On acceptable bundles

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Introduction

- This is a joint work with Taro Fujisawa and Takashi Ono.
- Our motivation is to understand acceptable bundles, which play a crucial role in the Simpson–Mochizuki theory.
- Our approach is more complex analytic. We aim to reformulate the theory within a standard framework, such as the one found in Demailly's textbook.

Acceptable bundles on Δ^*

We set $\Delta^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and let

$$\omega_P := \frac{\sqrt{-1}dz \wedge d\bar{z}}{|z|^2 \left(-\log |z|^2\right)^2}$$

denote the Poincaré metric on Δ^*.

Let (E, h) be a Hermitian holomorphic vector bundle on Δ^*. Let $D = D' + \overline{\partial}$ be the Chern connection of (E, h), and let $\Theta_h(E) := D^2$ be the curvature of (E, h).

Definition 1 (Acceptable bundles on Δ^*)

If there exists a constant $C > 0$ such that

$$|\Theta_h(E)|_{h,\omega_P} \leq C \quad \text{on} \quad \Delta^*,$$

then (E, h) is called an acceptable bundle on Δ^*.

Prolongation on Δ

Definition 2 (Prolongation by increasing orders on Δ)

Let (E, h) be an acceptable vector bundle on Δ^* and let a be any real number. For any open subset $U \subset \Delta := \{z \in \mathbb{C} \mid |z| < 1\}$, we define

$$ {}_aE(U) := \left\{ f \in E(U \setminus \{0\}) \,\middle|\, |f|_h = O\left(\frac{1}{|z|^{a+\varepsilon}}\right) \text{ for any } \varepsilon > 0 \right\}, $$

where $|f|_h$ denotes the norm of f with respect to the Hermitian metric h. Then we obtain a sheaf of $\mathcal{O}_\Delta$-modules, denoted by ${}_aE$.

Our goal is to understand the proof of the following theorem.

Theorem 3 (Simpson)

Let (E, h) be an acceptable vector bundle on Δ^*. Then ${}_aE$ is a holomorphic vector bundle on Δ for every $a \in \mathbb{R}$.

Acceptable bundles on $(\Delta^*)^l \times \Delta^{n-l}$

Definition 4 (Poincaré metric)

On $(\Delta^*)^l \times \Delta^{n-l} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| < 1 \text{ for all } i,\ z_j \neq 0 \text{ for } j \leq l\}$, the Poincaré metric is defined by

$$\omega_P := \sum_{j=1}^{l} \frac{\sqrt{-1}\, dz_j \wedge d\bar{z}_j}{|z_j|^2(-\log |z_j|^2)^2} + \sum_{k=l+1}^{n} \frac{\sqrt{-1}\, dz_k \wedge d\bar{z}_k}{(1 - |z_k|^2)^2}.$$

Definition 5 (Prolongation by increasing orders on Δ^n)

Let $\boldsymbol{a} = (a_1, \dots, a_l) \in \mathbb{R}^l$. Let (E, h) be a Hermitian holomorphic vector bundle on $(\Delta^*)^l \times \Delta^{n-l}$. We define an $\mathcal{O}_{\Delta^n}$-module ${}_{\boldsymbol{a}}E$ as follows. For any open set $U \subset \Delta^n$, set

$$\Gamma(U, {}_{\boldsymbol{a}}E) := \left\{ s \in \Gamma(U \cap ((\Delta^*)^l \times \Delta^{n-l}), E) \;\middle|\; |s|_h = O\left(\frac{1}{\prod_{j=1}^{l} |z_j|^{a_j + \varepsilon}} \right) \right.$$
$$\left. \text{for any } \varepsilon > 0 \right\}.$$

Definition 6 (Acceptable bundles on a partially punctured polydisk)

Let E be a holomorphic vector bundle on $(\Delta^*)^l \times \Delta^{n-l}$, equipped with a smooth Hermitian metric h.

We say that (E, h) is acceptable if its curvature $\sqrt{-1}\Theta_h(E)$, viewed as a smooth $\mathrm{Hom}(E, E)$-valued $(1, 1)$-form on $(\Delta^*)^l \times \Delta^{n-l}$, is bounded with respect to the Hermitian metric $(\cdot, \cdot)_{h,\omega_P}$, which is the natural Hermitian metric on $\mathrm{Hom}(E, E) \otimes \Omega^{1,1}$ induced by the metric h on E and the Poincaré metric ω_P.

In other words, there exists a constant $C > 0$ such that

$$\left| \sqrt{-1}\Theta_h(E) \right|_{h,\omega_P} \leq C \quad \text{on} \quad (\Delta^*)^l \times \Delta^{n-l}.$$

Prolongation on Δ^n

Our goal is to understand the proof of the following theorem.

Theorem 7 (Mochizuki)

Let (E, h) be an acceptable vector bundle on a partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$. Then ${}_{\boldsymbol{a}}E$ is a locally free sheaf on Δ^n for any $\boldsymbol{a} \in \mathbb{R}^l$. Moreover, the family $\left({}_{\boldsymbol{a}}E \mid \boldsymbol{a} \in \mathbb{R}^l\right)$ naturally forms a filtered bundle.

Due to time constraints, we omit the definition of filtered bundles in the sense of Mochizuki. For further details, please refer to the following preprints:

- O. Fujino, T. Fujisawa, T. Ono, On acceptable bundles I, preprint (2025). arXiv:2511.00760 [math.AG]
- O. Fujino, T. Fujisawa, T. Ono, On acceptable bundles II, preprint (2026). arXiv:2604.05233 [math.AG]

Part I

One-dimensional case

Elementary lemmas

The following elementary lemmas play a crucial role.

Lemma 8 (Harmonic functions on Δ^*)

Let f be a harmonic function on Δ^*. Then

$$f(z) = \mathrm{Re}\, g(z) + c \log |z|,$$

where g is a holomorphic function on Δ^* and c is a real number.

Lemma 9 (Removable singularities)

Let g be a holomorphic function on Δ^*. Assume that

$$\mathrm{Re}\, g(z) \leq C\,(-\log |z|)$$

holds on Δ^* for some $C > 0$. Then g is holomorphic on Δ.

Prolongation of line bundles on Δ^*

Theorem 10 (Prolongation of acceptable line bundles by increasing orders on Δ^*)

Let (L, h) be an acceptable line bundle on Δ^*. Then ${}_\alpha L$ is a holomorphic line bundle on Δ for any $\alpha \in \mathbb{R}$.

Proof. We divide the proof into 6 small steps.

Step 1 (Twisted metrics)

We put $\chi(N) := -N \log\left(-\log|z|^2\right)$. Then $\sqrt{-1}\partial\bar{\partial}\chi(N) = N\omega_P$. By assumption, $-C\omega_P \leq \sqrt{-1}\Theta_h(L) \leq C\omega_P$. Thus

$$\sqrt{-1}\Theta_{he^{-\chi(N)}}(L) = \sqrt{-1}\Theta_h(L) + N\omega_P \geq 0$$

$$\sqrt{-1}\Theta_{he^{-\chi(-N)}}(L) = \sqrt{-1}\Theta_h(L) - N\omega_P \leq 0$$

on Δ^* for $N \geq C$.

Step 2 (Trivialization)

We fix a trivialization $L = \mathcal{O}_{\Delta^*}$. Then we can write

$$h = |\cdot|^2 e^{-2\varphi}$$

for some smooth function φ on Δ^*. In this case, we have

$$\sqrt{-1}\Theta_h(L) = \sqrt{-1}\partial\bar{\partial}2\varphi.$$

Step 3 (Closed positive $(1,1)$ currents)

We have $\int_{\{z\in\Delta\,|\,|z|<r_0\}} \omega_P < \infty$ for $0 < r_0 < 1$. Thus, by taking the zero extensions,

$$\sqrt{-1}\Theta_{he^{-\chi(N)}}(L) \quad \text{and} \quad -\sqrt{-1}\Theta_{he^{-\chi(-N)}}(L)$$

are closed positive $(1,1)$ current on Δ.

Step 4

There exist subharmonic functions ψ_1 and ψ_2 on Δ such that

$$\sqrt{-1}\Theta_{he^{-\chi(N)}}(L) = \sqrt{-1}\partial\bar{\partial}2\psi_1$$

$$-\sqrt{-1}\Theta_{he^{-\chi(-N)}}(L) = \sqrt{-1}\partial\bar{\partial}2\psi_2$$

Then $2\varphi + \chi(N) - 2\psi_1$ and $-2\varphi + \chi(N) - 2\psi_2$ are harmonic on Δ^*. Then, by Lemma 8, we have

$$2\varphi + \chi(N) = 2\psi_1 + c_1 \log |z|^2 + 2\mathrm{Re} g_1(z) \tag{1}$$

$$-2\varphi + \chi(N) = 2\psi_2 + c_2 \log |z|^2 + 2\mathrm{Re} g_2(z) \tag{2}$$

where g_1 and g_2 are holomorphic on Δ^*, and c_1 and c_2 are real numbers.

Step 5 (Lelong numbers)

By changing the trivialization $L = \mathcal{O}_{\Delta^*}$ by multiplying $e^{g_1(z)}$, we may assume that $g_1 = 0$. By using (1), (2), and Lemma 9, we see that g_2 is holomorphic on Δ. By replacing ψ_2 with $\psi_2 - \mathrm{Re} g_2(z)$, we may further assume that $g_2 = 0$.

Then, by calculating Lelong numbers, we obtain

$$\liminf_{z\to 0} \frac{\varphi(z)}{\log|z|} = \nu_1 + c_1 =: \gamma$$

and

$$\liminf_{z\to 0} \frac{-\varphi(z)}{\log|z|} = \nu_2 + c_2 = -\gamma$$

where $\nu_i = \nu(\psi_i, 0)$.

Hence

$$\gamma = \lim_{z\to 0} \frac{\varphi(z)}{\log|z|}. \tag{3}$$

Step 6 (Final step)

By using (3), we can directly check that

$$ {}_{\alpha}L = \mathcal{O}_{\Delta} \cdot z^{-\lfloor \alpha - \gamma \rfloor}. $$

We finish the proof of Theorem 10.

Prolongation of vector bundles on Δ^*

Theorem 11 (Prolongation of acceptable vector bundles by increasing orders on Δ^*)

Let (E, h) be an acceptable vector bundle on Δ^*. Then ${}_{\alpha}E$ is a holomorphic vector bundle on Δ for any $\alpha \in \mathbb{R}$.

Idea of Proof. This part is mainly due to Cornalba–Griffiths.

For $N \gg 0$,

(A) $\sqrt{-1}\Theta_{he^{-\chi(N)}}(E)$ is (Nakano) semipositive. Then we can use Hörmander's L^2-estimates to solve $\overline{\partial}$-equations.

(B) $\sqrt{-1}\Theta_{he^{-\chi(-N)}}(E)$ is (Griffiths) seminegative. Then $\log |s|_h^2$ is subharmonic for any holomorphic section s of E.

We first apply the L^2 extension theorem thanks to (A). Then, applying the mean value inequality by (B), we can prove Theorem 11 based on the line bundle case (see Theorem 10).

This part is more or less standard, so we skip the details.

New invariant

Let (E, h) be an acceptable vector bundle over Δ^* with $\mathrm{rank} E = r$ and let $\boldsymbol{v} = \{v_1, \dots, v_r\}$ be a local frame of ${}_{\alpha}E$ around the origin. Let

$$H(h, \boldsymbol{v}) := \Big(h(v_i, v_j)\Big)_{i,j}$$

be the $r \times r$ Hermitian matrix-valued function on Δ^*.

We put

$$\gamma({}_{\alpha}E) := -\frac{1}{2} \liminf_{z \to 0} \frac{\log \det H(h, \boldsymbol{v})}{\log |z|}.$$

Then we have:

Theorem 12

$$\gamma({}_{\alpha}E) = -\frac{1}{2} \lim_{z \to 0} \frac{\log \det H(h, \boldsymbol{v})}{\log |z|} \quad \textit{and} \quad \det({}_{\alpha}E) = {}_{\gamma({}_{\alpha}E)} \det E.$$

The proof of Theorem 12 is similar to that of Theorem 10.

Parabolic weights

We set

$$\mathcal{P}ar^{\mathrm{red}}_{\alpha}(E,h) := \{\lambda \in (\alpha-1,\alpha] \,|\, {}_{\lambda}E/{}_{<\lambda}E \neq 0\},$$

where

$${}_{<\lambda}E := \bigcup_{\mu<\lambda} {}_{\mu}E \subset {}_{\lambda}E.$$

Then we obtain

$$\mathcal{P}ar^{\mathrm{red}}_{\alpha}(E,h) = \{\lambda_1,\dots,\lambda_k\},$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$. We set

$$l_i := \dim_{\mathbb{C}}\left({}_{\lambda_i}E/{}_{<\lambda_i}E\right).$$

Then we can check

$$\sum_{i=1}^{k} l_i = r = \mathrm{rank}E.$$

We define

$$\mathcal{P}ar_{\alpha}(E,h) := \{\underbrace{\lambda_1,\dots,\lambda_1}_{l_1 \text{ times}},\dots,\underbrace{\lambda_k,\dots,\lambda_k}_{l_k \text{ times}}\}.$$

Key equality

The following is one of the main theorems of Part I.

Theorem 13

$$\gamma({}_{\alpha}E) = \sum_{\lambda_i \in \mathcal{P}ar^{\mathrm{red}}({}_{\alpha}E)} \lambda_i \dim_{\mathbb{C}} \left({}_{\lambda_i}E / {}_{<\lambda_i}E\right) = \sum_{\lambda_i \in \mathcal{P}ar({}_{\alpha}E)} \lambda_i .$$

The first part of our joint paper is primarily devoted to the proof of Theorem 13.

A key point of the proof is that

$$\gamma({}_{\alpha}E) = -\frac{1}{2} \lim_{z \to 0} \frac{\log \det H(h, \boldsymbol{v})}{\log |z|} \in \mathbb{R}.$$

Note that $\gamma({}_{\alpha}E)$ is independent of the choice of the local frame $\boldsymbol{v}$. Crucially, it is established as a true limit $\lim_{z\to 0}$ rather than a limit inferior $\liminf_{z\to 0}$.

Part II

Higher-dimensional case

Ohsawa–Takegoshi L^2 extension theorem

We formulate the Ohsawa–Takegoshi L^2 extension theorem.

Let $0 < R < 1$. Define

$$X^*(R) := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} 0 < |z_i| < R \quad \text{for } 1 \leq i \leq l, \\ |z_i| < R \quad \text{for } l+1 \leq i \leq n \end{array} \right\}.$$

Then $X^*(R)$ is a bounded Stein open subset of $\mathbb{C}^n$. Let $(\mathcal{E}, h)$ be a Nakano semipositive vector bundle over $X^*(R)$. We set

$$z := z_1 \quad \text{and} \quad w_i := \begin{cases} z_i - z_1 & \text{for } 2 \leq i \leq l, \\ z_i & \text{for } l+1 \leq i \leq n. \end{cases}$$

Consider the submanifold

$$Y^*(R) := \left\{ (z_1, \ldots, z_n) \in X^*(R) \;\middle|\; w_2 = \cdots = w_n = 0 \right\}.$$

Set the weight functions

$$\psi := \frac{1}{l}\sum_{i=1}^{l}\log|z_i|^2,\ \phi := -\left(1-\frac{1}{l}\right)\sum_{i=1}^{l}\log|z_i|^2,\ \phi_{\boldsymbol{a}} := -\sum_{i=1}^{l}a_i\log|z_i|^2.$$

Let f be a holomorphic section of $\mathcal{E}|_{Y^*(R)}$ satisfying

$$\int_{Y^*(R)}|f|_h^2e^{-\psi-\phi_{\boldsymbol{a}}}\cdot\frac{\sqrt{-1}}{2}\,dz\wedge d\overline{z}<\infty.$$

Then there exists a holomorphic section F of $\mathcal{E}$ on $X^*(R)$ such that

$$F|_{Y^*(R)}=f,\qquad \int_{X^*(R)}|F|_h^2e^{-\psi-\phi_{\boldsymbol{a}}}\,d\lambda_n<\infty.$$

Consequently, we also have

$$\int_{X^*(R)}|F|_h^2e^{-\phi-\phi_{\boldsymbol{a}}}\frac{\omega_P^n}{n!}<\infty.$$

Note that $\phi\equiv 0$ when $l=1$.

Acceptable bundles on $\Delta^* \times \Delta^{n-1}$

Let $\pi\colon \Delta^* \times \Delta^{n-1} \to \Delta^{n-1}$ be the projection.

Lemma 14 (Key result)

Let (L, h) be an acceptable line bundle on $\Delta^* \times \Delta^{n-1}$. Then ${}_{\alpha}L$ is a line bundle on Δ^n for any $\alpha \in \mathbb{R}$. Moreover,

$$\gamma\left({}_{\alpha}(L|_{\pi^{-1}(Q)})\right)$$

is independent of $Q \in \Delta^{n-1}$.

Idea of Proof. Applying the Ohsawa–Takegoshi L^2 extension theorem to L and $L^\vee$ shows that ${}_{\alpha}L$ is a line bundle on Δ^n.
By Siu's theorem on the upper semicontinuity of Lelong numbers, we can prove that

$$\gamma\left({}_{\alpha}(L|_{\pi^{-1}(Q)})\right)$$

is independent of $Q \in \Delta^{n-1}$.

Theorem 15

Let (E, h) be an acceptable vector bundle on $\Delta^* \times \Delta^{n-1}$. Then ${}_{\alpha}E$ is a locally free sheaf on Δ^n for any $\alpha \in \mathbb{R}$. Moreover,

$$\mathcal{P}ar_{\alpha}\left(E|_{\pi^{-1}(Q)}, h|_{\pi^{-1}(Q)}\right)$$

is independent of $Q \in \Delta^{n-1}$.

Idea of Proof. We apply the Ohsawa–Takegoshi L^2 extension theorem to E and $E^\vee$. By considering $\det E$ and $\det E^\vee$, we see that $\gamma\left({}_{\alpha}(E|_{\pi^{-1}(Q)})\right)$ is independent of $Q \in \Delta^{n-1}$ for almost all $\alpha \in \mathbb{R}$ by Lemma 14. This implies that

$$\mathcal{P}ar_{\alpha}\left(E|_{\pi^{-1}(Q)}, h|_{\pi^{-1}(Q)}\right)$$

is independent of $Q \in \Delta^{n-1}$ for any $\alpha \in \mathbb{R}$.

Acceptable bundles on $(\Delta^*)^l \times \Delta^{n-1}$

We finally obtain the following theorem.

Theorem 16 (Prolongation of acceptable vector bundles by increasing orders on $(\Delta^*)^l \times \Delta^{n-l}$, Mochizuki)

Let (E, h) be an acceptable vector bundle on $(\Delta^*)^l \times \Delta^{n-l}$. Then ${}_{\boldsymbol{a}}E$ is a locally free sheaf on Δ^n for any $\boldsymbol{a} \in \mathbb{R}^l$.

The proof of Theorem 16 in our joint paper is essentially based on Mochizuki's original approach. Since we utilize the Ohsawa–Takegoshi L^2 extension theorem, our proof is slightly simpler; however, the core ideas are conceptually identical. Hence, we omit the details here.

Thank you very much!