}^1}$ by Theorem 3.16, $f(\mathbb{A}^1)$ is a point. □

4. Basic properties of quasi-abelian varieties

In this section, we collect some basic properties of quasi-abelian varieties for the reader's convenience. We will use them in the proof of Theorem 1.2.

4.1. Let G be a quasi-abelian variety and let

$$0 \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow \mathcal{A} \longrightarrow 0 \tag{4.1}$$

be the Chevalley decomposition such that $\mathcal{G} = \mathbb{G}_m^d$. We put $\dim \mathcal{A} = q$ and $n = \dim G = q + d$. Then there exists a $(2q + d) \times n$ matrix M with

$$M = \begin{pmatrix} P & Q \\ 0 & I_d \end{pmatrix}$$

where P is a $2q \times q$ matrix and I_d is the $d \times d$ unit matrix. The lattice spanned by the row vectors of M (resp. P) is denoted by L (resp. L_1). Then we have the short exact sequence of complex Lie groups:

$$0 \longrightarrow \mathbb{G}_m^d \longrightarrow \mathbb{C}^n/L \longrightarrow \mathbb{C}^q/L_1 \longrightarrow 0. \tag{4.2}$$

Note that

$$\mathbb{C}^d/\mathbb{Z}^d \simeq \mathbb{G}_m^d$$

by

$$(z_{q+1}, \cdots, z_n) \mapsto (\exp 2\pi\sqrt{-1}z_{q+1}, \cdots, \exp 2\pi\sqrt{-1}z_n),$$

where $(z_1, \cdots, z_n)$ is the standard coordinate system of $\mathbb{C}^n$. By the descriptions in Section 3, the short exact sequence of complex Lie groups (4.2) is isomorphic to the short exact sequence (4.1). The description $\mathbb{C}^n/L$ for G is useful for various computations in the following theorems.

We will repeatedly use the following theorem implicitly.

Theorem 4.2. *Let G be a quasi-abelian variety. Assume that $\pi\colon G' \to G$ is a finite étale morphism from a variety G'. Then G' is a quasi-abelian variety.*

Proof. We use the notation in 4.1. By 4.1, $G = \mathbb{C}^n/L$. Then we have a sublattice L' of L such that $[L : L'] < \infty$ and that $G' = \mathbb{C}^n/L'$. By a translation of G, we may assume that $\pi(0) = 0$. Then we can easily construct a commutative diagram of complex Lie groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m^d & \longrightarrow & \mathbb{C}^n/L' & \longrightarrow & \mathbb{C}^q/L'_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \\ 0 & \longrightarrow & \mathbb{G}_m^d & \longrightarrow & \mathbb{C}^n/L & \longrightarrow & \mathbb{C}^q/L_1 & \longrightarrow & 0 \end{array}$$

such that π, π_1, and π_2 are finite. Since π_1 is finite, $\mathbb{C}^q/L'_1$ is an abelian variety. Note that $G' = \mathbb{C}^n/L'$ is a principal $\mathbb{G}_m^d$-bundle over $\mathbb{C}^q/L'_1$ as a complex manifold. By the proof of Lemma 3.8, the group law of $G' = \mathbb{C}^n/L'$ as a complex Lie group is algebraic. This means that G' is a quasi-abelian variety and that $\pi\colon G' \to G$ is a group homomorphism between quasi-abelian varieties. □

Theorem 4.3 ([I1, 10.]). *Let G be a quasi-abelian variety. Then we have $\overline{\kappa}(G) = 0$ and $\overline{q}(G) = \dim G$.*

Proof. Note that G is a principal $\mathbb{G}_m^d$-bundle over an abelian variety $\mathcal{A}$ as a complex manifold. As in the proof of Lemma 3.8, we have a $\mathbb{P}^d$-bundle $\overline{G}$ over $\mathcal{A}$ such that $\overline{G}$ is a $\mathbb{G}_m^d$-equivariant embedding of G over $\mathcal{A}$. We put $D = \overline{G} - G$. Then D is a simple normal crossing divisor on $\overline{G}$. We can easily check that

$$\Omega^1_{\overline{G}}(\log D) \simeq \oplus \mathcal{O}_{\overline{G}}.$$

More explicitly, $\Omega^1_{\overline{G}}(\log D)$ is isomorphic to $\oplus_{i=1}^n \mathcal{O}_{\overline{G}} dz_i$ in the notation of 4.1. Therefore, we obtain that $\overline{q}(G) = \dim G$ and $K_{\overline{G}} + D \sim 0$. In particular, we have $\overline{\kappa}(G) = \kappa(\overline{G}, K_{\overline{G}} + D) = 0$. □

Theorem 4.4 (cf. [I1, Theorem 4.1]). *Let G be a quasi-abelian variety. Let W be a closed subvariety of G. Then $\overline{\kappa}(W) \geq 0$. Moreover, $\overline{\kappa}(W) = 0$ if and only if W is a translation of a quasi-abelian subvariety of G.*

Proof. We take a general point $p \in W$, around which we take a system of local analytic coordinates $(\zeta_1, \cdots, \zeta_n)$ such that

$$W = (\zeta_{r+1} = \cdots = \zeta_n = 0).$$

Let $\pi\colon \mathbb{C}^n \to G$ be the universal cover. We take $q \in \pi^{-1}(p)$ and assume that $z_1(q) = \cdots = z_n(q) = 0$, where $(z_1, \cdots, z_n)$ is a system of global coordinates of $\mathbb{C}^n$. Note that $(\zeta_1, \cdots, \zeta_n)$ can be regarded as a system of local analytic coordinates around q. By taking a suitable linear transformation of $\mathbb{C}^n$, we have

$$\zeta_j = z_j - \varphi_j(z_1, \cdots, z_n),$$

where $\varphi_j(0) = 0$ and

$$\frac{\partial \varphi_j}{\partial z_k}(0) = 0$$

for every j and k around q. The dz_j defines a logarithmic 1-form on G, that is, $dz_j \in T_1(G)$ for every j (see the proof of Theorem 4.3). Let $f\colon V \to W$ be a resolution and let $\overline{V}$ be a smooth projective variety such that $\Delta = \overline{V} - V$ is a simple normal crossing divisor on $\overline{V}$. Without loss of generality, we may assume that f is an isomorphism over a neighborhood of p. Thus, we see that $f^*(dz_j|_W)$ is an element of $T_1(W)$. Since $d\zeta_1, \cdots, d\zeta_r$ are linearly independent holomorphic 1-forms on W around p, $f^*(dz_1|_W), \cdots, f^*(dz_r|_W)$ are also linearly independent. Thus we have

$$0 \neq f^*(dz_1 \wedge \cdots \wedge dz_r)|_W \in H^0(\overline{V}, \mathcal{O}_{\overline{V}}(K_{\overline{V}} + \Delta)).$$

This means that $\overline{\kappa}(W) \geq 0$. For $r+1 \leq j \leq n$, we have

$$dz_j|_{\pi^{-1}(W)} - \sum_{k=1}^{n} \left.\frac{\partial \varphi_j}{\partial z_k}\right|_{\pi^{-1}(W)} \cdot dz_k|_{\pi^{-1}(W)} = 0$$

around q. Therefore, we obtain

$$\sum_{k=r+1}^{n} \left(\delta_{jk} - \left.\frac{\partial \varphi_j}{\partial z_k}\right|_{\pi^{-1}(W)}\right) dz_k|_{\pi^{-1}(W)} = \sum_{i=1}^{r} \left.\frac{\partial \varphi_j}{\partial z_i}\right|_{\pi^{-1}(W)} \cdot dz_i|_{\pi^{-1}(W)}$$

in a neighborhood of q. Thus, for $r+1 \leq j \leq n$, we have

$$dz_j|_W = \sum_{i=1}^{r} A_{ji}(\zeta_1, \cdots, \zeta_r) \cdot dz_i|_W$$

around p, where A_{ji} is a holomorphic function for every i and j such that $A_{ji}(0) = 0$. Note that

$$f^*(dz_1|_W), \cdots, f^*(dz_r|_W) \in T_1(W),$$

which are linearly independent. Assume that $\overline{\kappa}(W) = 0$. Then we have $\kappa(\overline{V}, K_{\overline{V}} + \Delta) = 0$. We note that $f^*(dz_1 \wedge \cdots \wedge dz_r)|_W$ is a nonzero element of $H^0(\overline{V}, \mathcal{O}_{\overline{V}}(K_{\overline{V}} + \Delta))$. Therefore, $H^0(\overline{V}, \mathcal{O}_{\overline{V}}(K_{\overline{V}} + \Delta)) = \mathbb{C}$ is spanned by $f^*(dz_1 \wedge \cdots \wedge dz_r)|_W$. Thus, we obtain

$$f^*(dz_2 \wedge \cdots \wedge dz_r \wedge dz_j)|_W = \alpha_{1j} f^*(dz_1 \wedge \cdots \wedge dz_r)|_W,$$

where $\alpha_{1j} \in \mathbb{C}$ for every j. On the other hand,

$$f^*(dz_2 \wedge \cdots \wedge dz_r \wedge dz_j)|_W = \pm f^* \left(A_{j1}(dz_1 \wedge \cdots \wedge dz_r)|_W \right)$$

over a neighborhood of p. Hence we obtain $\pm f^* A_{j1} = \alpha_{1j}$ for every j. Note that f is an isomorphism over a neighborhood of p. From this, $A_{j1} = 0$ because $A_{j1}(0) = 0$ for every j. By the same arguments, we get $A_{ji} = 0$ for $1 \leq i \leq r$ and every j. Thus, we obtain that

$$dz_{r+1}|_W = \cdots = dz_n|_W = 0$$

around p. This means that

$$\pi^{-1}(W) \subset \{z_{r+1} = \cdots = z_n = 0\}$$

near q. Note that $\{z_{r+1} = \cdots = z_n = 0\}$ is of dimension r and is irreducible. Thus

$$\pi^{-1}(W) = \{z_{r+1} = \cdots = z_n = 0\}.$$

Therefore, W is a quasi-abelian subvariety of G. On the other hand, if W is a translation of a quasi-abelian subvariety of G, then $\overline{\kappa}(W) = 0$ by Theorem 4.3. □

The following theorem is almost obvious by the description in 4.1.

Theorem 4.5. *Let G be a quasi-abelian variety. Then there are at most countably many quasi-abelian subvarieties of G.*

Proof. Let H be a quasi-abelian subvariety of G. Then we obtain

$$\iota\colon H = \mathbb{C}^{\dim H}/H_1(H, \mathbb{Z}) \hookrightarrow \mathbb{C}^{\dim G}/H_1(G, \mathbb{Z}),$$

where ι is the natural inclusion. Anyway, ι is determined by the subgroup $\mathrm{Im}\iota_*$ of $H_1(G, \mathbb{Z})$, where $\iota_*\colon H_1(H, \mathbb{Z}) \to H_1(G, \mathbb{Z})$. Therefore, there are at most countably many quasi-abelian subvarieties of G. □

5. Characterizations of abelian varieties and complex tori

In this section, we quickly recall an important property of the Albanese map of varieties of the Kodaira dimension zero for the reader's convenience. It is well known that Kawamata established the following theorem in [Ka2], which is his doctoral thesis.

Theorem 5.1 (see [Ka2, Theorem 1]). *Let X be a smooth projective variety with $\kappa(X) = 0$. Then the Albanese map*

$$\alpha\colon X \to A$$

is surjective and has connected fibers.

As an obvious corollary of Theorem 5.1, we obtain a birational characterization of abelian varieties.

Corollary 5.2. *Let X be a smooth projective variety. Then X is birationally equivalent to an abelian variety if and only if the Kodaira dimension $\kappa(X) = 0$ and the irregularity $q(X) = \dim X$.*

In this paper, we will use Theorem 5.1 and Corollary 5.2 for the proof of Theorem 1.2. For compact Kähler manifolds, we have:

Theorem 5.3 (see [Ka2, Theorem 24]). *Let X be a compact Kähler manifold with $\kappa(X) = 0$. Then the Albanese map*

$$\alpha\colon X \to A$$

is surjective and has connected fibers.

Therefore, we have:

Corollary 5.4. *Let X be a compact Kähler manifold. Then X is bimeromorphic to a complex torus if and only if the Kodaira dimension $\kappa(X) = 0$ and the irregularity $q(X) = \dim X$.*

Kawamata's original arguments in [Ka2] heavily depend on the theory of variations of Hodge structure (see Section 7 below). In [EL, Section 2], Ein and Lazarsfeld give a new proof of the above results. Their arguments are based on the generic vanishing theorem due to Green–Lazarsfeld. Anyway, the results in this section can be proved without using [Ka2] now. Note that Theorem 1.2 is a generalization of Theorem 5.1. We will give a detailed proof of Theorem 1.2 in Section 10 (see Theorem 10.1) following [Ka2]. The author does not know any proofs of Theorem 1.2 which are independent of Theorem 6.1 below and only depend on the generic vanishing theorem due to Green–Lazarsfeld.

6. On subadditivity of the logarithmic Kodaira dimensions

In this section, we explain some known results on the subadditivity of the logarithmic Kodaira dimensions.

Theorem 6.1. *Let $f\colon X \to Y$ be a dominant morphism between smooth varieties with irreducible general fibers. Assume that the logarithmic Kodaira dimension $\overline{\kappa}(Y) = \dim Y$. Then we have*

$$\begin{aligned}\overline{\kappa}(X) &= \overline{\kappa}(F) + \overline{\kappa}(Y)\\ &= \overline{\kappa}(F) + \dim Y\end{aligned}$$

where F is a sufficiently general fiber of $f\colon X \to Y$.

Remark 6.2. Theorem 6.1 is a generalization of [Ka2, Theorem 30]. In [Ka2], Kawamata claimed Theorem 6.1 under the extra assumption that $\overline{\kappa}(X) \geq 0$. Theorem 6.1 was first obtained by Maehara (see [Ma, Corollary 2]). Note that the arguments in [Ka2] and [Ma] heavily depend on [Ka2, Theorem 32]. Since the author has been unable to follow [Ka2, Theorem 32], he gave a proof of Theorem 6.1 which is independent of [Ka2, Theorem 32]. For the details, see [F5, Theorem 1.9] (see also Section 7).

Theorem 6.3 is the main theorem of [Ka1] (see [Ka1, Theorem 1]). For the proof, we recommend the reader to see [F8, Chapter 5].

Theorem 6.3. *Let $f\colon X \to Y$ be a dominant morphism between smooth varieties whose general fibers are irreducible curves. Then we have*

$$\overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(Y)$$

where F is a general fiber of $f\colon X \to Y$.

Theorems 6.1 and 6.3 will play a crucial role in Sections 9, 10, and 11. In general, we have:

Conjecture 6.4. *Let $f\colon X \to Y$ be a dominant morphism between smooth varieties whose general fibers are irreducible. Then we have*

$$\overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(Y)$$

where F is a sufficiently general fiber of $f\colon X \to Y$.

By [F6] and [F7], we see that Conjecture 6.4 follows from the minimal model program and the abundance conjecture. For the details, see [F6], [F7], and [F9].

7. Remarks on semipositivity theorems

In this section, we make some comments on the semipositivity theorems in [Ka2] for the reader's convenience. We recommend the reader to skip this section if he is only interested in Theorem 1.2. The arguments in Section 8 are sufficient for the proof of Theorem 1.2 and are more elementary. Let us recall Kawamata's famous result in [Ka2]. It is one of the main ingredients of Kawamata's proof of Theorem 5.1.

Theorem 7.1 ([Ka2, Theorem 5=Main Lemma])**.** *Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers which satisfies the following conditions:*

(i) *There is a Zariski open dense subset Y_0 of Y such that $\Sigma = Y - Y_0$ is a simple normal crossing divisor on Y.*
(ii) *Put $X_0 = f^{-1}(Y_0)$ and $f_0 = f|_{X_0}$. Then f_0 is smooth.*
(iii) *The local monodromies of $R^n f_{0*}\mathbb{C}_{X_0}$ around Σ are unipotent, where $n = \dim X - \dim Y$.*

Then $f_*\mathcal{O}_X(K_{X/Y})$ is a locally free sheaf and semipositive, where $K_{X/Y} = K_X - f^*K_Y$.

Remark 7.2. In [Ka2, §4. Semi-positivity (1)], Kawamata proved that $f_*\mathcal{O}_X(K_{X/Y})$ coincides with the canonical extension of the bottom Hodge filtration $\mathcal{F}$. This part was generalized by Nakayama and Kollár independently (see [Nak, Theorem 1] and [Ko, Theorem 2.6]). They proved that $R^i f_*\mathcal{O}_X(K_{X/Y})$ is locally free and can be characterized as the (upper) canonical extension of the bottom Hodge filtration of a suitable variation of Hodge structure.

Remark 7.3. In [Ka2, §4. Semi-positivity (2)], Kawamata proved that the canonical extension of the bottom Hodge filtration $\mathcal{F}$ is semipositive. This part is not so easy to follow. Kawamata's proof seems to be insufficient. Note that Kawamata could and did use only [D1], [Gri], and [Sc] for the Hodge theory when [Ka2] was written around 1980. Fortunately, [FF1, Theorem 1.3] and [FFS, Theorem 3] completely generalize [Ka2, §4. Semi-positivity (2)] for admissible variations of mixed Hodge structure and clarify Kawamata's proof simultaneously (see also [FF2, Theorem 1.1 and Corollary 1.2]). For Morihiko Saito's comments on Kawamata's arguments in [Ka2, §4. Semi-positivity (2)], see [FFS, 4.6. Remarks].

Remark 7.4. An approach to the semipositivity of $R^i f_*\mathcal{O}_X(K_{X/Y})$ which dose not use [Ka2, §4. Semi-positivity (2)] can be found in [F2, Section 4].

Anyway, [Ka2, Theorem 5] is now clearly understood. Let us go to a mixed generalization of Theorem 7.1, which was used in the proof of Theorem 6.1 in [F5]. In [F1], we obtain:

Theorem 7.5 (see [F1, Theorems 3.1, 3.4, and 3.9])**.** *Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties and let D be a simple normal crossing divisor on X such that every stratum of D is dominant onto Y. Let Σ be a simple normal crossing divisor on Y. If f is smooth and D is relatively normal crossing over $Y_0 = Y \setminus \Sigma$ and the local monodromies of $R^{n+i}f_{0*}\mathbb{C}_{X_0\setminus D_0}$ around Σ are unipotent, where $X_0 = f^{-1}(Y_0)$, $D_0 = D|_{X_0}$, $f_0 = f|_{X_0}$, and $n = \dim X - \dim Y$, then $R^if_*\mathcal{O}_X(K_{X/Y} + D)$ is locally free and semipositive.*

Theorem 7.5 is obviously a generalization of Theorem 7.1.

Remark 7.6. In [F1], we characterize $R^if_*\mathcal{O}_X(K_{X/Y} + D)$ as the canonical extension of the bottom Hodge filtration of a suitable variation of mixed Hodge structure. The proof of the semipositivity of $R^if_*\mathcal{O}_X(K_{X/Y} + D)$ in [F1] used [Ka2, §4. Semi-positivity (2)]. Now we can use [FF1, Theorem 1.3] or [FFS, Theorem 3] for the semipositivity of $R^if_*\mathcal{O}_X(K_{X/Y} + D)$ in place of [Ka2, §4. Semi-positivity (2)] (see also [FF2, Theorem 1.1 and Corollary 1.2]).

Remark 7.7. As we pointed out in [F5, Remark 6.5], Kawamata seems to misuse Schmid's nilpotent orbit theorem in [Ka3] and [Ka4]. Therefore, we do not use the papers [Ka3] and [Ka4]. Moreover, the main theorem of [Ka3] (see [Ka3, Theorem 1.1]) is weaker than [FF1, Theorem 1.1].

Remark 7.8. The main theorem in [Ka3] (see [Ka3, Theorem 1.1]) does not cover Theorem 7.5 nor [Ka2, Theorem 32]. We also note that [Ka5] does not cover Theorem 7.5 nor [Ka2, Theorem 32]. In [Ka5], Kawamata treats *well prepared fiber spaces*. For the details, see [Ka5]. The author did not find any proofs of [Ka2, Theorem 32] in the literature except the original one in [Ka2].

8. Weak positivity theorems revisited

In this section, we explain how to avoid using the theory of variations of mixed Hodge structure for the proof of Theorem 6.1. Let us recall the definition of weakly positive sheaves. Note that the theory of weakly positive sheaves is due to Viehweg. Roughly speaking, Viehweg treated only the *pure* case. For the details of the *mixed* case, we recommend the reader to see [F5]. For a slightly different approach, see [F9].

Definition 8.1 (Weak positivity)**.** Let W be a smooth projective variety and let $\mathcal{F}$ be a torsion-free coherent sheaf on W. We call $\mathcal{F}$ *weakly positive*, if for every ample line bundle $\mathcal{H}$ on W and every positive integer α there exists some positive integer β such that $\widehat{S}^{\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$ is generically generated by global sections. This means that the natural map

$$H^0(W, \widehat{S}^{\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta})\otimes\mathcal{O}_W\to\widehat{S}^{\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$$

is generically surjective.

Remark 8.2. In Definition 8.1, let $\widehat{W}$ be the largest Zariski open subset of W such that $\mathcal{F}|_{\widehat{W}}$ is locally free. Then we put

$$\widehat{S}^k(\mathcal{F})=i_*S^k(i^*\mathcal{F})$$

where $i\colon\widehat{W}\to W$ is the natural open immersion and S^k denotes the k-th symmetric product. Note that $\operatorname{codim}_W(W\setminus\widehat{W})\geq 2$ since $\mathcal{F}$ is torsion-free.

The following theorem, which is due to Viehweg, Campana, and others, is useful and is very important.

Theorem 8.3 (Twisted weak positivity, see [F5, Theorem 1.1])**.** *Let X be a normal projective variety and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X,Δ) is log canonical. Let $f\colon X\to Y$ be a surjective morphism onto a smooth projective variety Y with connected fibers. Assume that $k(K_X+\Delta)$ is Cartier. Then, for every positive integer m,*

$$f_*\mathcal{O}_X(mk(K_{X/Y}+\Delta))$$

is weakly positive.

Once we establish Theorem 8.3, we can prove Theorem 6.1 without any difficulties. For the details, see [F5, Section 10]. Theorem 8.3 is sufficient for [F5, Sections 9 and 10]. A key ingredient of Theorem 8.3 is the following result.

Theorem 8.4 (see [F5, Corollary 7.11])**.** *Let $f\colon V\to W$ be a surjective morphism between smooth projective varieties. Let D be a simple normal crossing divisor on V. Then*

$$f_*\mathcal{O}_V(K_{V/W}+D)$$

is weakly positive.

By Theorem 8.4, the arguments in [F5, Section 8] work without any modifications and produce Theorem 8.3. We recommend the reader to see [F5, Section 8]. In [F5, Section 7], we give a proof of Theorem 8.4

based on the theory of variations of mixed Hodge structure (cf. Theorem 7.5). Here, we give a more elementary proof based on the following easy observation.

Lemma 8.5. *Let $f\colon X \to Y$ be a surjective morphism from a smooth projective variety X to a projective variety Y and let D be a simple normal crossing divisor on X. Let $\mathcal{A}$ be an ample line bundle on Y such that $|\mathcal{A}|$ is free and let $\mathcal{B}$ be a line bundle on Y such that $\mathcal{A}^{\otimes a} \otimes \mathcal{B}$ is nef for some positive integer a. Then*

$$R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes m}$$

is generated by global sections for every i and every positive integer $m \geq \dim Y + 1 + a$.

Proof of Lemma 8.5. By [F1, Theorem 2.6] (see also [F3, Theorem 6.3 (ii)]), we obtain that

$$H^p(Y, R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes a} \otimes \mathcal{A}^{m-a-p}) = 0$$

for $p > 0$. By Castelnuovo–Mumford regularity, we see that $R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes m}$ is generated by global sections for every i and $m \geq \dim Y + 1 + a$. □

Let us start the proof of Theorem 8.4.

Proof of Theorem 8.4. In Step 1, we reduce the problem to a simpler case. In Step 2, we use Viehweg's clever trick and obtain the desired weak positivity.

Step 1. By replacing D with its horizontal part, we may assume that every irreducible component of D is dominant onto W (see [F5, Lemma 7.7]). If there is a log canonical center C of (V, D) such that $f(C) \subsetneq W$, then we take the blow-up $h\colon V' \to V$ along C. We put

$$K_{V'} + D' = h^*(K_V + D).$$

Then D' is a simple normal crossing divisor on V' and

$$f_* \mathcal{O}_V(K_{V/W} + D) \simeq (f \circ h)_* \mathcal{O}_{V'}(K_{V'/W} + D').$$

Therefore, we can replace (V, D) with (V', D'). Then we replace D with its horizontal part (see [F5, Lemma 7.7]). By repeating this process finitely many times, we may assume that every stratum of D is dominant onto W. Now we take a closed subset Σ of W such that f is smooth over $W \setminus \Sigma$ and that D is relatively normal crossing over $W \setminus \Sigma$. Let $g\colon W' \to W$ be a birational morphism from a smooth projective variety W' such that $\Sigma' = g^{-1}(\Sigma)$ is a simple normal crossing divisor. By taking some suitable blow-ups of V in $f^{-1}(\Sigma)$ and replacing D with its strict transform, we may further assume the following conditions:

(i) $f' = g^{-1} \circ f \colon V \to W'$ is a morphism,

(ii) f' is smooth over $W' \setminus \Sigma'$ and D is relatively normal crossing over $W' \setminus \Sigma'$, and

(iii) every irreducible component of D is dominant onto W and $\operatorname{Supp}(f'^*\Sigma' + D)$ is a simple normal crossing divisor on V.

$$\begin{array}{ccc} V & & \\ {\scriptstyle f'}\downarrow & \searrow^{f} & \\ W' & \xrightarrow[g]{} & W \end{array}$$

Here we used Szabó's resolution lemma. We assume that $f'_*\mathcal{O}_V(K_{V/W'}+ D)$ is weakly positive. Note that

$$f'_*\mathcal{O}_V(K_{V/W} + D) \simeq f'_*\mathcal{O}_V(K_{V/W'} + D) \otimes \mathcal{O}_{W'}(E)$$

where E is a g-exceptional effective divisor such that $K_{W'} = g^*K_W + E$. Thus $f'_*\mathcal{O}_V(K_{V/W} + D)$ is weakly positive. We note that

$$g_*f'_*\mathcal{O}_V(K_{V/W} + D) \simeq f_*\mathcal{O}_V(K_{V/W} + D).$$

We can take an effective g-exceptional divisor F on W' such that $-F$ is g-ample. Let H be an ample Cartier divisor on W. Then there exists a positive integer k such that $kg^*H - F$ is ample. Let α be a positive integer. Since $f'_*\mathcal{O}_V(K_{V/W} + D)$ is weakly positive,

$$\widehat{S}^{k\alpha\beta}(f'_*\mathcal{O}_V(K_{V/W} + D)) \otimes \mathcal{O}_{W'}(\beta(kg^*H - F))$$

is generically generated by global sections for some positive integer β. By taking g_*,

$$\widehat{S}^{\alpha k\beta}(f_*\mathcal{O}_V(K_{V/W} + D)) \otimes \mathcal{O}_W(k\beta H)$$

is generically generated by global sections. This means that $f_*\mathcal{O}_V(K_{V/W}+ D)$ is weakly positive. Therefore, all we have to do is to prove that $f'_*\mathcal{O}_V(K_{V/W'} + D)$ is weakly positive.

Step 2. By replacing W with W', we may assume that $W' = W$. Note that $f_*\omega_{V/W}$ and $f_*(\omega_{V/W} \otimes \mathcal{O}_V(D))$ are locally free on $W_0 = W \setminus \Sigma$. Let s be an arbitrary positive integer. We take the s-fold fiber product

$$V^s = V \times_W V \times_W \cdots \times_W V.$$

We put $f^s \colon V^s \to W$. Let $p_i \colon V^s \to V$ be the i-th projection for $1 \leq i \leq s$. Let $W^\dagger$ be a Zariski open set of W such that f is flat over $W^\dagger$ and that $\operatorname{codim}_W(W \setminus W^\dagger) \geq 2$. We may assume that $W_0 \subset W^\dagger \subset W$. We put $V^\dagger = f^{-1}(W^\dagger)$. We may further assume that $f_*\omega_{V^\dagger/W^\dagger}$ and $f_*(\omega_{V^\dagger/W^\dagger} \otimes \mathcal{O}_{V^\dagger}(D))$ are locally free. By the flat base change theorem (see, for example, [Mo, Section 4] and [F8, Section 3.1]), we obtain an isomorphism

$$f^s_*\omega_{V^{\dagger s}/W^\dagger} \simeq \bigotimes_{i=1}^{s} f_*\omega_{V^\dagger/W^\dagger},$$

where $V^{\dagger s}$ is the s-fold fiber product

$$V^\dagger \times_{W^\dagger} \cdots \times_{W^\dagger} V^\dagger.$$

We put $D^s = \sum_{i=1}^{s} p_i^* D$. Then, by the same argument, we have an isomorphism

$$f^s_*(\omega_{V^{\dagger s}/W^\dagger} \otimes \mathcal{O}_{V^{\dagger s}}(D^s)) \simeq \bigotimes_{i=1}^{s} f_*(\omega_{V^\dagger/W^\dagger} \otimes \mathcal{O}_{V^\dagger}(D)). \tag{8.1}$$

Let $\pi\colon V^{(s)} \to V^s$ be a resolution such that π is an isomorphism over $(f^s)^{-1}(W_0)$ with the following properties:

(i) $V^{(s)}$ is a smooth projective variety,
(ii) $f^{(s)} = f^s \circ \pi\colon V^{(s)} \to W$ is smooth over W_0,
(iii) $D^{(s)}$ is a simple normal crossing divisor on $V^{(s)}$,
(iv) $\mathrm{Supp}(D^{(s)} + (f^{(s)})^*\Sigma)$ is a simple normal crossing divisor on $V^{(s)}$,
(v) every irreducible component of $D^{(s)}$ is dominant onto W, and
(vi) $D^{(s)}$ coincides with D^s over W_0.

Note that $V^{\dagger s}$ is Gorenstein. We have

$$\pi_*\mathcal{O}_{V^{\dagger (s)}}(K_{V^{\dagger (s)}}) \subset \omega_{V^{\dagger s}},$$

where $V^{\dagger (s)} = \pi^{-1}(V^{\dagger s})$. Therefore, we obtain

$$\pi_*\mathcal{O}_{V^{\dagger (s)}}(K_{V^{\dagger (s)}} + D^{(s)} - \pi^* D^s) \subset \omega_{V^{\dagger s}} \tag{8.2}$$

since $D^{(s)} - \pi^* D^s \leq 0$. Thus we have a natural inclusion

$$f^{(s)}_*\mathcal{O}_{V^{(s)}}(K_{V^{(s)}/W} + D^{(s)}) \hookrightarrow \left(\bigotimes_{i=1}^{s} f_*\mathcal{O}_V(K_{V/W} + D)\right)^{**}$$

which is an isomorphism over W_0 by (8.1) and (8.2). Let $\mathcal{H}$ be an ample line bundle on W. Then

$$f^{(s)}_*\mathcal{O}_{V^{(s)}}(K_{V^{(s)}/W} + D^{(s)}) \otimes \mathcal{H}^{\otimes m}$$

is generated by global sections for every positive integer s and for every $m \geq b(\dim W + 1) + a$, where a is a positive integer such that

$\mathcal{O}_W(-K_W)\otimes\mathcal{H}^{\otimes a}$ is nef and b is a positive integer such that $|\mathcal{H}^{\otimes b}|$ is free by Lemma 8.5. Therefore, we obtain that

$$\left(\bigotimes_{i=1}^{s} f_*\mathcal{O}_V(K_{V/W}+D)\right)^{**}\otimes\mathcal{H}^{\otimes m}$$

is generated by global sections over W_0, where s and m are as above. This means that

$$\widehat{S}^{\alpha\beta}(f_*\mathcal{O}_V(K_{V/W}+D))\otimes\mathcal{H}^{\otimes\beta}$$

is generated by global sections over W_0 for every $\alpha\geq 1$ and $\beta\geq b(\dim W+1)+a$. Therefore, $f_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive.

We complete the proof of Theorem 8.4. □

Remark 8.6. Note that $f_*\mathcal{O}_V(K_{V/W}+D)$ is locally free in Step 2 in the proof of Theorem 8.4. This is because $f_*\mathcal{O}_V(K_{V/W}+D)$ is the upper canonical extension of the bottom Hodge filtration of a suitable variation of mixed Hodge structure (cf. Theorem 7.5).

Anyway, by this section, Theorem 6.1 is now released from the deep results of the theory of variations of mixed Hodge structure. This means that Theorem 1.2 is also independent of the theory of variations of mixed Hodge structure.

9. Finite covers of quasi-abelian varieties

In this section, we discuss finite covers of abelian and quasi-abelian varieties. Let us start with the following well-known theorem due to Kawamata–Viehweg.

Theorem 9.1 (see [KV, Main Theorem] and [Ka2, Theorem 4])**.** *Let $f\colon X\to A$ be a finite surjective morphism from a normal complete variety X to an abelian variety A. Assume that the Kodaira dimension $\kappa(X)$ of X is zero. Then f is an étale morphism.*

Proof. Let $\pi\colon\widetilde{X}\to X$ be a resolution of singularities from a smooth projective variety $\widetilde{X}$. Then $q(\widetilde{X})\geq\dim\widetilde{X}$ since $f\circ\pi\colon\widetilde{X}\to A$ is surjective. Therefore, $\widetilde{X}$ is birationally equivalent to an abelian variety by Theorem 5.1 and Corollary 5.2 since $\kappa(\widetilde{X})=0$. We consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \alpha\ } & A \\ {\scriptstyle\alpha_{\widetilde{X}}}\downarrow & \nearrow{\scriptstyle g} & \\ \mathcal{A}_{\widetilde{X}} & & \end{array}$$

where $\alpha_{\widetilde{X}} \colon \widetilde{X} \to \mathcal{A}_{\widetilde{X}}$ is the Albanese map of $\widetilde{X}$. Of course, $\alpha_{\widetilde{X}}$ is birational and g is a finite étale morphism between abelian varieties. Note that both X and $\mathcal{A}_{\widetilde{X}}$ are the normalization of A in $\mathbb{C}(\widetilde{X})$. Therefore, X is isomorphic to $\mathcal{A}_{\widetilde{X}}$ over A. This means that $f \colon X \to A$ is an étale morphism. □

Remark 9.2. Kawamata's original proof of Theorem 5.1, which is [Ka2, Theorem 1], in [Ka2] uses Theorem 9.1 (see [Ka2, Theorem 4] and [KV, Main Theorem]). However, Ein–Lazarsfeld's approach in [EL, Section 2] does not need Theorem 9.1 (see [Ka2, Theorem 4] and [KV, Main Theorem]) for the proof of Theorem 5.1 (see [Ka2, Theorem 1]). Therefore, there are no problems if we use Theorem 5.1 for the proof of Theorem 9.1.

We can generalize Theorem 9.1 as follows. Theorem 9.3 is nothing but [Ka2, Theorem 26].

Theorem 9.3 (Finite covers of quasi-abelian varieties)**.** *Let $f \colon X \to A$ be a finite surjective morphism from a normal variety X to a quasi-abelian variety A. Assume that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then f is an étale morphism.*

Proof. Let

$$0 \longrightarrow \mathcal{G}_A \longrightarrow A \longrightarrow \mathcal{A}_A \longrightarrow 0$$

be the Chevalley decomposition. We will prove that f is étale by induction on $d = \dim \mathcal{G}_A$. If $d = 0$, then it is Theorem 9.1. So, we assume that $d > 0$. We take a subgroup

$$G_1 = \mathbb{G}_m \times \{1\} \times \cdots \times \{1\} \subset \mathbb{G}_m^d = \mathcal{G}_A.$$

We consider

$$0 \longrightarrow G_1 \longrightarrow A \xrightarrow{\pi_1} A_1 \longrightarrow 0.$$

Note that A is a principal G_1-bundle over A_1 as an algebraic variety in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]). As in the proof of Lemma 3.10, we can construct a $\mathbb{P}^1$-bundle $\overline{\pi_1} \colon \overline{A} \to A_1$ which is a partial compactification of $\pi_1 \colon A \to A_1$. Let $\overline{X}$ be the normalization of $\overline{A}$ in $\mathbb{C}(X)$ and $\overline{f} \colon \overline{X} \to \overline{A}$ is the natural map. Let $\overline{X} \to X_1 \to A_1$ be the Stein factorization of $\overline{\pi_1} \circ \overline{f} \colon \overline{X} \to A_1$. Then we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{p_1} & X_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ A & \xrightarrow[\pi_1]{} & A_1, \end{array}$$

where $f_1\colon X_1 \to A_1$ is a finite morphism from a normal variety X_1. Since f_1 is finite and A_1 is a quasi-abelian variety, we have $\overline{\kappa}(X_1) \geq 0$. On the other hand, by Theorem 6.3, we have

$$0 = \overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(X_1),$$

where F is a general fiber of p_1. Note that $\overline{\kappa}(F) \geq 0$ since $\overline{\kappa}(X) = 0$. Therefore, we obtain $\overline{\kappa}(X_1) = \overline{\kappa}(F) = 0$. By induction on d, f_1 is étale. By replacing A_1 (resp. A) with X_1 (resp. $A \times_{A_1} X_1$), we may assume that f_1 is the identity.

$$\begin{array}{ccc} X & \xrightarrow{p_1} & A_1 \\ {\scriptstyle f}\downarrow & & \| \\ A & \xrightarrow[\pi_1]{} & A_1, \end{array}$$

Let x be a general point of A_1. Then

$$f|_{X_x}\colon X_x \simeq \mathbb{G}_m \to A_x \simeq \mathbb{G}_m$$

is étale. We put $e = \deg f$. By construction, there are prime divisors H_1 and H_2 on $\overline{A}$ such that $H_1, H_2 \subset \overline{A} - A$, $H_1 \sim_{\overline{\pi_1}} H_2$, $H_1 \neq H_2$, and H_i is a section of $\overline{\pi_1}$ for $i = 1, 2$. We can take a nonempty Zariski open set U of A_1 such that

(i) $\overline{p_1}\colon \overline{X} \to A_1$ is smooth over U.
(ii) every fiber of $\overline{p_1}$ is $\mathbb{P}^1$ over U.
(iii) there are prime divisors D_1 and D_2 on $\overline{X}$ such that $D_1, D_2 \subset \overline{X} - X$, $D_1 \sim D_2$ over U, $D_1 \neq D_2$, and D_i is a section of $\overline{p_1}\colon \overline{X} \to A_1$ over U for $i = 1, 2$.
(iv) $\overline{f}^* H_i = eD_i$ over U for $i = 1, 2$.

Therefore, we see that $f\colon X \to A$ is

$$\mathbb{G}_m \times U \to \mathbb{G}_m \times U$$

given by

$$(a, b) \mapsto (a^e, b)$$

over U. On the other hand, we can construct a quasi-abelian variety A' such that

$$\begin{array}{ccc} A' & \longrightarrow & A_1 \\ {\scriptstyle h}\downarrow & & \| \\ A & \xrightarrow[\pi_1]{} & A_1, \end{array}$$

where h is étale with $\deg h = e$ (see the description of quasi-abelian varieties in 4.1) and that $h\colon A' \to A$ is

$$\mathbb{G}_m \times U \to \mathbb{G}_m \times U$$

given by

$$(a, b) \mapsto (a^e, b)$$

over U, that is, h coincides with f over U. Note that X is normal and both f and h are finite. Thus X is isomorphic to A' over A. Hence, we obtain that $f\colon X \to A$ is étale. $\square$

We will use Theorem 9.3 in the proof of Theorem 1.2 (see the proof of Theorem 10.1 in Section 10).

10. Quasi-Albanese maps for varieties with $\overline{\kappa} = 0$

In this section, we give a detailed proof of Kawamata's theorem on quasi-Albanese maps for varieties with $\overline{\kappa} = 0$.

Theorem 10.1 (see [Ka2, Theorem 28])**.** *Let X be a smooth variety such that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then the quasi-Albanese map $\alpha\colon X \to A$ is dominant and has irreducible general fibers.*

As an easy consequence of Theorem 10.1, we have:

Corollary 10.2 (see [Ka2, Corollary 29] and [I3, Theorem I])**.** *Let X be a smooth variety such that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then we have $\overline{q}(X) \leq \dim X$, where $\overline{q}(X)$ is the logarithmic irregularity of X. Moreover, the equality holds if and only if the quasi-Albanese map $\alpha\colon X \to A$ is birational.*

For the proof of Theorem 10.1, we start with a useful lemma.

Lemma 10.3. *Let A be a quasi-abelian variety and let B be a quasi-abelian subvariety of A. Let $B^\dagger$ be a variety and let $B^\dagger \to B$ be a finite étale cover. Then we can construct a finite étale cover $A^\dagger \to A$ such that $B^\dagger$ is a quasi-abelian subvariety of $A^\dagger$ satisfying*

$$\begin{array}{ccc} B^\dagger & \hookrightarrow & A^\dagger \\ \downarrow & & \downarrow \\ B & \hookrightarrow & A. \end{array}$$

Proof of Lemma 10.3. By Theorem 4.2, $B^\dagger$ is a quasi-abelian variety. We consider the Chevalley decompositions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_B & \longrightarrow & B & \longrightarrow & \mathcal{A}_B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}_A & \longrightarrow & A & \longrightarrow & \mathcal{A}_A & \longrightarrow & 0 \end{array}$$

and

$$0 \longrightarrow \mathcal{G}_{B^\dagger} \longrightarrow B^\dagger \longrightarrow \mathcal{A}_{B^\dagger} \longrightarrow 0.$$

By Poincaré reducibility (see, for example, [Mu]), we have an étale morphism

$$a\colon \mathcal{A}_{B^\dagger} \times \mathcal{A}' \to \mathcal{A}_A$$

for some abelian variety $\mathcal{A}'$. By taking the base change of $A \to \mathcal{A}_A$ by a, we obtain an étale cover $A_1 \to A$ and

$$\begin{array}{ccc} B^\dagger & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ B & \hookrightarrow & A. \end{array}$$

We note the Chevalley decompositions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_{B^\dagger} & \longrightarrow & B^\dagger & \longrightarrow & \mathcal{A}_{B^\dagger} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}_{A_1} & \longrightarrow & A_1 & \longrightarrow & \mathcal{A}_{B^\dagger} \times \mathcal{A}' & \longrightarrow & 0. \end{array}$$

By replacing the lattice corresponding to $\mathcal{G}_{A_1}$ with a suitable sublattice, we can construct a finite étale morphism

$$A^\dagger \to A_1$$

over $\mathcal{A}_{B^\dagger} \times \mathcal{A}'$ such that

$$\begin{array}{ccc} B^\dagger & \hookrightarrow & A^\dagger \\ \downarrow & & \downarrow \\ B & \hookrightarrow & A \end{array}$$

(see the description of quasi-abelian varieties in 4.1). We obtain a desired finite étale cover $A^\dagger \to A$. □

Before we prove Theorem 10.1, we have to prove the following important lemma.

Lemma 10.4 ([Ka2, Theorem 27])**.** *Let X be a normal algebraic variety, let A be a quasi-abelian variety, and let $f\colon X \to A$ be a finite morphism. Then $\overline{\kappa}(X) \geq 0$ and there are a quasi-abelian subvariety B of A, finite étale covers $\widetilde{X}$ and $\widetilde{B}$ of X and B respectively, and a normal algebraic variety $\widetilde{Y}$ such that:*

(i) *$\widetilde{Y}$ is finite over A/B.*
(ii) *$\widetilde{X}$ is a principal $\widetilde{B}$-bundle over $\widetilde{Y}$ as a complex manifold.*
(iii) *$\overline{\kappa}(\widetilde{Y}) = \dim \widetilde{Y} = \overline{\kappa}(X)$.*

In [Ka2], Kawamata claims this statement without proof. Hence we give a detailed proof for the sake of completeness. When we treat Iitaka fibrations in the proof of Lemma 10.4, we have to take care of the difference between general fibers and sufficiently general fibers.

Remark 10.5 (Sufficiently general fibers and points)**.** Let $f\colon V \to W$ be a morphism between varieties. Then a *sufficiently general fiber* (resp. *general fiber*) F of $f\colon V \to W$ means that $F = f^{-1}(w)$, where w is any closed point contained in a countable intersection of nonempty Zariski open sets (resp. a nonempty Zariski open set) of W. A sufficiently general fiber is sometimes called a *very general fiber* in the literature. A *sufficiently general point* (resp. *general point*) w of W is any closed point contained in a countable intersection of nonempty Zariski open sets (resp. a nonempty Zariski open set) of W.

Let us prove Lemma 10.4.

Proof of Lemma 10.4. We divide the proof into several steps.

Step 1. Let

$$\begin{array}{ccc} Z & \xrightarrow{\Phi} & Y \\ {\scriptstyle g}\big\downarrow & \nearrow & \\ X & & \end{array}$$

be the logarithmic Iitaka fibration of X, that is, we take a smooth complete variety $\overline{X}$ such that $D = \overline{X} - X$ is a simple normal crossing divisor, $\overline{X} \dashrightarrow Y$ is a dominant rational map to a normal projective variety Y associated to $|m(K_{\overline{X}}+D)|$ for a sufficiently large and divisible positive integer m, and $\overline{g}\colon \overline{Z} \to \overline{X}$ is an elimination of indeterminacy of $\overline{X} \dashrightarrow Y$ such that $Z := \overline{g}^{-1}(X)$, $\overline{Z} - Z$ is a simple normal crossing divisor on $\overline{Z}$, and $g := \overline{g}|_Z$. Let y be a sufficiently general point of Y. Then we have $\overline{\kappa}(Z_y) = 0$ by construction, where $Z_y := \Phi^{-1}(y)$. Since $f\circ g\colon Z_y \to f\circ g(Z_y)$ is proper and generically finite and $f\circ g(Z_y) \subset A$, $B_y := f \circ g(Z_y)$ is a translation of a quasi-abelian subvariety of A by Theorem 4.4. By Theorem 9.3, Z'_y is a quasi-abelian variety where $Z_y \to Z'_y \to B_y$ is the Stein factorization. In particular, the proper birational morphism $Z_y \to Z'_y$ is the quasi-Albanese map for sufficiently general $y \in Y$. Let y be a sufficiently general point of Y. Note that

$$H_1(Z_y, \mathbb{Z}) \to H_1(A, \mathbb{Z})$$

does not depend on y by discreteness. Therefore, the image of Z_y by $f\circ g$ in A does not depend on y up to translation. Therefore, we obtain a quasi-abelian subvariety B of A such that B_y is a translation of B for sufficiently general $y \in Y$. Without loss of generality, we may assume that

$$Z \xrightarrow{f\circ g} A \longrightarrow A/B$$

extends to

$$\overline{Z} \xrightarrow{\overline{f\circ g}} \overline{A} \longrightarrow \overline{A/B},$$

where $\overline{A}$ and $\overline{A/B}$ are smooth compactifications of A and A/B, respectively. By applying the rigidity lemma (see, for example, [BS, Lemma 4.2.13. (Rigidity Lemma)]) to $\overline{Z} \to Y$ and $\overline{Z} \to \overline{A/B}$, we see that $B_y = f \circ g(Z_y) \subset A$ is a translation of B for general $y \in Y$. For general $y \in Y$, the image of

$$H_1(Z_y, \mathbb{Z}) \to H_1(B_y, \mathbb{Z}) \simeq H_1(B, \mathbb{Z})$$

does not depend on y by discreteness. Thus, we have an étale cover $\widetilde{B}$ of B such that

$$f \circ g \colon Z_y \xrightarrow{\alpha_{Z_y}} \mathcal{A}_{Z_y} = \widetilde{B} \longrightarrow B_y,$$

where $\alpha_{Z_y} \colon Z_y \to \mathcal{A}_{Z_y}$ is the quasi-Albanese map of Z_y, for general $y \in Y$ and that $\alpha_{Z_y} \colon Z_y \to \mathcal{A}_{Z_y}$ is a proper birational morphism for general $y \in Y$. In particular, we have $\overline{\kappa}(Z_y) = 0$ for general $y \in Y$.

Step 2. In this step, we will construct $\widetilde{X}$, $\widetilde{Y}$, and $\widetilde{B}$ satisfying (i) and (ii).

By Lemma 10.3, we take a finite étale cover $\widetilde{A} \to A$ such that

$$\begin{array}{ccc} \widetilde{B} & \hookrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ B & \hookrightarrow & A. \end{array}$$

By taking base changes, we obtain $\widetilde{X}$, $\widetilde{Z}$, and the following commutative diagram:

$$\begin{array}{ccccccc} \widetilde{B} & \hookrightarrow & \widetilde{A} & \xleftarrow{\widetilde{f}} & \widetilde{X} & \xleftarrow{\widetilde{g}} & \widetilde{Z} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ B & \hookrightarrow & A & \xleftarrow[f]{} & X & \xleftarrow[g]{} & Z. \end{array}$$

We first assume that $\widetilde{X}$ is irreducible. We can construct a logarithmic Iitaka fibration $\widetilde{\Phi} \colon \widetilde{Z} \to Y'$ as follows. Without loss of generality, we may assume that there are a smooth projective variety $\overline{Z}$ such that $\overline{Z} - Z$ is a simple normal crossing divisor on $\overline{Z}$ and a morphism

$a\colon \overline{Z} \to Y$ such that $\Phi = a|_Z\colon Z \to Y$ in Step 1. Let $Z^\dagger$ be the normalization of $\overline{Z}$ in $\mathbb{C}(\widetilde{Z})$ and let $b\colon Z^\dagger \to \overline{Z}$ be the natural map. Let H be a very ample Cartier divisor on Y. We consider $\Phi_{|mb^*a^*H|}\colon Z^\dagger \to Y'$ for a sufficiently large and divisible positive integer m. We put $\widetilde{\Phi} := \Phi_{|mb^*a^*H|}|_{\widetilde{Z}}\colon \widetilde{Z} \to Y'$. As we saw in Step 1, there exists a quasi-abelian subvariety B' of $\widetilde{A}$ such that $\widetilde{f} \circ \widetilde{g}(\widetilde{Z}_{y'})$ is a translation of B' for general $y' \in Y'$. On the other hand, for general $y \in Y$, we can take $V \subset \widetilde{Z}$ such that $p\colon V \to Z_y$ is an isomorphism since there exists $\alpha_{Z_y}\colon Z_y \to \mathcal{A}_{Z_y} = \widetilde{B}$. This implies that $B' = \widetilde{B}$ and $\widetilde{Z}_{y'} \to \widetilde{f} \circ \widetilde{g}(\widetilde{Z}_{y'})$ is proper birational for general $y' \in Y'$. Therefore, there is a rational map $Y' \dashrightarrow \widetilde{A}/\widetilde{B}$ such that

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\ \widetilde{\Phi}\ } & Y' \\ {\scriptstyle \widetilde{f}\circ\widetilde{g}}\downarrow & & \downarrow \\ \widetilde{A} & \longrightarrow & \widetilde{A}/\widetilde{B} \end{array}$$

(see, for example, [Ka2, Lemma 14]). Let $\widetilde{Y}$ be the normalization of $\widetilde{A}/\widetilde{B}$ in $\mathbb{C}(Y')$. We put $X' = \widetilde{A} \times_{\widetilde{A}/\widetilde{B}} \widetilde{Y}$. Then X' is normal and is birationally equivalent to $\widetilde{X}$. We note that $\widetilde{X}$ and X' are both finite over $\widetilde{A}$. Thus X' is isomorphic to $\widetilde{X}$ over $\widetilde{A}$. We also note that $\widetilde{Y}$ is finite over A/B since $\widetilde{A}/\widetilde{B}$ is finite over A/B. By construction, $\widetilde{X}$ is a principal $\widetilde{B}$-bundle over $\widetilde{Y}$. When $\widetilde{X}$ is reducible, we replace $\widetilde{X}$ with a suitable irreducible component of $\widetilde{X}$. Then the above argument works. Anyway, we can construct $\widetilde{X}$, $\widetilde{Y}$, and $\widetilde{B}$ satisfying (i) and (ii).

Step 3. All we have to show is $\overline{\kappa}(\widetilde{Y}) = \dim \widetilde{Y} = \overline{\kappa}(X)$. Since $\widetilde{Y}$ is finite over $\widetilde{A}/\widetilde{B}$, we have $\overline{\kappa}(\widetilde{Y}) \geq 0$. We assume that $\overline{\kappa}(Y) < \dim \widetilde{Y}$. By applying the results obtained in Steps 1 and 2 to $\widetilde{Y} \to A/B$, we obtain an étale cover $\widetilde{Y}'$ with the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{X}' = \widetilde{X} \times_{\widetilde{Y}} \widetilde{Y}' & \longrightarrow & \widetilde{Y}' & \longrightarrow & W \\ \downarrow & & \downarrow & & \\ \widetilde{X} & \longrightarrow & \widetilde{Y} & & \downarrow \\ \downarrow & & \downarrow & & \\ A & \longrightarrow & A/B & \longrightarrow & A/C, \end{array}$$

where C is a quasi-abelian subvariety of A such that $B \subset C$. Note that W is finite over A/C and that $\dim W = \overline{\kappa}(Y)$. We can easily see that every fiber of $\widetilde{X}' \to W$ is a quasi-abelian variety and

$$\overline{\kappa}(\widetilde{X}') = \overline{\kappa}(\widetilde{X}) = \overline{\kappa}(X).$$

By the easy addition formula, we obtain

$$\overline{\kappa}(\widetilde{X}') \leq \dim W < \dim Y = \overline{\kappa}(X).$$

This is a contradiction. Therefore, we have $\dim \widetilde{Y} = \overline{\kappa}(\widetilde{Y})$.

We have desired $\widetilde{X}$, $\widetilde{B}$, and $\widetilde{Y}$. We finish the proof of Lemma 10.4. □

Let us start the proof of Theorem 10.1.

Proof of Theorem 10.1. By using the Stein factorization, we obtain

$$\alpha\colon X \xrightarrow{q} Z \xrightarrow{p} A$$

where q is dominant, q has irreducible general fibers, p is finite, and Z is normal. It is sufficient to prove that p is an isomorphism. We assume that $\overline{\kappa}(Z) > 0$. Then, by Lemma 10.4, we obtain an étale cover $\widetilde{Z} \to Z$ such that $\widetilde{Z} \to W$ is a principal G-bundle for some quasi-abelian variety G with $\overline{\kappa}(W) = \dim W = \overline{\kappa}(Z) > 0$. We consider $r\colon \widetilde{X} = X \times_Z \widetilde{Z} \to \widetilde{Z} \to W$. Since $\overline{\kappa}(\widetilde{X}) = \overline{\kappa}(X) = 0$, $\overline{\kappa}(F) \geq 0$ for a sufficiently general fiber F of r. By Theorem 6.1, we obtain

$$0 = \overline{\kappa}(X) = \overline{\kappa}(\widetilde{X}) \geq \overline{\kappa}(W) + \overline{\kappa}(F) \geq \overline{\kappa}(Z) > 0.$$

This is a contradiction. Therefore, we obtain $\overline{\kappa}(Z) = 0$. By Theorem 4.4, $\overline{\kappa}(p(Z)) = 0$ and $p(Z)$ is a quasi-abelian variety. By Theorem 9.3, we obtain that $p\colon Z \to p(Z)$ is étale. In particular, Z is a quasi-abelian variety (see Theorem 4.2). This means that p is an isomorphism since $\alpha\colon X \to A$ is a quasi-Albanese map of X. Thus, we obtain that $\alpha\colon X \to A$ is dominant and has irreducible general fibers. □

We close this section with the proof of Corollary 10.2.

Proof of Corollary 10.2. Let $\alpha\colon X \to A$ be a quasi-Albanese map. By Theorem 10.1, α is dominant. Note that $\dim A = \overline{q}(X)$. Therefore, we have $\overline{q}(X) \leq \dim X$. By Theorem 10.1, the general fibers of α are irreducible. Thus, α is birational if and only if $\dim X = \dim A = \overline{q}(X)$. □

11. Proof of Theorem 1.3 and Corollaries 1.4 and 1.5

In this final section, we give a proof of Theorem 1.3 following [FMPT]. Then we prove Corollaries 1.4 and 1.5 as easy applications.

Proof of Theorem 1.3. Let $\alpha\colon X \to A$ be the quasi-Albanese map (see Theorem 1.1). By Corollary 10.2, we see that α is birational.

Step 1. Let

$$0 \longrightarrow \mathbb{G}_m^d \longrightarrow A \overset{\pi}{\longrightarrow} B \longrightarrow 0$$

be the Chevalley decomposition as in Definition 2.8. Then A is a principal $\mathbb{G}_m^d$-bundle over an abelian variety B in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]) and there is a natural completion $\overline{\pi}\colon \overline{A} \to B$ of $\pi\colon A \to B$ where $\overline{A}$ is a $\mathbb{P}^d$-bundle over B (see the proof of Lemma 3.8). We set $\Delta_{\overline{A}} := \overline{A} - A$. Then $\Delta_{\overline{A}}$ is a simple normal crossings divisor on $\overline{A}$. In particular, $(\overline{A}, \Delta_{\overline{A}})$ is a log canonical pair. Let $\overline{\alpha}\colon \overline{X} \to \overline{A}$ be a compactification of $\alpha\colon X \to A$, that is, $\overline{X}$ is a smooth complete algebraic variety containing X, $\Delta_{\overline{X}} := \overline{X} - X$ is a simple normal crossing divisor on $\overline{X}$, and $\overline{\alpha}$ is a morphism extending α.

Claim. *Let D be an irreducible component of $\Delta_{\overline{X}}$ such that $\overline{\alpha}(D)$ is a divisor. Then $\overline{\pi}\colon \overline{\alpha}(D) \to B$ is dominant.*

Proof of Claim. We set $D_1 := \overline{\alpha}(D)$. If $\overline{\pi}\colon D_1 \to B$ is not dominant, then we can write $D_1 = \overline{\pi}^* D_2$ for some prime divisor D_2 on B. Since every log canonical center of $(\overline{A}, \Delta_{\overline{A}})$ dominates B, D_1 does not contain any log canonical centers. Hence, in particular, it is not a component of $\Delta_{\overline{A}}$. Thus, $(\overline{A}, \Delta_{\overline{A}} + \varepsilon D_1)$ is log canonical for every $0 < \varepsilon \ll 1$ since the support of D_1 does not contain any log canonical centers of $(\overline{A}, \Delta_{\overline{A}})$. Therefore, we have

$$K_{\overline{X}} + \Delta_{\overline{X}} - \overline{\alpha}^*(K_{\overline{A}} + \Delta_{\overline{A}}) \geq \varepsilon \overline{\alpha}^* D_1$$

for some $0 < \varepsilon \ll 1$. By construction, we have $K_{\overline{A}} + \Delta_{\overline{A}} \sim 0$ (see the proof of Lemma 3.8). Hence we obtain

$$0 = \overline{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + \Delta_{\overline{X}}) \geq \kappa(\overline{X}, \overline{\alpha}^* D_1) = \kappa(\overline{A}, D_1) = \kappa(B, D_2) > 0,$$

where the last inequality follows from the fact that D_2 is a nonzero effective divisor on the abelian variety B. This contradiction proves the claim. □

Step 2. We assume that there exists an irreducible component D of $\Delta_{\overline{X}}$ such that $\overline{\alpha}(D)$ is a divisor with $\overline{\alpha}(D) \not\subset \overline{A} - A$. We set $D' := \overline{\alpha}(D) \cap A$.

By Claim in Step 1, D' dominates B. Therefore, we can find a subgroup $\mathbb{G}_m$ of A such that $\varphi|_{D'}\colon D' \to A_1$ is dominant, where

$$0 \longrightarrow \mathbb{G}_m \longrightarrow A \xrightarrow{\varphi} A_1 \longrightarrow 0.$$

Note that A is a principal $\mathbb{G}_m$-bundle over A_1 in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]). We take a compactification

$$f^\dagger\colon X^\dagger \xrightarrow{\alpha^\dagger} A^\dagger \xrightarrow{\varphi^\dagger} A_1^\dagger$$

of

$$f\colon X \xrightarrow{\alpha} A \xrightarrow{\varphi} A_1,$$

where $X^\dagger$, $A^\dagger$, and $A_1^\dagger$ are smooth complete algebraic varieties such that $X^\dagger - X$, $A^\dagger - A$, and $A_1^\dagger - A_1$ are simple normal crossing divisors. Let F be a general fiber of f. Then $\overline{\kappa}(F) = 1$ since $\varphi|_{D'}\colon D' \to A_1$ is dominant. Note that A_1 is a quasi-abelian variety. Hence we have $\overline{\kappa}(A_1) = 0$. By Theorem 6.3, we obtain

$$0 = \overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(A_1) = 1.$$

This is a contradiction. Thus, every irreducible component of $\Delta_{\overline{X}}$ which is not contracted by $\overline{\alpha}$ is mapped to $\Delta_{\overline{A}}$.

Step 3. Let Δ' be the union of $\Delta_{\overline{X}}$ and the exceptional locus $\mathrm{Exc}(\overline{\alpha})$ of $\overline{\alpha}\colon \overline{X} \to \overline{A}$. Note that $\mathrm{Exc}(\overline{\alpha})$ is of pure codimension one by [Sh, Chapter 2, Section 4.4, Theorem 2.16]. Therefore, Δ' is a divisor on $\overline{X}$ with $\Delta' \geq \Delta_{\overline{X}}$. We put $Z := \overline{\alpha}(\Delta') \cap A$. By Step 2, we see that Z is a closed subset of A with $\mathrm{codim}_A Z \geq 2$. By definition, $\overline{\alpha}\colon \overline{X} \to \overline{A}$ is an isomorphism over $A \setminus Z$. Over A, we can easily check that $\overline{\alpha}^{-1}(Z) = \mathrm{Exc}(\overline{\alpha}) = \Delta'$ holds by [Sh, Chapter 2, Section 4.4, Theorem 2.16]. Hence, $\alpha\colon X \setminus \alpha^{-1}(Z) \to A \setminus Z$ is an isomorphism and $\alpha^{-1}(Z)$ is of pure codimension one. This is what we wanted.

We finish the proof of Theorem 1.3. □

Corollary 1.4 easily follows from Theorem 1.3.

Proof of Corollary 1.4. If $X \simeq \mathbb{G}_m^n$, then $\overline{\kappa}(X) = 0$ and $\overline{q}(X) = n$ hold by Theorem 4.3. Hence it is sufficient to prove that $X \simeq \mathbb{G}_m^n$ holds under the assumption that $\overline{\kappa}(X) = 0$ and $\overline{q}(X) = n$. Let $\alpha\colon X \to A$ be the quasi-Albanese map. By Theorem 1.3, we can take a closed subset Z of A such that $\mathrm{codim}_A Z \geq 2$, $\alpha^{-1}(Z)$ is of pure codimension one, and $\alpha\colon X \setminus \alpha^{-1}(Z) \to X \setminus A$ is an isomorphism. We note that $X \setminus \alpha^{-1}(Z)$ is affine since X is a smooth affine variety and $\alpha^{-1}(Z)$ is of pure codimension one. This implies that $A \setminus Z$ is also affine. Thus we obtain $Z = \emptyset$ since $\mathrm{codim}_A Z \geq 2$ (see, for example, [I2, Lemma 6]). Hence $\alpha\colon X \to A$ is an isomorphism. In particular, $X \simeq A \simeq \mathbb{G}_m^n$. Note that A is quasi-abelian and affine. This is what we wanted. $\square$

Although Corollary 1.5 is almost obvious, we prove it for the sake of completeness.

Proof of Corollary 1.5. If $\mathrm{codim}_A(A \setminus X) \geq 2$, then we have $\overline{\kappa}(X) = 0$ by Lemma 2.5. We assume that $\overline{\kappa}(X) = 0$ holds. Then $X \hookrightarrow A$ is nothing but the quasi-Albanese map and $\mathrm{codim}_A(A \setminus X) \geq 2$ by Theorem 1.3. We finish the proof of Corollary 1.5. $\square$

References

[AK] Y. Abe, K. Kopfermann, *Toroidal groups. Line bundles, cohomology and quasi-abelian varieties*, Lecture Notes in Mathematics, **1759**. Springer-Verlag, Berlin, 2001.

[B] A. Beauville, *Complex algebraic surfaces*, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. Second edition, London Mathematical Society Student Texts, **34**, Cambridge University Press, Cambridge, 1996.

[BS] M. C. Beltrametti, A. J. Sommese, *The adjunction theory of complex projective varieties*, De Gruyter Expositions in Mathematics, **16**. Walter de Gruyter & Co., Berlin, 1995.

[BCM] A. Białynicki-Birula, J. B. Carrell, W. M. McGovern, *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, Encyclopaedia of Mathematical Sciences, **131**. Invariant Theory and Algebraic Transformation Groups, II. Springer-Verlag, Berlin, 2002.

[BSU] M. Brion, P. Samuel, V. Uma, *Lectures on the structure of algebraic groups and geometric applications*, CMI Lecture Series in Mathematics, **1**. Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013.

[C] B. Conrad, A modern proof of Chevalley's theorem on algebraic groups, J. Ramanujan Math. Soc. **17** (2002), no. 1, 1–18.

[D1] P. Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. No. **40** (1971), 5–57.

[D2] P. Deligne, Théorie de Hodge. III, Inst. Hautes Études Sci. Publ. Math. No. **44** (1974), 5–77.

[EL] L. Ein, R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, J. Amer. Math. Soc. **10** (1997), no. 1, 243–258.

[F1] O. Fujino, Higher direct images of log canonical divisors, J. Differential Geom. **66** (2004), no. 3, 453–479.

[F2] O. Fujino, Remarks on algebraic fiber spaces, J. Math. Kyoto Univ. **45** (2005), no. 4, 683–699.

[F3] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[F4] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[F5] O. Fujino, Notes on the weak positivity theorems, *Algebraic varieties and automorphism groups*, 73–118, Adv. Stud. Pure Math., **75**, Math. Soc. Japan, Tokyo, 2017.

[F6] O. Fujino, On subadditivity of the logarithmic Kodaira dimension, J. Math. Soc. Japan **69** (2017), no. 4, 1565–1581.

[F7] O. Fujino, Corrigendum to "On subadditivity of the logarithmic Kodaira dimension", J. Math. Soc. Japan **72** (2020), no. 4, 1181–1187.

[F8] O. Fujino, *Iitaka conjecture—an introduction*, SpringerBriefs in Mathematics. Springer, Singapore, 2020.

[F9] O. Fujino, On mixed-ω-sheaves, to appear in Asian J. Math.

[FF1] O. Fujino, T. Fujisawa, Variations of mixed Hodge structure and semipositivity theorems, Publ. Res. Inst. Math. Sci. **50** (2014), no. 4, 589–661.

[FF2] O. Fujino, T. Fujisawa, On semipositivity theorems, Math. Res. Lett. **26** (2019), no. 5, 1359–1382.

[FFS] O. Fujino, T. Fujisawa, M. Saito, Some remarks on the semipositivity theorems, Publ. Res. Inst. Math. Sci. **50** (2014), no. 1, 85–112.

[FMPT] O. Fujino, M. Mendes Lopes, R. Pardini, S. Tirabassi, Erratum to "A footnote to a theorem of Kawamata", preprint (2024).

[GR] H. Grauert, R. Remmert, *Coherent analytic sheaves*, Grundlehren der Mathematischen Wissenschaften, **265**, Springer-Verlag, Berlin, 1984.

[Gri] P. Griffiths, Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping, Inst. Hautes Études Sci. Publ. Math. No. **38** (1970), 125–180.

[GH] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[Har] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. **156**. Springer-Verlag, Berlin-New York 1970.

[Hat] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II, Ann. of Math. (2) **79** (1964), 109–203; **79** (1964), 205–326.

[I1] S. Iitaka, Logarithmic forms of algebraic varieties, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 3, 525–544.

[I2] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, *Complex analysis and algebraic geometry*, 175–189. Iwanami Shoten, Tokyo, 1977.

[I3] S. Iitaka, A numerical criterion of quasi-abelian surfaces, Nagoya Math. J. **73** (1979), 99–115.

[I4] S. Iitaka, *Birational geometry for open varieties*, Séminaire de Mathématiques Supérieures, **76**, Presses de l'Université de Montréal, Montreal, Que., 1981.

[I5] S. Iitaka, *Algebraic geometry. An introduction to birational geometry of algebraic varieties*, Graduate Texts in Mathematics, **76**, North-Holland Mathematical Library, 24. Springer-Verlag, New York-Berlin, 1982.

[Ka1] Y. Kawamata, Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one, *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, 207–217, Kinokuniya Book Store, Tokyo, 1978.

[Ka2] Y. Kawamata, Characterization of abelian varieties, Compositio Math. **43** (1981), no. 2, 253–276.

[Ka3] Y. Kawamata, Semipositivity theorem for reducible algebraic fiber spaces, Pure Appl. Math. Q. **7** (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1427–1447.

[Ka4] Y. Kawamata, Hodge theory on generalized normal crossing varieties, Proc. Edinb. Math. Soc. (2) **57** (2014), no. 1, 175–189.

[Ka5] Y. Kawamata, Variation of mixed Hodge structures and the positivity for algebraic fiber spaces, *Algebraic geometry in east Asia—Taipei 2011*, 27–57, Adv. Stud. Pure Math., **65**, Math. Soc. Japan, Tokyo, 2015.

[KV] Y. Kawamata, E. Viehweg, On a characterization of an abelian variety in the classification theory of algebraic varieties, Compositio Math. **41** (1980), no. 3, 355–359.

[Ko] J. Kollár, Higher direct images of dualizing sheaves. II, Ann. of Math. (2) **124** (1986), no. 1, 171–202

[Ma] K. Maehara, The weak 1-positivity of direct image sheaves, J. Reine Angew. Math. **364** (1986), 112–129.

[MPT] M. Mendes Lopes, R. Pardini, S. Tirabassi, A footnote to a theorem of Kawamata, Math. Nachr. **296** (2023), no. 10, 4739–4744.

[Mo] S. Mori, Classification of higher-dimensional varieties, *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, 269–331, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[Mu] D. Mumford, *Abelian varieties*, With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition, Tata Institute of Fundamental Research Studies in Mathematics, **5**, Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008.

[Nag] M. Nagata, Imbedding of an abstract variety in a complete variety, J. Math. Kyoto Univ. **2** (1962), 1–10.

[Nak] N. Nakayama, Hodge filtrations and the higher direct images of canonical sheaves, Invent. Math. **85** (1986), no. 1, 217–221.

[NW] J. Noguchi, J. Winkelmann, *Nevanlinna theory in several complex variables and Diophantine approximation*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **350**. Springer, Tokyo, 2014.

[Sc] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Invent. Math. **22** (1973), 211–319.

[SGA1] A. Grothendieck, M. Raynaud, *Revêtements étales et groupe fondamental, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1)*. Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud. Lecture Notes in Mathematics, Vol. **224**. Springer-Verlag, Berlin–New York, 1971.

[Sh] I. R. Shafarevich, *Basic algebraic geometry. 1. Varieties in projective space*, Third edition. Translated from the 2007 third Russian edition. Springer, Heidelberg, 2013.

[U] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, Vol. **439**, Springer-Verlag, Berlin-New York, 1975.

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