PERMISSIBLE DIVISORS
(PRELIMINARY VERSION)

OSAMU FUJINO

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1.1. **Divisors.** [1] In this subsection, we quickly recall basic definitions of divisors. We note that we have to deal with reducible and non-reduced algebraic schemes in this paper. For details, see [Mu, Lecture 9] and [Fu, Appendix B.4].

1.1. Let X be a noetherian scheme with structure sheaf $\mathcal{O}_X$ and let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$, that is, for every affine open set $U \subset X$, $\Gamma(U, \mathcal{K}_X)$ is the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. Let $\mathcal{K}^*_X$ denote the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$, and $\mathcal{O}^*_X$ the sheaf of invertible elements in $\mathcal{O}_X$. We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}^*_X \subset \mathcal{K}^*_X$.

1.2 (Cartier, $\mathbb{Q}$-Cartier, and $\mathbb{R}$-Cartier divisors)**.** A *Cartier divisor* D on X is a global section of $\mathcal{K}^*_X/\mathcal{O}^*_X$, that is, D is an element of $H^0(X, \mathcal{K}^*_X/\mathcal{O}^*_X)$. A *$\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor* (resp. *$\mathbb{R}$-Cartier $\mathbb{R}$-divisor*) is an element of $H^0(X, \mathcal{K}^*_X/\mathcal{O}^*_X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $H^0(X, \mathcal{K}^*_X/\mathcal{O}^*_X) \otimes_{\mathbb{Z}} \mathbb{R}$).

1.3 (Linearly, $\mathbb{Q}$-linearly, and $\mathbb{R}$-linearly equivalence)**.** [2] Let D_1 and D_2 be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X. Then D_1 is *linearly* (resp. *$\mathbb{Q}$-linearly*, or *$\mathbb{R}$-linearly*) *equivalent* to D_2, denoted by $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$, or $D_1 \sim_{\mathbb{R}} D_2$) if

$$D_1 = D_2 + \sum_i r_i (f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}^*_X)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by $\Gamma(X, \mathcal{K}^*_X) \to \Gamma(X, \mathcal{K}^*_X/\mathcal{O}^*_X)$.

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[1] I will add this subsection to my book.

[2] I have to make these definitions compatible with other definitions.

1.4. Let D be a Cartier divisor on X. The *support* of D, denoted by Supp$D$, is the subset of X consisting of points x such that a local equation for D is not in $\mathcal{O}^*_{X,x}$. The support of D is a closed subset of X.

1.5. Let X be an equi-dimensional reduced separated algebraic scheme. We note that X is not necessarily regular in codimension one. A (*Weil*) *divisor* D on X is a finite formal sum

$$\sum_{i=1}^{n} d_i D_i$$

where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is an integer for every i.

If $d_i \in \mathbb{Q}$ (resp. $d_i \in \mathbb{R}$) for every i, then D is called a $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*).

2. Preliminaries

[3] We explain basic notion according to ambro [?, Section 2].

01 **Definition 2.1** (Normal and simple normal crossing varieties)**.** A variety X has *normal crossing* singularities if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k)}$$

for some $0 \leq k \leq N$, where $N = \dim X$. Furthermore, if each irreducible component of X is smooth, X is called a *simple normal crossing* variety. If X is a normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. So, we can define[4] the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well defined up to linear equivalence.

011 **Definition 2.2** (Mayer–Vietoris simplicial resolution)**.** Let X be a simple normal crossing variety with the irreducible decomposition $X = \bigcup_{i \in I} X_i$. Let I_n be the set of strictly increasing sequences $(i_0, \cdots, i_n)$ in I and $X^n = \coprod_{I_n} X_{i_0} \cap \cdots \cap X_{i_n}$ the disjoint union of the intersections of X_i. Let $\varepsilon_n : X^n \to X$ be the disjoint union of the natural inclusions. Then $\{X^n, \varepsilon_n\}_n$ has a natural semi-simplicial scheme structure. The face operator is induced by $\lambda_{j,n}$, where $\lambda_{j,n} : X_{i_0} \cap \cdots \cap X_{i_n} \to X_{i_0} \cap \cdots \cap X_{i_{j-1}} \cap X_{i_{j+1}} \cap \cdots \cap X_{i_n}$ is the natural closed embedding

[3]This is a revised version of the first half of Section 2.2 of my book.

[4]Is it OK? Can I always define K_X?

for $j \leq n$ (cf. elzein2 [?, 3.5.5]). We denote it by $\varepsilon : X^\bullet \to X$ and call it the *Mayer–Vietoris simplicial resolution* of X. The complex

$$0 \to \varepsilon_{0*}\mathcal{O}_{X^0} \to \varepsilon_{1*}\mathcal{O}_{X^1} \to \cdots \to \varepsilon_{k*}\mathcal{O}_{X^k} \to \cdots ,$$

where the differential $d_k : \varepsilon_{k*}\mathcal{O}_{X^k} \to \varepsilon_{k+1*}\mathcal{O}_{X^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda^*_{j,k+1}$ for any $k \geq 0$, is denoted by $\mathcal{O}_{X^\bullet}$. It is easy to see that $\mathcal{O}_{X^\bullet}$ is quasi-isomorphic to $\mathcal{O}_X$. By tensoring $\mathcal{L}$, any line bundle on X, to $\mathcal{O}_{X^\bullet}$, we obtain a complex

$$0 \to \varepsilon_{0*}\mathcal{L}^0 \to \varepsilon_{1*}\mathcal{L}^1 \to \cdots \to \varepsilon_{k*}\mathcal{L}^k \to \cdots ,$$

where $\mathcal{L}^n = \varepsilon_n^*\mathcal{L}$. It is denoted by $\mathcal{L}^\bullet$. Of course, $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$. We note that $H^q(X^\bullet, \mathcal{L}^\bullet)$ is $\mathbb{H}^q(X, \mathcal{L}^\bullet)$ by definition and it is obviously isomorphic to $H^q(X, \mathcal{L})$ for any $q \geq 0$ because $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$.

0111 **Definition 2.3.** Let X be a simple normal crossing variety. A *stratum* of X is the image on X of some irreducible component of $X^\bullet$. Note that an irreducible component of X is a stratum of X.

02 **Definition 2.4** (Permissible and normal crossing divisors)**.** Let X be a simple normal crossing variety.

A Cartier divisor D on X is called *permissible* if D contains no strata of X in its support. In this case, D induces a Cartier divisor $D^\bullet$ on $X^\bullet$. This means that $D^n = \varepsilon_n^* D$ is a Cartier divisor on X^n for every n. A $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) divisor on X is *permissible* if it can be written as a $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisors.

We say that a permissible Cartier divisor D is a *normal crossing* divisor on X if, in the notation of Definition 01 2.1, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \cdots, i_l\} \subset \{k+1, \cdots, N\}$. It is equivalent to the condition that D^n is a normal crossing divisor on X^n for every n in the usual sense. Furthermore, let D be a normal crossing divisor on a simple normal crossing variety X. If D^n is a simple normal crossing divisor on X^n for every n, then D is called a *simple normal crossing divisor* on X.

2.5. Let X be a simple normal crossing variety. Let $\mathrm{PerDiv}(X)$ be the abelian group generated by permissible Cartier divisors on X and $\mathrm{Weil}(X)$ the abelian group generated by Weil divisors on X. Then we can define natural injective homomorphisms of abelian groups

$$\psi : \mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} K \to \mathrm{Weil}(X) \otimes_{\mathbb{Z}} K,$$

where $K = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. Let $\nu : \widetilde{X} \to X$ be the normalization. Then we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Div}(\widetilde{X}) \otimes_{\mathbb{Z}} K & \xrightarrow[\widetilde{\psi}]{\sim} & \mathrm{Weil}(\widetilde{X}) \otimes_{\mathbb{Z}} K \\ {\scriptstyle \nu^*} \uparrow & & \downarrow {\scriptstyle \nu_*} \\ \mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} K & \xrightarrow[\psi]{} & \mathrm{Weil}(X) \otimes_{\mathbb{Z}} K \end{array}$$

Note that $\mathrm{Div}(\widetilde{X})$ is the abelian group generated by Cartier divisors on $\widetilde{X}$ and that $\widetilde{\psi}$ is an isomorphism since $\widetilde{X}$ is smooth.

By ψ, every permissible Cartier (resp. $\mathbb{Q}$-Cartier or $\mathbb{R}$-Cartier) divisor can be considered as a Weil divisor (resp. $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor). Therefore, various operations, for example, $\llcorner D \lrcorner$, $D^{<1}$, and so on, make sense for a permissible $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X.

We note the following easy example.

Example 2.6. Let X be a simple normal crossing variety in $\mathbb{C}^3 = \mathrm{Spec}\mathbb{C}[x, y, z]$ defined by $xy = 0$. We put $D_1 = (x + z = 0) \cap X$ and $D_2 = (x - z = 0) \cap X$. Then $D = \frac{1}{2}D_1 + \frac{1}{2}D_2$ is a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. In this case, $\llcorner D \lrcorner = (x = z = 0)$ on X. Therefore, $\llcorner D \lrcorner$ is not a Cartier divisor on X.

The following lemma is easy but important. We will repeatedly use it in Sections ?? and ??.

7 **Lemma 2.7.** *Let X be a simple normal crossing variety and B a permissible $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\llcorner B \lrcorner = 0$. Let A be a Cartier divisor on X. Assume that $A \sim_{\mathbb{R}} B$. Then there exists a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\llcorner C \lrcorner = 0$, and* $\mathrm{Supp}C = \mathrm{Supp}B$.

Proof. We can write $B = A + \sum_i r_i(f_i)$, where $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{R}$ for every i. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$. Let $P \in X$ be a scheme theoretic point corresponding to some stratum of X. We consider the following affine map

$$K^k \to H^0(X_P, \mathcal{K}_{X_P}^*/\mathcal{O}_{X_P}^*) \otimes_{\mathbb{Z}} K$$

given by $(a_1, \cdots, a_k) \mapsto A + \sum_i a_i(f_i)$, where $X_P = \mathrm{Spec}\mathcal{O}_{X,P}$ and $K = \mathbb{Q}$ or $\mathbb{R}$. Then we can check that

$$\mathcal{P} = \{(a_1, \cdots, a_k) \in \mathbb{R}^k \,|\, A + \sum_i a_i(f_i) \text{ is permissible}\} \subset \mathbb{R}^k$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Therefore, we see that

$$\mathcal{S} = \{(a_1, \cdots, a_k) \in \mathcal{P} \,|\, \mathrm{Supp}(A + \sum_i a_i(f_i)) = \mathrm{Supp}B\} \subset \mathcal{P}$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Since $(r_1, \cdots, r_k) \in \mathcal{S}$, we know that $\mathcal{S} \neq \emptyset$. We take a point $(s_1, \cdots, s_k) \in \mathcal{S} \cap \mathbb{Q}^k$ which is very close to $(r_1, \cdots, r_k)$ and put $C = A + \sum_i s_i(f_i)$. By construction, C is a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $C \sim_{\mathbb{Q}} A$, $\llcorner C \lrcorner = 0$, and $\mathrm{Supp} C = \mathrm{Supp} B$. □

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujino@math.kyoto-u.ac.jp