On injectivity, vanishing and torsion-free theorems

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Abstract: We give a short and almost self-contained proof of generalizations of Kollár's vanishing and torsion-free theorems. Although they are contained in Ambro's much more general results on embedded normal crossing pairs, we give an alternate and direct reduction argument to the mixed Hodge theory. In this sense, this paper gives a more readable account of the application to the log minimal model program for log canonical pairs.

Key words: Vanishing theorem; Torsion-freeness; Injectivity theorem; Hodge Theory.

1. Introduction

The main purpose of this paper is to give a short and almost self-contained proof of the following theorem.

Theorem 1.1 (Torsion-free and vanishing theorems)**.** *Let Y be a smooth projective variety and B a boundary $\mathbb{Q}$-divisor such that* Supp$B$ *is simple normal crossing. Let $f : Y \to X$ be a projective morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{Q}} L - (K_Y + B)$ is f-semi-ample.*

(i) *Every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B), where a* stratum *of (Y, B) denotes Y itself or an lc center of (Y, B).*
(ii) *Assume that $H \sim_{\mathbb{Q}} f^*H'$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H' on X. Then $H^p(X, R^q f_* \mathcal{O}_Y(L)) = 0$ for any $p > 0$ and $q \geq 0$.*

Although this theorem is a very special case of [A, Theorem 3.2], it will play important roles in the log minimal model program for log canonical pairs. In [A], Ambro proved the above theorem for *embedded normal crossing pairs*. His proof is rather difficult involving a highly technical notion of normal crossing pairs. For a systematic and thorough treatment, we refer the reader to [F1, Chapter 2].

The author has found a straightforward proof of the following cone theorem for log canonical pairs, which does not use *quasi-log varieties*. The proof will be published in the forthcoming [F2].

Theorem 1.2 (Cone theorem)**.** *Let (X, B) be a projective log canonical pair. Then we have*

(i) *There are* (*countably many*) *rational curves $C_j \subset X$ such that*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$
(ii) *For any $\varepsilon > 0$ and ample $\mathbb{Q}$-divisor H,*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B+\varepsilon H)\geq 0} + \sum_{\textit{finite}} \mathbb{R}_{\geq 0}[C_j].$$
(iii) *Let $F \subset \overline{NE}(X)$ be a $(K_X + B)$-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ such that $(\varphi_F)_* \mathcal{O}_X \simeq \mathcal{O}_Z$, Z is projective, and an irreducible curve $C \subset X$ is mapped to a point by φ_F if and only if $[C] \in F$. The map φ_F is called the* contraction *of F.*
(iv) *Let F and φ_F be as in* (iii). *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Z on Z such that $L \simeq \varphi_F^* L_Z$.*

Theorem 1.2 is the starting point of the log minimal model program for log canonical pairs. Being free from resolution of singularities and perturbation of coefficients, the proof of Theorem 1.2 in [F2] will be even easier than the original proof of the cone theorem for Kawamata log terminal pairs. Both [A, Chapter 3] and [F1, Chapter 2] were intended for the experts and rather involved. Although it was the feature of [A] to prove Theorem 1.2 in the context of *quasi-log varieties*, the proof of Theorem 1.2 without *quasi-log varieties* was not available. Thus Theorem 1.1 had to be proved for embedded normal crossing pairs, which was the most difficult part in [A]. Both of [A, Chapter 3] and [F1, Chapter 2] adopted Esnault–Viehweg's framework explained in [EV]. Here, we give a short proof of Theorem 1.1 after Kollár's philosophy explained in, for example,

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[KM, §2.4]. It is the first time that we use Kollár's philosophy to treat Theorem 1.1 in the literature. Hopefully, the approach adopted here will clarify the nature of Theorem 1.1.

We summarize the contents of this paper. Section 2 is a short review of the Hodge theoretic aspect of the injectivity theorem. We would like to emphasize that the E_1-degeneration in [D] is sufficient for our purposes. We do not know whether the E_1-degeneration discussed in [EV, (3.2, c)] follows from the one in [D] if $A \neq 0$ in [EV, (3.2, c)] (cf. [EV, 3.18. Remarks. a)]). In Section 3, we give a short proof of Theorem 1.1. It is a standard argument once the fundamental injectivity theorem is given in Section 2. In Section 4, we will explain two applications of Theorem 1.1. The first one contains the extension theorem from log canonical centers. It is very strong, hard to reach by the Kawamata–Viehweg–Nadel vanishing theorem, and intended for use in the log minimal model program for log canonical pairs. Although [A] proved it in the context of quasi-log varieties, this paper gives a more accessible account. The final theorem is the Kodaira vanishing theorem for log canonical pairs, which was not explicitly stated in [A].

Notation. Let X be a normal variety and B an effective $\mathbb{Q}$-divisor such that K_X+B is $\mathbb{Q}$-Cartier. Then we can define the discrepancy $a(E,X,B) \in \mathbb{Q}$ for any prime divisor E *over* X. If $a(E,X,B) \geq -1$ for any E, then (X,B) is called *log canonical*. We sometimes abbreviate log canonical to *lc*. Assume that (X,B) is log canonical. If E is a prime divisor over X such that $a(E,X,B) = -1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X,B), where $c_X(E)$ is the closure of the image of E on X. A *stratum* of (X,B) denotes X itself or an lc center of (X,B).

Let r be a rational number. The integral part $\llcorner r \lrcorner$ is the largest integer $\leq r$ and the fractional part $\{r\}$ is defined by $r - \llcorner r \lrcorner$. We put $\ulcorner r \urcorner = -\llcorner -r \lrcorner$ and call it the round-up of r. For a $\mathbb{Q}$-divisor $D = \sum_{i=1}^{r} d_i D_i$, where D_i is a prime divisor for any i and $D_i \neq D_j$ for $i \neq j$, we call D a *boundary* $\mathbb{Q}$-divisor if $0 \leq d_i \leq 1$ for any i. We note that $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. We put $\llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i$, $\ulcorner D \urcorner = \sum \ulcorner d_i \urcorner D_i$, $\{D\} = \sum \{d_i\} D_i$, $D^{<1} = \sum_{d_i<1} d_i D_i$, and $D^{=1} = \sum_{d_i=1} D_i$.

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

2. Hodge theoretic aspect In this section, we will prove the following injectivity theorem, which is essentially the same as [EV, 3.2. Theorem. c), 5.1. b)]. We use the classical topology throughout this section.

Proposition 2.1 (Fundamental injectivity theorem)**.** *Let X be a projective smooth variety and $S+B$ a boundary $\mathbb{Q}$-divisor on X such that the support of $S+B$ is simple normal crossing and that $\llcorner S+B \lrcorner = S$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in* Supp$B$. *Assume that $L \sim_{\mathbb{Q}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Before we prove Proposition 2.1, let us recall some results on the Hodge theory.

2.2. Let V be a smooth projective variety and Σ a simple normal crossing divisor on V. Let $\iota : V \setminus \Sigma \to V$ be the natural open immersion. Then $\iota_! \mathbb{C}_{V\setminus\Sigma}$ is quasi-isomorphic to the complex $\Omega^{\bullet}_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)$. By this quasi-isomorphism, we can construct the following spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(V, \Omega^p_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)) \\ &\Rightarrow H_c^{p+q}(V \setminus \Sigma, \mathbb{C}). \end{aligned}$$

By the Serre duality, the right hand side

$$H^q(V, \Omega^p_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma))$$

is dual to $H^{n-q}(V, \Omega^{n-p}_V(\log \Sigma))$, where $n = \dim V$. By the Poincaré duality, $H_c^{p+q}(V \setminus \Sigma, \mathbb{C})$ is dual to $H^{2n-(p+q)}(V \setminus \Sigma, \mathbb{C})$. Therefore,

$$\begin{aligned} &\dim H_c^k(V \setminus \Sigma, \mathbb{C}) \\ &= \sum_{p+q=k} \dim H^q(V, \Omega^p_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)) \end{aligned}$$

by Deligne (cf. [D, Corollaire (3.2.13) (ii)]). Thus, the above spectral sequence degenerates at E_1. We will use this E_1-degeneration in the proof of Proposition 2.1. By the above E_1-degeneration, we obtain

$$H_c^k(V \setminus \Sigma, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(V, \Omega^p_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)).$$

In particular, the natural inclusion $\iota_! \mathbb{C}_{V\setminus\Sigma} \subset \mathcal{O}_V(-\Sigma)$ induces surjections

$$H_c^p(V \setminus \Sigma, \mathbb{C}) \simeq H^p(V, \iota_! \mathbb{C}_{V\setminus\Sigma}) \to H^p(V, \mathcal{O}_V(-\Sigma))$$

for any p.

Proof of Proposition 2.1. We put $\mathcal{L} = \mathcal{O}_X(L - K_X - S)$. Let ν be the smallest positive integer such that $\nu L \sim \nu(K_X + S + B)$. In particular, νB is an integral Weil divisor. We take the ν-fold cyclic cover $\pi' : Y' = \mathrm{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i} \to X$ associated to the section $\nu B \in |\mathcal{L}^\nu|$. More precisely, let $s \in H^0(X, \mathcal{L}^\nu)$ be a section whose zero divisor is νB. Then the dual of $s : \mathcal{O}_X \to \mathcal{L}^\nu$ defines a $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i}$. Let $Y \to Y'$ be the normalization and $\pi : Y \to X$ the composition morphism. For the details, see [EV, 3.5. Cyclic covers]. We can take a finite cover $\varphi : V \to Y$ such that V is smooth and that T is a simple normal crossing divisor on V, where $\psi = \pi \circ \varphi$ and $T = \psi^* S$, by Kawamata's covering trick (cf. [EV, 3.17. Lemma]). Let $\iota' : Y \setminus \pi^* S \to Y$ be the natural open immersion and U the smooth locus of Y. We denote the natural open immersion $U \to Y$ by j. We put $\widetilde{\Omega}^p_Y(\log(\pi^* S)) = j_* \Omega^p_U(\log(\pi^* S))$ for any p. Then it can be checked easily that

$$\iota'_! \mathbb{C}_{Y \setminus \pi^* S} \xrightarrow{qis} \widetilde{\Omega}^\bullet_Y(\log(\pi^* S)) \otimes \mathcal{O}_Y(-\pi^* S)$$

is a direct summand of

$$\varphi_*(\iota_! \mathbb{C}_{V \setminus T}) \xrightarrow{qis} \varphi_*(\Omega^\bullet_V(\log T) \otimes \mathcal{O}_V(-T)),$$

where qis means a quasi-isomorphism. On the other hand, we can decompose $\pi_*(\widetilde{\Omega}^\bullet_Y(\log(\pi^* S)) \otimes \mathcal{O}_Y(-\pi^* S))$ and $\pi_*(\iota'_! \mathbb{C}_{Y \setminus \pi^* S})$ into eigen components. We have that

$$\mathcal{C} \xrightarrow{qis} \Omega^\bullet_X(\log(S + B)) \otimes \mathcal{L}^{-1}(-S)$$

is a direct summand of

$$\psi_*(\iota_! \mathbb{C}_{V \setminus T}) \xrightarrow{qis} \psi_*(\Omega^\bullet_V(\log T) \otimes \mathcal{O}_V(-T)).$$

The E_1-degeneration of the spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(V, \Omega^p_V(\log T) \otimes \mathcal{O}_V(-T)) \\ &\Rightarrow \mathbb{H}^{p+q}(V, \Omega^\bullet_V(\log T) \otimes \mathcal{O}_V(-T)) \\ &\simeq H^{p+q}(V, \iota_! \mathbb{C}_{V \setminus T}) \end{aligned}$$

implies the E_1-degeneration of

$$\begin{aligned} E_1^{pq} &= H^q(X, \Omega^p_X(\log(S + B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log(S + B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\simeq H^{p+q}(X, \mathcal{C}) \end{aligned}$$

Therefore, the inclusion $\mathcal{C} \subset \mathcal{L}^{-1}(-S)$ induces surjections

$$H^p(X, \mathcal{C}) \to H^p(X, \mathcal{L}^{-1}(-S)).$$

We can check the following simple property by seeing the monodromy action of the Galois group of $\pi : Y \to X$ on $\mathcal{C}$ around $\mathrm{Supp} B$.

Corollary 2.3 (cf. [KM, Corollary 2.54])**.** *Let $U \subset X$ be a connected open set such that $U \cap \mathrm{Supp} B \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

This property is utilized via the following fact. The proof is obvious.

Lemma 2.4 (cf. [KM, Lemma 2.55])**.** *Let F be a sheaf of Abelian groups on a topological space X and $F_1, F_2 \subset F$ subsheaves. Let $Z \subset X$ be a closed subset. Assume that*

(1) *$F_2|_{X \setminus Z} = F|_{X \setminus Z}$, and*
(2) *if U is connected, open and $U \cap Z \neq \emptyset$, then $H^0(U, F_1|U) = 0$.*

Then F_1 is a subsheaf of F_2.

As a corollary, we obtain:

Corollary 2.5 (cf. [KM, Corollary 2.56])**.** *Let $M \subset \mathcal{L}^{-1}(-S)$ be a subsheaf such that $M|_{X \setminus \mathrm{Supp} B} = \mathcal{L}^{-1}(-S)|_{X \setminus \mathrm{Supp} B}$. Then the injection*

$$\mathcal{C} \to \mathcal{L}^{-1}(-S)$$

factors as

$$\mathcal{C} \to M \to \mathcal{L}^{-1}(-S).$$

Therefore,

$$H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S))$$

is surjective for every i.

Proof. The first part is clear from Corollary 2.3 and Lemma 2.4. This implies that we have maps

$$H^i(X, \mathcal{C}) \to H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right. □

Therefore, we obtain that

$$H^q(X, \mathcal{L}^{-1}(-S - D)) \to H^q(X, \mathcal{L}^{-1}(-S))$$

is surjective for any q. By the Serre duality, we obtain

$$\begin{aligned} &H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S)) \\ &\quad \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S + D)) \end{aligned}$$

is injective for any q. This means that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

is injective for any q. □

3. Proof of the main theorem

In this section, we prove Theorem 1.1. First, we prove a generalization of Kollár's injectivity theorem (cf. [A, Theorem 3.1]). It is a straightforward consequence of

Proposition 2.1 and will produce the desired torsion-free and vanishing theorems.

Theorem 3.1 (Injectivity theorem)**.** *Let X be a smooth projective variety and $S+B$ a boundary $\mathbb{Q}$-divisor such that* $\mathrm{Supp}(S+B)$ *is simple normal crossing and that $\llcorner S+B \lrcorner = S$. Let L be a Cartier divisor on X and D an effective Cartier divisor that contains no lc centers of $(X, S+B)$. Assume the following conditions.*

(i) *$L \sim_{\mathbb{Q}} K_X + S + B + H$,*
(ii) *H is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, and*
(iii) *$tH \sim_{\mathbb{Q}} D + D'$ for some positive rational number t, where D' is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor that contains no lc centers of $(X, S+B)$.*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. We can take a resolution $f: Y \to X$ such that f is an isomorphism outside $\mathrm{Supp}(D+D'+B)$, and that the union of the support of $f^*(S+B+D+D')$ and the exceptional locus of f has a simple normal crossing support on Y. Let B' be the strict transform of B on Y. We write $K_Y + S' + B' = f^*(K_X + S + B) + E$, where S' is the strict transform of S, and E is f-exceptional. It is easy to see that $E_+ = \ulcorner E \urcorner \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is an effective $\mathbb{Q}$-Cartier divisor with $\llcorner E_- \lrcorner = 0$. Since f^*H is semi-ample, we can write $f^*H \sim_{\mathbb{Q}} aH'$, where $0 < a < 1$ and H' is a general Cartier divisor on Y. We put $B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D+D') + (1-\varepsilon)aH'$ for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{Q}} K_Y + S' + B''$. By the construction, $\llcorner B'' \lrcorner = 0$, the support of $S' + B''$ is simple normal crossing on Y, and $\mathrm{Supp}B'' \supset \mathrm{Supp}f^*D$. So, Proposition 2.1 implies that the homomorphisms $H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D))$ are injective for all q. By Lemma 3.2 below, $R^q f_* \mathcal{O}_Y(L') = 0$ for any $q > 0$ and it is easy to see that $f_*\mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ are injective for all q. $\square$

Let us recall the following well-known easy lemma.

Lemma 3.2. *Let V be a smooth projective variety and B a boundary $\mathbb{Q}$-divisor on V such that* $\mathrm{Supp}B$ *is simple normal crossing. Let $f: V \to W$ be a projective birational morphism onto a variety W. Assume that f is an isomorphism at the generic point of any lc center of (V, B) and that D is a Cartier divisor on V such that $D - (K_V + B)$ is nef. Then $R^i f_* \mathcal{O}_V(D) = 0$ for any $i > 0$.*

Proof. We use the induction on the number of irreducible components of $\llcorner B \lrcorner$ and on the dimension of V. If $\llcorner B \lrcorner = 0$, then the lemma follows from the Kawamata–Viehweg vanishing theorem (cf. [KM, Corollary 2.68]). Therefore, we can assume that there is an irreducible divisor $S \subset \llcorner B \lrcorner$. We consider the following short exact sequence

$$0 \to \mathcal{O}_V(D-S) \to \mathcal{O}_V(D) \to \mathcal{O}_S(D) \to 0.$$

By induction, we see that $R^i f_* \mathcal{O}_V(D-S) = 0$ and $R^i f_* \mathcal{O}_S(D) = 0$ for any $i > 0$. Thus, we have $R^i f_* \mathcal{O}_V(D) = 0$ for $i > 0$. $\square$

Let us go to the proof of the main theorem: Theorem 1.1.

Proof of Theorem 1.1. We can assume that H is semi-ample by replacing L (resp. H) with $L + f^*A'$ (resp. $H + f^*A'$), where A' is a very ample Cartier divisor on X. Assume that $R^q f_* \mathcal{O}_Y(L)$ has a local section whose support does not contain the image of any (Y, B)-stratum. Then we can find a very ample Cartier divisor A with the following properties.

(a) f^*A contains no lc centers of (Y, B), and
(b) $R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We can assume that $H - f^*A$ is semi-ample by replacing L (resp. H) with $L + f^*A$ (resp. $H + f^*A$). If necessary, we replace L (resp. H) with $L + f^*A''$ (resp. $H + f^*A''$), where A'' is a very ample Cartier divisor on X. Then, we have

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \simeq H^q(Y, \mathcal{O}_Y(L))$$

and

$$H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y, \mathcal{O}_Y(L + f^*A)).$$

We obtain that

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \to H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A))$$

is not injective by (b) if A'' is sufficiently ample. So,

$$H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + f^*A))$$

is not injective. It contradicts Theorem 3.1. We finish the proof of (i).

Let us go to the proof of (ii). We take a general member $A \in |mH'|$, where m is a sufficiently large and divisible integer, such that $A' = f^*A$ and

$R^q f_*\mathcal{O}_Y(L+A')$ is Γ-acyclic for all q. By (i), we have the following short exact sequences,

$$0 \to R^q f_*\mathcal{O}_Y(L) \to R^q f_*\mathcal{O}_Y(L+A') \to R^q f_*\mathcal{O}_{A'}(L+A') \to 0.$$

for any q. Note that $R^q f_*\mathcal{O}_{A'}(L+A')$ is Γ-acyclic by induction on $\dim X$ and $R^q f_*\mathcal{O}_Y(L+A')$ is also Γ-acyclic by the above assumption. We consider the spectral sequences

$$E_2^{pq} = H^p(X, R^q f_*\mathcal{O}_Y(L)) \to H^{p+q}(Y, \mathcal{O}_Y(L)),$$

and

$$\overline{E}_2^{pq} = H^p(X, R^q f_*\mathcal{O}_Y(L+A')) \to H^{p+q}(Y, \mathcal{O}_Y(L+A')).$$

Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{array}{ccc} E_2^{pq} & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L)) \\ {\scriptstyle \varphi^{pq}}\downarrow & & \downarrow{\scriptstyle \varphi^{p+q}} \\ \overline{E}_2^{pq} & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L+A')) \end{array}$$

We note that φ^{1+q} is injective by Theorem 3.1. We have $E_2^{1q} \to H^{1+q}(Y, \mathcal{O}_Y(L))$ is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$. We also have that $\overline{E}_2^{1q} = 0$ by the above assumption. Therefore, we obtain $E_2^{1q} = 0$ since the injection $E_2^{1q} \to H^{1+q}(Y, \mathcal{O}_Y(L+A'))$ factors through $\overline{E}_2^{1q}$. This implies that $H^p(X, R^q f_*\mathcal{O}_Y(L)) = 0$ for any $p > 0$. $\square$

4. Applications In this final section, we give two applications of Theorem 1.1. The next theorem is enough powerful and hard to reach by the classical approaches. We recommend the reader to see [F2] for some applications to the log minimal model program for log canonical pairs.

Theorem 4.1 (cf. [A, Theorem 4.4])**.** *Let X be a normal projective variety and B a boundary $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a Cartier divisor on X. Assume that $L - (K_X + B)$ is ample. Let $\{C_i\}$ be* any *set of lc centers of the pair (X, B). We put $W = \bigcup C_i$ with a reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) = 0, \quad H^i(X, \mathcal{O}_X(L)) = 0,$$

and

$$H^i(W, \mathcal{O}_W(L)) = 0$$

for any $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(W, \mathcal{O}_W(L))$$

is surjective. Therefore, if (X, B) has a zero-dimensional lc center, then the linear system $|L|$ is not empty and the base locus of $|L|$ contains no zero-dimensional lc centers of (X, B).

Proof. Let $f : Y \to X$ be a resolution such that $\mathrm{Supp} f_*^{-1}B \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor. We can further assume that $f^{-1}(W)$ is a simple normal crossing divisor on Y. We can write

$$K_Y + B_Y = f^*(K_X + B).$$

Let T be the union of the irreducible components of $B_Y^{=1}$ that are mapped into W by f. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A - T) \to \mathcal{O}_Y(A) \to \mathcal{O}_T(A) \to 0,$$

where $A = \ulcorner -(B_Y^{<1}) \urcorner$. Note that A is an effective f-exceptional divisor. We obtain the following long exact sequence

$$0 \to f_*\mathcal{O}_Y(A - T) \to f_*\mathcal{O}_Y(A) \to f_*\mathcal{O}_T(A) \xrightarrow{\delta} R^1 f_*\mathcal{O}_Y(A - T) \to \cdots.$$

Since

$$A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) = -(K_Y + B_Y) \sim_{\mathbb{Q}} -f^*(K_X + B),$$

any non-zero local section of $R^1 f_*\mathcal{O}_Y(A - T)$ contains in its support the f-image of some strata of $(Y, \{B_Y\} + B_Y^{=1} - T)$ by Theorem 1.1 (i). On the other hand, $W = f(T)$. Therefore, the connecting homomorphism δ is a zero map. Thus, we have a short exact sequence

$$0 \to f_*\mathcal{O}_Y(A - T) \to \mathcal{O}_X \to f_*\mathcal{O}_T(A) \to 0.$$

So, we obtain $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ and $f_*\mathcal{O}_Y(A - T) \simeq \mathcal{I}_W$, the defining ideal sheaf of W. The isomorphism $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ plays crucial roles in [F2]. So, we proved it here. Since

$$f^*L + A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) \sim_{\mathbb{Q}} f^*(L - (K_X + B)),$$

and

$$f^*L + A - (K_Y + \{B_Y\} + B_Y^{=1}) \sim_{\mathbb{Q}} f^*(L - (K_X + B)),$$

we have

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) \simeq H^i(X, f_*\mathcal{O}_Y(A - T) \otimes \mathcal{O}_X(L)) = 0,$$

and

$$H^i(X, \mathcal{O}_X(L)) \simeq H^i(X, f_*\mathcal{O}_Y(A) \otimes \mathcal{O}_X(L)) = 0$$

for any $i > 0$ by Theorem 1.1 (ii). By the following long exact sequence

$$\cdots \to H^i(X, \mathcal{O}_X(L)) \to H^i(W, \mathcal{O}_W(L)) \to H^{i+1}(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) \to \cdots,$$

we obtain $H^i(W, \mathcal{O}_W(L)) = 0$ for any $i > 0$. We finish the proof. □

Remark 4.2. We note that we do not need Ambro's vanishing theorem for embedded normal crossing pairs (cf. [A, Theorem 3.2]) to obtain $H^i(W, \mathcal{O}_W(L)) = 0$ for $i > 0$ in Theorem 4.1.

We close this paper with the Kodaira vanishing theorem for log canonical pairs, which was not explicitly stated in [A]. For a more general result containing the Kawamata–Viehweg vanishing theorem, see [F1, Theorem 2.48].

Theorem 4.3 (Kodaira vanishing theorem for lc pairs)**.** *Let X be a normal projective variety and B a boundary $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a $\mathbb{Q}$-Cartier Weil divisor on X such that $L - (K_X + B)$ is ample. Then $H^q(X, \mathcal{O}_X(L)) = 0$ for any $q > 0$.*

Proof. Let $f : Y \to X$ be a resolution of (X, B) such that $K_Y = f^*(K_X + B) + \sum_i a_i E_i$ with $a_i \geq -1$ for any i and that $\mathrm{Supp} \sum E_i$ is simple normal crossing. We can assume that $\sum_i E_i \cup \mathrm{Supp} f^*L$ is a simple normal crossing divisor on Y. We put $E = \sum_i a_i E_i$ and $F = \sum_{a_j=-1}(1 - b_j)E_j$, where $b_j = \mathrm{mult}_{E_j}\{f^*L\}$. We note that $A = L - (K_X + B)$ is ample by the assumption. So, we have $f^*A = f^*L - f^*(K_X+B) = \ulcorner f^*L+E+F\urcorner - (K_Y+F+\{-(f^*L+E+F)\})$. We can easily check that $f_*\mathcal{O}_Y(\ulcorner f^*L+E+F\urcorner) \simeq \mathcal{O}_X(L)$ and that $F + \{-(f^*L + E + F)\}$ has a simple normal crossing support and is a boundary $\mathbb{Q}$-divisor on Y. By Theorem 1.1 (ii), we obtain that $\mathcal{O}_X(L)$ is Γ-acyclic. Thus, we have $H^q(X, \mathcal{O}_X(L)) = 0$ for any $q > 0$. □

The reader can find more advanced topics and many other applications in [F1]. This paper is a gentle introduction to Chapter 2 in [F1]. We recommend the reader to see [F1].

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References

[A] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233

[D] P. Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. No. **40** (1971), 5–57.

[EV] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, **20**. Birkhäuser Verlag, Basel, 1992.

[F1] O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint (2008).

[F2] O. Fujino, Non-vanishing theorem for log canonical pairs, preprint (2009).

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the Minimal Model Problem, in *Algebraic Geometry, Sendai 1985,* Advanced Studies in Pure Math. **10**, (1987) Kinokuniya and North-Holland, 283–360.

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties,* Cambridge Tracts in Mathematics, Vol. **134**, 1998.

[L] R. Lazarsfeld, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], **49**. Springer-Verlag, Berlin, 2004.