# ON THE KODAIRA DIMENSION

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Abstract. We discuss the behavior of the Kodaira dimension under smooth morphisms.

Contents



## 1. Introduction

We will discuss the behavior of the Kodaira dimension under smooth morphisms. Throughout this paper, we will work over $\mathbb{C}$, the field of complex numbers. One of the motivations of this paper is to understand [Pa]. In [Pa], Sung Gi Park established the following striking and unexpected theorem.

**Theorem 1.1** (Park's logarithmic base change theorem, see [Pa, Theorem 1.2])**.** *Let $X$, $Y$, and $Y'$ be smooth quasi-projective varieties and let $E$, $D$, and $D'$ be simple normal crossing divisors on $X$, $Y$, and $Y'$, respectively. Let $f\colon X \to Y$ and $g\colon Y' \to Y$ be projective surjective morphisms such that $f$ and $g$ are smooth over $Y \setminus D$, $f^{-1}(D) \subset E$ and $g^{-1}(D) \subset D'$, and that $E$ and $D'$ are relatively normal crossing over $Y \setminus D$. Let $X'$ be the union of the irreducible components of $X \times_Y Y'$ dominating $Y$ and $E' := g'^{-1}(E) \cup f'^{-1}(D')$. We consider the following commutative diagram:*

$$
\begin{array}{ccccc}
(X, E) & \xleftarrow{\ g'\ } & (X', E') & \xleftarrow{\ \mu\ } & (X'', E'') \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & \swarrow{\scriptstyle f''} & \\
(Y, D) & \xleftarrow[\ g\ ]{} & (Y', D') & &
\end{array}
$$

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*where $\mu\colon X''\to X'$ is a projective resolution of singularities such that $\mu$ is an isomorphism over $Y\setminus D$, $E''$ is a simple normal crossing divisor on $X''$ such that $E''$ coincides with $E'$ over $Y\setminus D$ and that $(g\circ f'')^{-1}(D)\subset E''$. We put $\omega_{(X,E)}:=\omega_X\otimes\mathscr{O}_X(E)$, $\omega_{(Y,D)}:=\omega_Y\otimes\mathscr{O}_Y(D)$, $\omega_{(X,E)/(Y,D)}:=\omega_{(X,E)}\otimes f^*\omega_{(Y,D)}^{\otimes -1}$, and so on. Then, for every positive integer $N$, there exists a generically isomorphic inclusion*

$$\left(f_*\omega^{\otimes N}_{(X,E)/(Y,D)}\otimes g_*\omega^{\otimes N}_{(Y',D')/(Y,D)}\right)^{\vee\vee}\subset\left(h_*\omega^{\otimes N}_{(X'',E'')/(Y,D)}\right)^{\vee\vee}\tag{1.1}$$

*where $h:=g\circ f''$.*

Precisely speaking, Park treated only the case where $f^{-1}(D)=E$ and $g^{-1}(D)=D'$ hold. However, we can easily see that [Pa, Proposition 2.5] implies the inclusion (1.1). As a direct and easy consequence of Theorem 1.1, he obtained the following very important result.

**Corollary 1.2** (Park's logarithmic fiber product trick, see [Pa, Corollary 1.4])**.** *Let $f\colon X\to Y$ be a projective surjective morphism of smooth quasi-projective varieties. Let $E$ and $D$ be simple normal crossing divisors on $X$ and $Y$, respectively. Assume that $f$ is smooth over $Y\setminus D$, $E$ is relatively normal crossing over $Y\setminus D$, and $f^{-1}(D)\subset E$. Let $s$ be a positive integer and let $f^s\colon X^s:=X\times_Y\cdots\times_Y X\to Y$ be the $s$-fold fiber product of $f\colon X\to Y$. We put $E^s:=\sum_{i=1}^s p_i^*E$, where $p_i\colon X^s\to X$ is the $i$-th projection for every $i$. Let $\varphi\colon X^{(s)}\to X^s$ be a projective resolution of singularities of the union of the irreducible components of $X^s$ dominating $Y$ such that $\varphi$ is an isomorphism over $Y\setminus D$. Let $E^{(s)}$ be a simple normal crossing divisor on $X^{(s)}$ such that $E^{(s)}$ coincides with $E^s$ over $Y\setminus D$ and that $(f^{(s)})^{-1}(D)\subset E^{(s)}$, where $f^{(s)}:=f^s\circ\varphi$. Then we have the following generically isomorphic injection*

$$\left(\bigotimes^s f_*\left(\omega^{\otimes N}_{(X,E)/(Y,D)}\right)\right)^{\vee\vee}\hookrightarrow\left(f^{(s)}_*\left(\omega^{\otimes N}_{(X^{(s)},E^{(s)})/(Y,D)}\right)\right)^{\vee\vee}\tag{1.2}$$

*for every positive integer $N$. Note that $X^{(s)}$ is smooth, $f^{(s)}$ is smooth over $Y\setminus D$, $E^{(s)}$ is a simple normal crossing divisor on $X^{(s)}$ and is relatively normal crossing over $Y\setminus D$, and $(f^{(s)})^{-1}(D)\subset E^{(s)}$.*

We make a small remark on Corollary 1.2.

**Remark 1.3.** If $f\colon X\to Y$ has connected fibers in Corollary 1.2, then $X^{(s)}$ is a smooth variety, that is, $X^{(s)}$ is connected, by construction. In general, $X^{(s)}$ may have some connected components.

In this paper, we will establish the following theorem, which is a slight generalization of [Pa, Section 3], as an application of Corollary 1.2 and the theory of variations of mixed Hodge structure. In [Pa], Theorem 1.4 was treated under the assumption that $f^{-1}(D)=E$ holds.

**Theorem 1.4** (see [Pa, Section 3])**.** *Let $f\colon X\to Y$ be a surjective morphism of smooth projective varieties and let $E$ and $D$ be simple normal crossing divisors on $X$ and $Y$, respectively. Assume that $f$ is smooth over $Y\setminus D$, $E$ is relatively normal crossing over $Y\setminus D$, and $f^{-1}(D)\subset E$. Let $\mathscr{L}$ be a line bundle on $Y$ such that there exists a nonzero homomorphism*

$$\mathscr{L}^{\otimes N}\to\left(f_*\omega^{\otimes N}_{(X,E)/(Y,D)}\right)^{\vee\vee}$$

*for some positive integer $N$. Then there exists a pseudo-effective line bundle $\mathscr{P}$ on $Y$ and a nonzero homomorphism*

$$\mathscr{L}^{\otimes r} \otimes \mathscr{P} \to \left(\Omega^1_Y(\log D)\right)^{\otimes kr}$$

*for some $r > 0$ and $k \geq 0$.*

By using Theorem 1.4, we will prove the following results.

**Theorem 1.5** (see [Pa, Theorem 1.5])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let $E$ and $D$ be simple normal crossing divisors on $X$ and $Y$, respectively. Assume that $f^{-1}(D) \subset E$, $f$ is smooth over $Y \setminus D$, and $E$ is relatively normal crossing over $Y \setminus D$. We further assume that $\kappa(F, (K_X + E)|_F) \geq 0$ holds, where $F$ is a sufficiently general fiber of $f\colon X \to Y$. Then $\kappa(Y, K_Y + D) = \dim Y$ holds if and only if $\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$.*

**Remark 1.6.** In Theorem 1.5, it is well known that

$$\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

holds under the assumption that $\kappa(Y, K_Y + D) = \dim Y$. This is due to Maehara (see [Ma] and [Fn1]). Hence the opposite implication is new and nontrivial.

Corollary 1.7 is an obvious consequence of Theorem 1.5.

**Corollary 1.7.** *Let $f\colon X \to Y$, $D$, and $E$ be as in Theorem 1.5, that is, $f^{-1}(D) \subset E$, $f$ is smooth over $Y \setminus D$, and $E$ is relatively normal crossing over $Y \setminus D$. If $\kappa(X, K_X + E) = \dim X$, then $\kappa(Y, K_Y + D) = \dim Y$ and $\kappa(F, (K_X + E)|_F) = \dim F$ hold, where $F$ is a general fiber of $f\colon X \to Y$.*

**Theorem 1.8** (see [Pa, Theorem 1.7 (1)])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties and let $E$ and $D$ be simple normal crossing divisors on $X$ and $Y$, respectively. Assume that $f$ is smooth over $Y \setminus D$, $E$ is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. In this situation, if $\kappa(X, K_X + E - \varepsilon f^*D) \geq 0$ holds for some positive rational number $\varepsilon$, then there exists some positive rational number $\delta$ such that $K_Y + (1 - \delta)D$ is pseudo-effective.*

If $f^{-1}(D) = E$ in Theorems 1.5 and 1.8, then they are nothing but [Pa, Theorem 1.5] and [Pa, Theorem 1.7], respectively. In Theorem 1.8, if $\kappa(X, K_X + E) \geq 0$, then we can prove that $K_Y + D$ is pseudo-effective without using Theorem 1.4. It will be treated in Theorem 4.5.

Let us consider a conjecture on the behavior of the (logarithmic) Kodaira dimension under smooth morphisms.

**Conjecture 1.9** (see [Po, Conjecture 3.6] and [Pa, Conjecture 5.1])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let $E$ and $D$ be simple normal crossing divisors on $X$ and $Y$, respectively. Assume that $f$ is smooth over $Y \setminus D$, $E$ is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. Then*

$$\kappa(X, K_X + E) = \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F)$$

*holds, where $F$ is a sufficiently general fiber of $f\colon X \to Y$.*

We explain some related conjectures. Conjecture 1.10 is a special case of the generalized abundance conjecture, which is one of the most important conjectures in the theory of minimal models. For the details of Conjecture 1.10, see [Fn2, Section 4.1].

**Conjecture 1.10** (Generalized abundance conjecture for projective smooth pairs)**.** *Let $X$ be a smooth projective variety and let $E$ be a simple normal crossing divisor on $X$. Then*

$$\kappa(X, K_X + E) = \kappa_\sigma(X, K_X + E)$$

*holds, where $\kappa_\sigma$ denotes Nakayama's numerical dimension.*

Conjecture 1.10 is still widely open. Conjecture 1.10 contains Conjecture 1.11 as a special case. For the details of the nonvanishing conjecture, see, for example, [Fn2, Section 4.8] and [H1].

**Conjecture 1.11** (Nonvanishing conjecture for projective smooth pairs)**.** *Let $X$ be a smooth projective variety and let $E$ be a simple normal crossing divisor on $X$, Assume that $K_X + E$ is pseudo-effective. Then*

$$\kappa(X, K_X + E) \geq 0$$

*holds.*

On Conjecture 1.9, we have a partial result, which is obviously a generalization of [Pa, Theorem 1.12].

**Theorem 1.12** (Superadditivity, see [Pa, Theorem 1.12])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let $E$ and $D$ be simple normal crossing divisors on $X$ and $Y$, respectively. Assume that $f$ is smooth over $Y \setminus D$, $E$ is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. We further assume that $\kappa(Y, K_Y + D) \geq 0$ and that the generalized abundance conjecture holds for sufficiently general fibers of the Iitaka fibration of $Y$ with respect to $K_Y + D$. Then*

$$\kappa(X, K_X + E) \leq \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F)$$

*holds, where $F$ is a sufficiently general fiber of $f\colon X \to Y$.*

We have already known that Conjecture 1.10 holds true when $X \setminus E$ is affine. More generally, Conjecture 1.10 holds true under the assumption that there exists a projective birational morphism $X \setminus E \to V$ onto an affine variety $V$. Hence we have:

**Corollary 1.13.** *Let $f\colon X \to Y$, $E$, and $D$ be as in Conjecture 1.9. We assume that $Y \setminus D$ is affine. Then the following superadditivity*

$$\kappa(X, K_X + E) \leq \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F) \tag{1.3}$$

*holds.*

In Conjecture 1.9, we have already known that the subadditivity

$$\kappa(X, K_X + E) \geq \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F)$$

follows from the generalized abundance conjecture (see Conjecture 1.10). For the details, see [Fn3], [Fn4], and [H2]. Hence, if the generalized abundance conjecture holds true, then Conjecture 1.9 is also true. Roughly speaking, we have:

**Theorem 1.14.** *Let $f\colon X \to Y$, $E$, and $D$ be as in Conjecture 1.9. We assume that Conjecture 1.10 holds true. Then we have*

$$\kappa(X, K_X + E) = \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F),$$

*where $F$ is a sufficiently general fiber of $f\colon X \to Y$, that is, Conjecture 1.9 holds true.*

We look at the organization of this paper. In Section 2, we collect some basic definitions and results necessary for this paper for the reader's convenience. Subsection 2.1 collects some basic definitions. In Subsection 2.2, we recall various notions of positivities. In Subsection 2.3, we explain systems of Hodge bundles. In Section 3, we discuss Hodge theoretic weak positivity results. They seem to be more general than the usual ones slightly. In Section 4, we construct graded logarithmic Higgs sheaves and prove Theorem 1.4 following [Pa, Section 3.1] closely. Here we use variations of mixed Hodge structure. In Section 5, we prove results explained in Section 1. In Section 6, we discuss variations of mixed Hodge structure necessary for Section 4 for the sake of completeness.

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For the classical Iitaka subadditivity conjecture and some related topics, see [Fn5]. For the details of relatively new conjectures on the behavior of the Kodaira dimension under morphisms of smooth complex varieties, see [Po] and the references therein.

## 2. Preliminaries

In this section, we will collect some basic definitions and properties necessary for this paper.

2.1. **Basic definitions.** We will work over $\mathbb{C}$, the field of complex numbers. A *variety* means an irreducible and reduced separated scheme of finite type over $\mathbb{C}$.

**2.1** ($\kappa$ and $\kappa_\sigma$)**.** Let $X$ be a smooth projective variety. Then $\kappa(X, \bullet)$ and $\kappa_\sigma(X, \bullet)$ denote *the Iitaka dimension* and *Nakayama's numerical dimension* of $\bullet$, respectively, where $\bullet$ is a $\mathbb{Q}$-Cartier divisor or a line bundle on $X$. For the details of $\kappa$ and $\kappa_\sigma$, see [U], [Mo], [N], and so on.

**2.2** (Canonical bundles and log canonical bundles)**.** Let $X$ be a smooth variety and let $E$ be a simple normal crossing divisor on $X$. Then we put

$$\omega_X := \det \Omega^1_X$$

and

$$\omega_{(X,E)} := \omega_X \otimes \mathscr{O}_X(E) =: \omega_X(E).$$

We note that

$$\omega_{(X,E)} = \det \Omega^1_X(\log E).$$

Let $f\colon X \to Y$ be a surjective morphism of smooth varieties and let $D$ be a simple normal crossing divisor on $Y$. Then we put

$$\omega_{(X,E)/(Y,D)} := \omega_{(X,E)} \otimes f^*\omega^{\otimes -1}_{(Y,D)} = \omega_X(E) \otimes \left(f^*\omega_Y(D)\right)^{\otimes -1}.$$

Let $K_X$ be a Cartier divisor with $\mathscr{O}_X(K_X) \simeq \omega_X$. Then it is obvious that $\omega_{(X,E)} \simeq \mathscr{O}_X(K_X + E)$ holds.

**2.3** (Duals and double duals)**.** Let $\mathscr{F}$ be a coherent sheaf on a smooth variety $X$. Then we put

$$\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$$

and

$$\mathscr{F}^{\vee\vee} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee}, \mathscr{O}_X).$$

We further assume that $\mathscr{F}$ is torsion-free. Then $\widehat{\det}\mathscr{F}$ and $\widehat{S}^{\alpha}(\mathscr{F})$ denote $(\det \mathscr{F})^{\vee\vee}$ and $S^{\alpha}(\mathscr{F})^{\vee\vee}$, respectively, where $S^{\alpha}(\mathscr{F})$ is the $\alpha$-symmetric product of $\mathscr{F}$.

**2.4** (Sufficiently general fibers and general fibers)**.** Let $f\colon X \to Y$ be a surjective morphism between varieties. Then a *sufficiently general fiber* (resp. *general fiber*) $F$ of $f\colon X \to Y$ means that $F = f^{-1}(y)$, where $y$ is any closed point contained in a countable intersection of nonempty Zariski open sets (resp. a nonempty Zariski open set) of $Y$. A sufficiently general fiber is sometimes called a *very general fiber* in the literature.

2.2. **Weakly positive sheaves.** Let us recall the necessary definitions around various positivity. The following definition is well known and standard in the study of higher-dimensional algebraic varieties.

**Definition 2.5.** Let $X$ be a projective variety and let $\mathscr{L}$ be a line bundle on $X$.

(i) $\mathscr{L}$ is *big* if $\mathscr{L}^{\otimes k} \simeq \mathscr{H} \otimes \mathscr{O}_X(B)$ for some positive integer $k$, an ample line bundle $\mathscr{H}$, and an effective Cartier divisor $B$ on $X$.
(ii) $\mathscr{L}$ is *nef* if $\mathscr{L} \cdot C \geq 0$ holds for every curve $C$ on $X$.
(iii) $\mathscr{L}$ is *pseudo-effective* if $\mathscr{L}^{\otimes m} \otimes \mathscr{H}$ is big for every positive integer $m$ and every ample line bundle $\mathscr{H}$ on $X$.

We note that a nef line bundle is always pseudo-effective. Let $\mathscr{E}$ be a locally free sheaf of finite rank on a projective variety $X$.

(iv) $\mathscr{E}$ is *nef* if $\mathscr{O}_{\mathbb{P}_X(\mathscr{E})}(1)$ is a nef line bundle on $\mathbb{P}_X(\mathscr{E})$.

We note that a nef locally free sheaf is sometimes called a *semipositive* locally free sheaf.

We recall the definition of weakly positive sheaves, which was first introduced by Viehweg. For the basic properties of weakly positive sheaves, see [Fn5].

**Definition 2.6** (Weakly positive sheaves)**.** Let $X$ be a normal quasi-projective variety and let $\mathscr{A}$ be a torsion-free coherent sheaf on $X$. We say that $\mathscr{A}$ is *weakly positive* if, for every positive integer $\alpha$ and every ample line bundle $\mathscr{H}$ on $X$, there exists a positive integer $\beta$ such that $\widehat{S}^{\alpha\beta}(\mathscr{A}) \otimes \mathscr{H}^{\otimes\beta}$ is generically generated by global sections, where $\widehat{S}^{\alpha\beta}(\mathscr{A})$ denotes the double dual of the $\alpha\beta$-symmetric product of $\mathscr{A}$.

A line bundle on a normal projective variety is pseudo-effective if and only if it is weakly positive. In general, the weak positivity does not behave well under extensions.

**Remark 2.7.** In [EjFI], we constructed a short exact sequence

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$

of locally free sheaves on a smooth projective surface such that $\mathscr{E}'$ and $\mathscr{E}''$ are pseudo-effective line bundles but $\mathscr{E}$ is not weakly positive.

We make a small remark on [EjFI] for the reader's convenience. Professor Robert Lazarsfeld pointed out that the following example answers [EjFI, Question 3.2] negatively.

**Example 2.8** (Gieseker)**.** There exists a rank two ample vector bundle $E$ on $\mathbb{P}^2$ sitting in an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^2}(-1)^{\oplus 4} \to E \to 0,$$

where $d$ is a sufficiently large positive integer $d$ (see, for example, [L, Example 6.3.67]). Let $\pi\colon X \to \mathbb{P}^2$ be any generically finite surjective morphism from a smooth variety $X$. Then we see that $H^0(X, \pi^*E) = 0$ since $H^1(X, \pi^*\mathscr{O}_{\mathbb{P}^2}(-d)^{\oplus 2}) = 0$ by the Kawamata–Viehweg vanishing theorem. In particular, $\pi^*E$ is not generically globally generated.

2.3. **Systems of Hodge bundles.** Let $V_0 = (\mathbb{V}_0, W_0, F_0)$ be a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure on a complex manifold $X_0$, where $\mathbb{V}_0$ is a local system of finite-dimensional $\mathbb{R}$-vector spaces on $X_0$, $W_0$ is an increasing filtration of $\mathbb{V}_0$ by local subsystems, and $F_0 = \{F_0^p\}$ is the Hodge filtration. Then we obtain a Higgs bundle $(E_0, \theta_0)$ on $X_0$ by setting

$$E_0 = \operatorname{Gr}^{\bullet}_{F_0} \mathscr{V}_0 = \bigoplus_p F_0^p / F_0^{p+1}$$

where $\mathscr{V}_0 = \mathbb{V}_0 \otimes \mathscr{O}_{X_0}$. Note that $\theta_0$ is induced by the Griffiths transversality

$$\nabla\colon F_0^p \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} F_0^{p-1}.$$

More precisely, $\nabla$ induces

$$\theta_0^p\colon F_0^p/F_0^{p+1} \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} \left(F_0^{p-1}/F_0^p\right)$$

for every $p$. Then

$$\theta_0 = \bigoplus_p \theta_0^p\colon E_0 \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} E_0.$$

The pair $(E_0, \theta_0)$ is usually called the *system of Hodge bundles* associated to $V_0 = (\mathbb{V}_0, W_0, F_0)$ and $\theta_0$ is called the *Higgs field* of $(E_0, \theta_0)$.

We further assume that $X_0$ is a Zariski open subset of a complex manifold $X$ such that $D = X \setminus X_0$ is a normal crossing divisor on $X$. We note that the local monodromy of $\mathbb{V}_0$ around $D$ is quasi-unipotent because $V_0$ is admissible. Let ${}^{\ell}\mathscr{V}$ be the lower canonical extension of $\mathscr{V}_0$ on $X$, that is, the *Deligne extension* of $\mathscr{V}_0$ on $X$ such that the eigenvalues of the residue of the connection are contained in $[0, 1)$. Let ${}^{\ell}F^p$ be the lower canonical extension of $F_0^p$, that is,

$${}^{\ell}F^p = j_*F_0^p \cap {}^{\ell}\mathscr{V},$$

where $j\colon X_0 \hookrightarrow X$ is the natural open immersion, for every $p$. Since $V_0$ is admissible, ${}^{\ell}F^p$ is a subbundle of ${}^{\ell}\mathscr{V}$ for every $p$, and we can extend $(E_0, \theta_0)$ to $(E, \theta)$ on $X$, where

$$E = \operatorname{Gr}^{\bullet}_F {}^{\ell}\mathscr{V} = \bigoplus_p {}^{\ell}F^p / {}^{\ell}F^{p+1}$$

and

$$\theta\colon E \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} E.$$

When all the local monodromies of $V_0$ around $D$ are unipotent, we simply write $\mathscr{V}$ and $F^p$ to denote ${}^{\ell}\mathscr{V}$ and ${}^{\ell}F^p$, respectively. We say that $\mathscr{V}$ (resp. $F^p$) is the canonical extension of $\mathscr{V}_0$ (resp. $F_0^p$).

**Remark 2.9.** Although we formulated systems of Hodge bundles for graded polarizable *admissible* variations of $\mathbb{R}$-mixed Hodge structure, we do not need the relative monodromy weight filtration in this paper. We will only use Hodge bundles and their extensions.

## 3. Hodge theoretic weak positivity theorem

In this section, we will prove the following results. For related topics, see [Z], [B1], [PoW], [PoS1], [FnFs2], [B2], and so on. In this section, we will freely use the notation in Subsection 2.3.

**Theorem 3.1** (Hodge theoretic weak positivity theorem)**.** *Let $X$ be a smooth projective variety and let $X_0 \subset X$ be a Zariski open subset such that $D = X \setminus X_0$ is a simple normal crossing divisor on $X$. Let $V_0$ be a polarizable variation of $\mathbb{R}$-Hodge structure over $X_0$ with quasi-unipotent monodromies around $D$. If $\mathscr{A}$ is a coherent sheaf on $X$ such that $\mathscr{A}$ is contained in the kernel of the logarithmic Higgs field*

$$\theta\colon \operatorname{Gr}_F^{\bullet}\,{}^{\ell}\mathscr{V} \to \Omega^1_X(\log D) \otimes \operatorname{Gr}_F^{\bullet}\,{}^{\ell}\mathscr{V},$$

*then the dual coherent sheaf $\mathscr{A}^{\vee}$ is weakly positive.*

**Corollary 3.2** (Pseudo-effectivity)**.** *Let $X$ be a smooth projective variety and let $X_0 \subset X$ be a Zariski open subset such that $D = X \setminus X_0$ is a simple normal crossing divisor on $X$. Let $V_0$ be a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure over $X_0$. If $\mathscr{A}$ is a coherent sheaf on $X$ such that $\mathscr{A}$ is contained in the kernel of the logarithmic Higgs field*

$$\theta\colon \operatorname{Gr}_F^{\bullet}\,{}^{\ell}\mathscr{V} \to \Omega^1_X(\log D) \otimes \operatorname{Gr}_F^{\bullet}\,{}^{\ell}\mathscr{V},$$

*then $\widehat{\det}\mathscr{A}^{\vee}$ is a pseudo-effective line bundle on $X$.*

From now, we will prove Theorem 3.1 and Corollary 3.2. We do not use Saito's theory of mixed Hodge modules (see [PoS1] and [PoW]) nor any deep results due to Simpson and Mochizuki (see [B2]). Our approach here is traditional and classical and is based on the theory of variations of mixed Hodge structure (see [FnFs1], [FnFs2], and [Fs2]).

*Proof of Theorem 3.1.* Let $U$ be the largest Zariski open subset of $X$ such that $\mathscr{A}|_U$ is locally free. Since $\mathscr{A}$ is torsion-free, we have $\operatorname{codim}_X(X \setminus U) \geq 2$. By Kawamata's unipotent reduction theorem, we have a finite surjective flat morphism $f\colon Y \to X$ from a smooth projective variety such that $f^{-1}D$ is a simple normal crossing divisor on $Y$ and that $f^{-1}V_0$ is a polarizable variation of $\mathbb{R}$-Hodge structure on $Y_0 := Y \setminus f^{-1}D$ with unipotent monodromies around $f^{-1}D$. By considering the canonical extension of the system of Hodge bundles associated to $f^{-1}V_0$, we have

$$\theta_Y\colon \operatorname{Gr}_F^{\bullet}\mathscr{V}_Y \to \Omega^1_Y(\log f^{-1}D) \otimes \operatorname{Gr}_F^{\bullet}\mathscr{V}_Y,$$

where $\mathscr{V}_Y$ is the canonical extension of $f^{-1}\mathbb{V}_0 \otimes \mathscr{O}_{Y_0}$. Then $f^*\mathscr{A}$ is contained in the kernel of $\theta_Y$. If $(f^*\mathscr{A})^{\vee}$ is weakly positive, then it is obvious that $f^*(\mathscr{A}^{\vee}|_U)$ is weakly positive. Hence we see that $\mathscr{A}^{\vee}$ is also weakly positive. Therefore, by replacing $\mathscr{A}$ and $V_0$ with $f^*\mathscr{A}$ and $f^{-1}V_0$, respectively, we may assume that $V_0$ has unipotent monodromies around $D$. We apply the flattening theorem to $\mathscr{G}/\mathscr{A}$, where $\mathscr{G} := \operatorname{Gr}_F^{\bullet}\mathscr{V}$. Then we get a projective birational morphism $f\colon Y \to X$ from a smooth projective variety $Y$ such that $f^*(\mathscr{G}/\mathscr{A})/\text{torsion}$ is locally free and that $f^{-1}D$ is a simple normal crossing divisor on $Y$. By construction, we obtain a subbundle $\mathscr{E}$ of $f^*\mathscr{G}$ such that there exists a generically isomorphic injection $f^*\mathscr{A} \hookrightarrow \mathscr{E}$ on $f^{-1}(U)$. Thus $\mathscr{E}$ is contained in the kernel of $f^*\theta$, where

$$f^*\theta\colon \operatorname{Gr}_F^{\bullet} f^*\mathscr{V} \to \Omega^1_Y(\log f^{-1}D) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_F^{\bullet} f^*\mathscr{V}.$$

Hence $\mathscr{E}^{\vee}$ is a nef locally free sheaf on $Y$ by [FnFs2, Corollary 1.6] (see also [Fs2]). In particular, $\mathscr{E}^{\vee}$ is weakly positive. Since we have a generically isomorphic injection

$\mathscr{E}^\vee \hookrightarrow (f^*\mathscr{A})^\vee$ on $f^{-1}(U)$, $(f^*\mathscr{A})^\vee$ is weakly positive on $f^{-1}(U)$. This implies that $\mathscr{A}^\vee$ is weakly positive on $U$. Hence, $\mathscr{A}^\vee$ is a weakly positive reflexive sheaf on $X$ since $\operatorname{codim}_X(X \setminus U) \geq 2$. This is what we wanted. □

*Proof of Corollary 3.2.* Let $\{{}^\ell\mathscr{W}_i\}$ be the lower canonical extension of the weight filtration. Then we put $\mathscr{A}_i := \mathscr{A} \cap {}^\ell\mathscr{W}_i$ for every $i$. We consider

$$\operatorname{Gr}_k^W \theta \colon \operatorname{Gr}_F^\bullet \operatorname{Gr}_k^W {}^\ell\mathscr{V} \to \Omega_X^1(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}_F^\bullet \operatorname{Gr}_k^W {}^\ell\mathscr{V}$$

for every $k$. By construction, $\mathscr{A}_i/\mathscr{A}_{i-1}$ is contained in the kernel of $\operatorname{Gr}_i^W \theta$ for every $i$. Hence, $(\mathscr{A}_i/\mathscr{A}_{i-1})^\vee$ is weakly positive for every $i$ by Theorem 3.1. Let $U$ be the largest Zariski open subset of $X$ where $\mathscr{A}_i$, $\mathscr{A}_{i-1}$, and $\mathscr{A}_i/\mathscr{A}_{i-1}$ are locally free on $U$ for all $i$. Then we have $\operatorname{codim}_X(X \setminus U) \geq 2$. Moreover, on $U$, we have the following short exact sequence

$$0 \to (\mathscr{A}_i/\mathscr{A}_{i-1})^\vee \to \mathscr{A}_i^\vee \to \mathscr{A}_{i-1}^\vee \to 0$$

for every $i$. Note that $(\mathscr{A}_i/\mathscr{A}_{i-1})^\vee$ is weakly positive on $U$ for every $i$. Therefore, we can inductively check that $\widehat{\det}\mathscr{A}_i^\vee$ is weakly positive on $U$ for every $i$. Thus, $\widehat{\det}\mathscr{A}^\vee$ is weakly positive on $U$. This implies that $\widehat{\det}\mathscr{A}^\vee$ is a pseudo-effective line bundle since $X$ is projective. We finish the proof. □

By the proof of Corollary 3.2, the following remark is obvious.

**Remark 3.3.** The proof of Corollary 3.2 says that there exists a finite sequence of coherent sheaves

$$0 \subset \cdots \subset \mathscr{A}_{i-1} \subset \mathscr{A}_i \subset \mathscr{A}_{i+1} \subset \cdots \subset \mathscr{A}_l =: \mathscr{A}$$

such that $\mathscr{A}_i/\mathscr{A}_{i-1}$ is torsion-free and $(\mathscr{A}_i/\mathscr{A}_{i-1})^\vee$ is weakly positive for every $i$.

Note that, by Remark 2.7, the general theory of weakly positive sheaves does not imply the weak positivity of $\mathscr{A}^\vee$ in the proof of Corollary 3.2. We do not know whether $\mathscr{A}^\vee$ is weakly positive or not in Corollary 3.2.

## 4. Graded logarithmic Higgs sheaves

This section is a direct generalization of [Pa, Section 3.1]. The original idea goes back to Viehweg and Zuo (see [VZ1] and [VZ2]). Let us recall the definition of graded logarithmic Higgs sheaves following [Pa] (see also [VZ1], [VZ2], [PoS1], and so on).

**Definition 4.1** (Graded logarithmic Higgs sheaves)**.** Let $Y$ be a smooth variety and let $D$ be a simple normal crossing divisor on $Y$. A graded $\mathcal{O}_Y$-module $\mathscr{F}_\bullet = \bigoplus_k \mathscr{F}_k$ is a *graded logarithmic Higgs sheaf* with poles along $D$ if there exists a logarithmic Higgs structure

$$\phi \colon \mathscr{F}_\bullet \to \mathscr{F}_\bullet \otimes_{\mathscr{O}_Y} \Omega_Y^1(\log D)$$

such that $\phi = \bigoplus_k \phi_k$,

$$\phi_k \colon \mathscr{F}_k \to \mathscr{F}_{k+1} \otimes_{\mathscr{O}_Y} \Omega_Y^1(\log D)$$

for every $k$, and

$$\phi \wedge \phi \colon \mathscr{F}_\bullet \to \mathscr{F}_\bullet \otimes_{\mathscr{O}_Y} \Omega_Y^2(\log D)$$

is zero. We put

$$\mathscr{K}_k(\phi) := \ker\left(\phi_k \colon \mathscr{F}_k \to \mathscr{F}_{k+1} \otimes_{\mathscr{O}_Y} \Omega_Y^1(\log D)\right),$$

that is, $\mathscr{K}_k(\phi)$ is the kernel of the generalized Kodaira–Spencer map $\phi_k$ for every $k$.

Theorem 4.2 is a slight generalization of [Pa, Theorem 3.2]. One of the motivations of this paper is to understand [Pa, Theorem 3.2].

**Theorem 4.2.** *Let $f\colon X \to Y$ be a projective surjective morphism of smooth quasi-projective varieties. Let $D$ (resp. $E$) be a simple normal crossing divisor on $Y$ (resp. $X$). Assume that $f$ is smooth over $Y \setminus D$, $E$ is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. Let $\mathscr{L}$ be a line bundle on $Y$ such that*

$$\kappa(X, \omega_{(X,E)/(Y,D)} \otimes f^*\mathscr{L}^{\otimes -1}) = \kappa(X, \omega_X(E) \otimes f^*(\omega_Y(D))^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}) \geq 0.$$

*Then there exists a graded logarithmic Higgs sheaf $\mathscr{F}_\bullet$ with poles along $D$ satisfying the following properties:*

(i) *$\mathscr{L} \subset \mathscr{F}_0$ and $\mathscr{F}_k = 0$ for every $k < 0$.*
(ii) *There exists a positive integer $d$ such that $\mathscr{F}_k = 0$ for every $k > d$.*
(iii) *$\mathscr{F}_k$ is a reflexive coherent sheaf on $Y$ for every $k$.*
(iv) *Let $\mathscr{A}$ be a coherent subsheaf of $\mathscr{F}_\bullet$ contained in the kernel of $\phi$. Then $\widehat{\det}\mathscr{A}^\vee$ is a pseudo-effective line bundle on $Y$. In particular, $\widehat{\det}\mathscr{K}_k(\phi)^\vee$ is a pseudo-effective line bundle on $Y$.*

*We further assume that $E = 0$ over $Y \setminus D$. Then we have:*

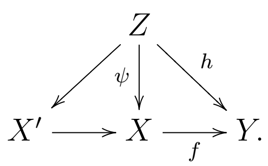
(v) *Let $\mathscr{A}$ be a coherent subsheaf of $\mathscr{F}_\bullet$ contained in the kernel of $\phi$. Then $\mathscr{A}^\vee$ is weakly positive. In particular, $\mathscr{K}_k(\phi)^\vee$ is weakly positive.*

Although the proof of Theorem 4.2 is essentially the same as that of [Pa, Theorem 3.2], we explain it in detail for the sake of completeness.

*Proof of Theorem 4.2.* We will closely follow the argument in [Pa]. Since it is sufficient to construct $\mathscr{F}_\bullet$ on the complement of a codimension two closed subset in $Y$, we will freely remove suitable codimension two closed subsets from $Y$ throughout this proof. We put

$$\mathscr{N} := \omega_X(E) \otimes f^*\left(\omega_Y(D)\right)^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}.$$

Since $\kappa(X, \mathscr{N}) \geq 0$ by assumption, we can take a section $s$ of $\mathscr{N}^{\otimes m}$ for some positive integer $m$. Let $X' \to X$ be the cyclic cover of $X$ associated to $s$ and let $Z \to X'$ be a suitable resolution of singularities. We put $\psi\colon Z \to X$ and $h := f \circ \psi\colon Z \to Y$. Hence we have the following commutative diagram:




Then there exists a natural inclusion $\psi^*\mathscr{N}^{\otimes -1} \hookrightarrow \mathscr{O}_Z$. Let $E_Z$ be a simple normal crossing divisor on $Z$ such that $\psi^{-1}(E) \subset E_Z$ and that $\psi^{-1}(E) = E_Z$ holds over the generic point of $Y$. After removing a suitable codimension two closed subset from $Y$ and taking a birational modification of $Z$ suitably, we may further assume that there exists a smooth divisor $D'$ on $Y$ such that $h\colon Z \to Y$ is smooth over $Y \setminus D'$, $E_Z$ is a relatively normal crossing over $Y \setminus D'$, and $h^{-1}(D') \subset E_Z$. As usual, we put

$$\Omega^1_{X/Y}(\log E) := \operatorname{Coker}\left(f^*\Omega^1_Y(\log D) \to \Omega^1_X(\log E)\right)$$

and

$$\Omega^1_{Z/Y}(\log E_Z) := \operatorname{Coker}\left(h^*\Omega^1_Y(\log D') \to \Omega^1_Z(\log E_Z)\right).$$

Without loss of generality, we may assume that $D$ is smooth and that $\Omega^1_{X/Y}(\log E)$ and $\Omega^1_{Z/Y}(\log E_Z)$ are both locally free sheaves. By construction, we see that $D \le D'$ holds. We consider the Koszul filtration

$$\mathrm{Koz}^q\Omega^i_X(\log E) := \mathrm{Image}\left(f^*\Omega^q_Y(\log D)\otimes \Omega^{i-q}_X(\log E) \to \Omega^i_X(\log E)\right). \tag{4.1}$$

Then we have

$$\mathrm{Koz}^q/\mathrm{Koz}^{q+1}\Omega^i_X(\log E) \simeq f^*\Omega^q_Y(\log D)\otimes \Omega^{i-q}_{X/Y}(\log E)$$

and get the following short exact sequence:

$$0 \to f^*\Omega^1_Y(\log D)\otimes \Omega^{i-1}_{X/Y}(\log E) \to \mathrm{Koz}^0/\mathrm{Koz}^2\Omega^i_X(\log E) \to \Omega^i_{X/Y}(\log E) \to 0,$$

which is denoted by $\mathscr{C}^i_{X/Y}(\log E)$. Similarly, we can define $\mathrm{Koz}^q\Omega^i_Z(\log E_Z)$ and obtain $\mathscr{C}^i_{Z/Y}(\log E_Z)$. By construction, we have a natural map $\psi^*\mathscr{C}^i_{X/Y}(\log E) \to \mathscr{C}^i_{Z/Y}(\log E_Z)$ for every $i$. By tensoring with the natural injection $\psi^*\mathscr{N}^{\otimes -1} \hookrightarrow \mathscr{O}_Z$, we have

$$\psi^*\left(\mathscr{C}^i_{X/Y}(\log E)\otimes \mathscr{N}^{\otimes -1}\right) \to \mathscr{C}^i_{Z/Y}(\log E_Z).$$

Then, by using the edge homomorphism of the Leray spectral sequence, we obtain the natural homomorphism

$$R^{d-i}f_*\left(\Omega^i_{X/Y}(\log E)\otimes \mathscr{N}^{\otimes -1}\right) \to R^{d-i}h_*\left(\psi^*\left(\Omega^i_{X/Y}(\log E)\otimes \mathscr{N}^{\otimes -1}\right)\right),$$

where $d = \dim X - \dim Y$. Thus we get the following commutative diagram of the connecting homomorphisms.

$$\begin{array}{ccc}
R^{d-i}f_*\left(\Omega^i_{X/Y}(\log E)\otimes \mathscr{N}^{\otimes -1}\right) & \longrightarrow & R^{d-i+1}f_*\left(\Omega^{i-1}_{X/Y}(\log E)\otimes \mathscr{N}^{\otimes -1}\right)\otimes \Omega^1_Y(\log D) \\
\downarrow{\scriptstyle \rho_{d-i}} & & \downarrow{\scriptstyle \rho_{d-i+1}\otimes \iota} \\
R^{d-i}h_*\left(\Omega^i_{Z/Y}(\log E_Z)\right) & \xrightarrow[\phi'_{d-i}]{} & R^{d-i+1}h_*\left(\Omega^{i-1}_{Z/Y}(\log E_Z)\right)\otimes \Omega^1_Y(\log D')
\end{array}$$

We get a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure over $Y\setminus D'$ from $h\colon (Z, E_Z) \to (Y, D')$. By Theorem 6.1 below,

$$\bigoplus_{k=0}^{d} R^k h_*\left(\Omega^{d-k}_{Z/Y}(\log E_Z)\right)$$

is the lower canonical extension of the system of Hodge bundles associated to the above variation of $\mathbb{R}$-mixed Hodge structure. Then we put

$$\mathscr{F}_k := \left(\mathrm{Image}\left(\rho_k\colon R^kf_*\left(\Omega^{d-k}_{X/Y}(\log E)\otimes \mathscr{N}^{\otimes -1}\right) \to R^kh_*\left(\Omega^{d-k}_{Z/Y}(\log E_Z)\right)\right)\right)^{\vee\vee}$$

for $0 \le k \le d$. We put $\mathscr{F}_k = 0$ if $k < 0$ or $k > d = \dim X - \dim Y$. Then we see that $\mathscr{F}_\bullet$ is a graded logarithmic Higgs sheaf with poles along $D$. Since $\rho_0$ is the pushforward $f_*$ of the inclusion

$$f^*\mathscr{L} = \omega_X(E)\otimes f^*\left(\omega_Y(D)\right)^{\otimes -1}\otimes \mathscr{N}^{\otimes -1} \to \psi_*\left(\omega_Z(E_Z)\otimes h^*\left(\omega_Y(D')\right)^{\otimes -1}\right).$$

This implies that $\mathscr{F}_0 = (\mathcal{L}\otimes f_*\mathscr{O}_X)^{\vee\vee}$. Hence $\mathscr{L} \subset \mathscr{F}_0$ holds. Therefore, (i), (ii), and (iii) hold. By taking a suitable compactification and applying Corollary 3.2, we obtain that $\widehat{\det}\mathscr{A}^\vee$ is a pseudo-effective line bundle on $Y$, which is (iv). When $E = 0$ over $Y\setminus D$, we can apply Theorem 3.1. Hence we obtain (v). We finish the proof. $\square$

**Remark 4.3** (see [Pa, Remark 3.6])**.** In Theorem 4.2, we can replace the assumption

$$\kappa(X, \omega_X(E) \otimes f^*(\omega_Y(D))^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}) \geq 0$$

with the existence of a nonzero homomorphism

$$\mathscr{L}^{\otimes N} \to \left(f_*\omega^{\otimes N}_{(X,E)/(Y,D)}\right)^{\vee\vee} \tag{4.2}$$

for some positive integer $N$. Note that (4.2) implies the existence of a nonzero section of

$$\left(\omega_X(E) \otimes f^*(\omega_Y(D))^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}\right)^{\otimes N}$$

over the complement of some codimension two closed subset $\Sigma$ in $Y$. Hence we can construct a desired graded logarithmic Higgs sheaf $\mathscr{F}_\bullet$ on $Y \setminus \Sigma$. By taking the reflexive hull of $\mathscr{F}_\bullet$, we can extend $\mathscr{F}_\bullet$ over $Y$.

For geometric applications, the following lemma is crucial.

**Lemma 4.4** ([Pa, Lemma 3.7])**.** *Let $Y$ be a smooth projective variety and let $D$ be a simple normal crossing divisor on $Y$. Let $\mathscr{F}_\bullet$ be a graded logarithmic Higgs sheaf with poles along $D$ satisfying* (i)*,* (ii)*,* (iii)*, and* (iv) *in Theorem 4.2. Then we have a pseudo-effective line bundle $\mathscr{P}$ and a nonzero homomorphism*

$$\mathscr{L}^{\otimes r} \otimes \mathscr{P} \to \left(\Omega^1_Y(\log D)\right)^{\otimes kr}$$

*for some $r > 0$ and $k \geq 0$.*

*Proof of Lemma 4.4.* We have a sequence of homomorphisms

$$\phi_k \otimes \mathrm{id}\colon \mathscr{F}_k \otimes \left(\Omega^1_Y(\log D)\right)^{\otimes k} \to \mathscr{F}_{k+1} \otimes \left(\Omega^1_Y(\log D)\right)^{\otimes k+1}.$$

Note that $\mathscr{F}_k$ is zero for $k \gg 0$ and that $\mathscr{L} \subset \mathscr{F}_0$. Hence, the line bundle $\mathscr{L}$ is contained in the kernel of $\phi_k \otimes \mathrm{id}$ for some $k \geq 0$, that is,

$$\mathscr{L} \subset \mathscr{K}_k(\phi) \otimes \left(\Omega^1_Y(\log D)\right)^{\otimes k}.$$

This implies the existence of a nonzero homomorphism

$$\mathscr{K}_k(\phi)^\vee \to \left(\Omega^1_Y(\log D)\right)^{\otimes k} \otimes \mathscr{L}^{\otimes -1}.$$

Let $\mathscr{Q}$ be the image of the above homomorphism and let $r$ be the rank of $\mathscr{Q}$. By considering the split surjection

$$\mathscr{Q}^{\otimes r} \to \det \mathscr{Q}$$

outside some suitable codimension two closed subset of $Y$, we have a nonzero homomorphism

$$\widehat{\det}\mathscr{Q} \to \left(\left(\Omega^1_Y(\log D)\right)^{\otimes k} \otimes \mathscr{L}^{\otimes -1}\right)^{\otimes r}.$$

Since $\mathscr{P} := \widehat{\det}\mathscr{Q}$ is a pseudo-effective line bundle by Theorem 4.2 (iv), we obtain a desired nonzero homomorphism. We finish the proof. □

Let us prove Theorem 1.4, which is one of the main results of this paper.

*Proof of Theorem 1.4.* By Remark 4.3, we can construct a graded logarithmic Higgs sheaf with poles along $D$ satisfying (i), (ii), (iii), and (iv) in Theorem 4.2. Then, by Lemma 4.4, we obtain a desired pseudo-effective line bundle and a nonzero homomorphism. We finish the proof. □

As a direct consequence of Lemma 4.4, we have:

**Theorem 4.5** (see [Pa, Theorem 1.7 (2)])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties and let $E$ and $D$ be simple normal crossing divisors on $X$ and $Y$, respectively. Assume that $f$ is smooth over $Y \setminus D$, $E$ is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. In this situation, if $\kappa(X, K_X + E) \geq 0$ holds, then $K_Y + D$ is pseudo-effective.*

*Proof of Theorem 4.5.* The proof of [Pa, Theorem 1.7 (2)] works. We put $\mathscr{L} := \omega_{(Y,D)}^{\otimes -1}$. By assumption, we have

$$\kappa(X, \omega_{(X,E)/(Y,D)} \otimes f^*\mathscr{L}^{\otimes -1}) = \kappa(X, K_X + E) \geq 0.$$

By Lemma 4.4, this implies that there exists a pseudo-effective line bundle $\mathscr{P}$ and a nonzero homomorphism

$$\omega_{(Y,D)}^{\otimes -r} \otimes \mathscr{P} \to \left(\Omega_Y^1(\log D)\right)^{\otimes kr}$$

for some $r > 0$ and $k \geq 0$. Thus $K_Y + D$ is pseudo-effective by [Pa, Theorem 3.9], which is due to [CP, Theorem 7.6]. We finish the proof. □

In Section 5, we will use Theorem 4.5 in the proofs of Corollary 1.13 and Theorem 1.14.

## 5. Proofs

In this section, we will prove results in Section 1. Let us start with the proof of Theorem 1.1.

*Proof of Theorem 1.1.* In [Pa, Section 2], this theorem is proved under the extra assumption that $f^{-1}(D) = E$ and $g^{-1}(D) = D'$ hold. However, we can easily see that [Pa, Proposition 2.5] implies the desired inclusion (1.1). Moreover, by the proof in [Pa, Section 2], it is easy to see that the inclusion (1.1) is an isomorphism over some nonempty Zariski open subset of $Y$. □

*Proof of Corollary 1.2.* This corollary is an easy consequence of Theorem 1.1. All we have to do is to apply Theorem 1.1 repeatedly. Note that the inclusion (1.2) is an isomorphism over some nonempty Zariski open subset of $Y$. □

We have already proved Theorem 1.4 in Section 4. Thus, let us prove Theorem 1.5.

*Proof of Theorem 1.5.* If $\kappa(Y, K_Y + D) = \dim Y$, that is, $K_Y + D$ is big, then

$$\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

holds by Maehara's theorem (see [Ma] and [Fn1]). On the other hand, if the equality

$$\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

holds, then there exists a positive integer $N$ and an ample Cartier divisor $A$ on $Y$ such that

$$f^*\mathscr{O}_Y(A) \subset \omega_{(X,E)}^{\otimes N}$$

by [Mo, Proposition 1.14]. We put $\mathscr{L} := \mathscr{O}_Y(A) \otimes \omega_{(Y,D)}^{\otimes -N}$. Hence we have

$$\mathscr{L}^{\otimes N} \subset \left(\bigotimes^N f_*\omega_{(X,E)/(Y,D)}^{\otimes N}\right)^{\vee\vee} \subset \left(f_*^{(N)}\omega_{(X^{(N)},E^{(N)})/(Y,D)}^{\otimes N}\right)^{\vee\vee}.$$

Here we used Corollary 1.2 with $s = N$. By Theorem 1.4, there exist a pseudo-effective line bundle $\mathscr{P}$ on $Y$ and a nonzero homomorphism

$$\mathscr{L}^{\otimes r} \otimes \mathscr{P} \to \left(\Omega^1_Y(\log D)\right)^{\otimes kr}$$

for some $r > 0$ and $k \geq 0$. Hence we have a nonzero homomorphism

$$\mathscr{O}_Y(A)^{\otimes r} \otimes \mathscr{P} \to \left(\Omega^1_Y(\log D)\right)^{\otimes kr} \otimes \omega^{\otimes r}_{(Y,D)}.$$

Then, by [Pa, Theorem 3.9], which is due to [CP, Theorem 7.6], $K_Y + D$ is pseudo-effective. Since we have $\omega_{(Y,D)} \subset (\Omega^1_Y(\log D))^{\otimes \dim Y}$ by definition, we have a nonzero homomorphism

$$\mathscr{O}_Y(A)^{\otimes r} \otimes \mathscr{P} \to \left(\Omega^1_Y(\log D)\right)^{\otimes N'}$$

for some positive integer $N'$. Therefore, by [Pa, Theorem 3.8], which is due to [CP, Theorem 1.3], $K_Y + D$ is the sum of an ample divisor and a pseudo-effective divisor, so it is big as desired. We finish the proof. $\square$

*Proof of Corollary 1.7.* By the easy addition formula, we have

$$\dim X = \kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

and

$$\kappa(F, (K_X + E)|_F) = \dim F,$$

where $F$ is a general fiber of $f\colon X \to Y$. By Theorem 1.5, we obtain $\kappa(Y, K_Y + D) = \dim Y$. We finish the proof. $\square$

*Proof of Theorem 1.8.* We take a sufficiently large and divisible positive integer $N$ such that

$$f^*\mathscr{O}_Y(D) \subset \omega^{\otimes N}_{(X,E)}.$$

As in the proof of Theorem 1.5 above, by Theorem 1.4, the proof of [Pa, Theorem 1.7 (1)] works. Then we obtain a positive rational number $\delta$ such that $K_X + (1-\delta)D$ is pseudo-effective. We finish the proof. $\square$

For the proof of Theorem 1.12, we prepare the following lemma. Note that we need Corollary 1.2 in the proof of Lemma 5.1.

**Lemma 5.1.** *In Conjecture 1.9, we assume $\kappa_\sigma(Y, K_Y + D) = 0$. Then we have*

$$\kappa(X, K_X + E) \leq \kappa(F, (K_X + E)|_F),$$

*where $F$ is a sufficiently general fiber of $f\colon X \to Y$.*

*Proof of Lemma 5.1.* We note that $K_Y + D$ is pseudo-effective by the assumption $\kappa_\sigma(Y, K_Y + D) = 0$. As in the proofs of Theorems 1.5 and 1.8, the proof of [Pa, Proposition 5.2] works. We finish the proof. $\square$

Let us prove Theorem 1.12.

*Proof of Theorem 1.12.* Let $\mu\colon Y' \to Y$ be a projective birational morphism from a smooth variety $Y'$ such that $D' := \mu^{-1}(D)$ is a simple normal crossing divisor on $Y'$. We replace $(Y, D)$ with $(Y', D')$ and take suitable birational modifications. Then we may assume that the Iitaka fibration $\Phi := \Phi_{|m(K_Y+D)|}\colon Y \to Z$ is a morphism onto a normal projective variety $Z$ with connected fibers, where $m$ is a sufficiently large positive integer. Let $(G, D|_G)$ (resp. $(H, E|_H)$) be a sufficiently general fiber of $\Phi$ (resp. $\Phi \circ f$), that is,

$G=\Phi^{-1}(z)$ and $H=(\Phi\circ f)^{-1}(z)$, where $z$ is a sufficiently general point of $Z$. Note that $G$ and $H$ are smooth projective varieties and $D|_G$ and $E|_H$ are simple normal crossing divisors. By construction, we see that $f|_H\colon H\to G$ satisfies that $f|_H$ is smooth over $G\setminus(D|_G)$, $E|_H$ is relatively normal crossing over $G\setminus(D|_G)$, and $(f|_H)^{-1}(D|_G)\subset E|_H$. By assumption,

$$\kappa_\sigma(G, K_G+D|_G)=\kappa(G, K_G+D|_G)=0.$$

By Lemma 5.1,

$$\kappa(H, K_H+E|_H)\leq \kappa(F, (K_X+E)|_F)$$

holds. Therefore, by applying the easy addition formula to $\Phi\circ f\colon X\to Z$,

$$\begin{aligned}\kappa(X, K_X+E)&\leq \kappa(H, K_H+E|_H)+\dim Z\\ &\leq \kappa(F, (K_X+E)|_F)+\kappa(Y, K_Y+D).\end{aligned}$$

We finish the proof. □

We need Gongyo's theorem for the proof of Corollary 1.13.

**Theorem 5.2** (see [Fn3, Proposition 4.1])**.** *Let $X$ be a smooth projective variety and let $E$ be a simple normal crossing divisor on $X$. Assume that there exists a projective birational morphism $\varphi\colon X\setminus E\to V$ onto an affine variety $V$. Then $\kappa_\sigma(X, K_X+E)=\kappa(X, K_X+E)$ holds, that is, the generalized abundance conjecture holds for $(X, E)$. In particular, if $K_X+E$ is pseudo-effective, then $\kappa(X, K_X+E)\geq 0$.*

*Proof of Theorem 5.2.* This theorem is an easy application of the minimal model program. For the details, see the proof of [Fn3, Proposition 4.1]. □

*Proof of Corollary 1.13.* If $\kappa(X, K_X+E)=-\infty$, then (1.3) is obvious. Hence we may assume that $\kappa(X, K_X+E)\geq 0$. By Theorems 4.5 and 5.2, we have $\kappa(Y, K_Y+D)\geq 0$. We can apply Theorem 5.2 to a sufficiently general fiber of the Iitaka fibration of $Y$ with respect to $K_Y+D$. Therefore, by Theorem 1.12, we obtain the desired inequality (1.3). □

Finally, we prove Theorem 1.14.

*Proof of Theorem 1.14.* The subadditivity

$$\kappa(X, K_X+E)\geq \kappa(Y, K_Y+D)+\kappa(F, (K_X+E)|_F)$$

follows from $\kappa_\sigma(X, K_X+E)=\kappa(X, K_X+E)$ (see [Fn3] and [Fn4]) or $\kappa_\sigma(F, (K_X+E)|_F)=\kappa(F, (K_X+E)|_F)$ (see [H2]). Therefore, from now on, we will prove the superadditivity

$$\kappa(X, K_X+E)\leq \kappa(Y, K_Y+D)+\kappa(F, (K_X+E)|_F). \tag{5.1}$$

If $\kappa(X, K_X+E)=-\infty$, then (5.1) is obviously true. Hence we may assume that $\kappa(X, K_X+E)\geq 0$ holds. By Theorem 4.5, $K_Y+D$ is pseudo-effective. By Conjecture 1.11, which is a special case of Conjecture 1.10, we have $\kappa(Y, K_Y+D)\geq 0$. Thus, by Theorem 1.12 and Conjecture 1.10, we obtain the desired superadditivity (5.1). We finish the proof. □

## 6. On variations of mixed Hodge structure

In this final section, for the sake of completeness, we will explain the following theorem, which is more or less well known to the experts (see [StZ], [El], and so on). Theorem 6.1 has already been used in the proof of Theorem 4.2 and is one of the main ingredients of Theorem 4.2.

**Theorem 6.1.** *Let $f\colon X \to Y$ be a proper surjective morphism from a Kähler manifold $X$ to a complex manifold $Y$ with $d = \dim X - \dim Y$ and let $\Sigma_X$ and $\Sigma_Y$ be reduced simple normal crossing divisors on $X$ and $Y$, respectively. We set $Y_0 = Y \setminus \Sigma_Y, X_0 = f^{-1}(Y_0), f_0 = f|_{X_0}\colon X_0 \to Y_0$ and $U = X \setminus \Sigma_X$ and assume that $f_0$ is a smooth morphism, $\Sigma_X \cap X_0$ is relatively normal crossing over $Y_0$, and $\operatorname{Supp} f^*\Sigma_Y \subset \operatorname{Supp}\Sigma_X$ holds. Then the local system $R^i(f_0|_U)_*\mathbb{R}_U$ underlies a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure on $Y_0$ for every $i$ such that the Hodge filtration $F$ on $\mathscr{V} = \mathscr{O}_{Y_0} \otimes R^i(f_0|_U)_*\mathbb{R}_U$ extends to a filtration $F$ on the lower canonical extension ${}^{\ell}\mathscr{V}$ satisfying the following conditions:*

(6.1.1) $\operatorname{Gr}_F^p \operatorname{Gr}_m^W({}^{\ell}\mathscr{V})$ *is locally free of finite rank for every $m, p$, and*

(6.1.2) $\operatorname{Gr}_F^p({}^{\ell}\mathscr{V})$ *coincides with $R^{i-p}f_*\Omega^p_{X/Y}(\log \Sigma_X)$ for every $p$, after removing some suitable codimension two closed subset from $Y$.*

*In the above statement, if $\Sigma_X \cap X_0 = 0$, then $R^i(f_0|_U)_*\mathbb{R}_U = R^i(f_0)_*\mathbb{R}_{X_0}$ underlies a polarizable variation of $\mathbb{R}$-Hodge structure on $Y_0$ for every $i$.*

In this section, we adopt the same approach as in Sections 3, 4, and 7 in [FnFs3]. Before starting the proof of the theorem above, we recall several facts concerning on the Koszul complex.

**6.2.** In the situation above, we set $E = (f^*\Sigma_Y)_{\mathrm{red}}$ and $D = \Sigma_X - E$. Then $D$ and $E$ are reduced simple normal crossing divisors on $X$ with no common irreducible components. The open immersions $X \setminus D \hookrightarrow X$ and $U \hookrightarrow X_0$ are denoted by $j$ and $j_0$ respectively. The situation is summarized in the commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & X \setminus D & & \\ \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle j} & & \\ X_0 & \longrightarrow & X & \longleftarrow & E \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow \\ Y_0 & \longrightarrow & Y & \longleftarrow & \Sigma_Y \end{array}$$

where the left two squares are Cartesian.

We denote by $D = \sum_{i=1}^{l} D_i$ the irreducible decomposition of $D$ and set

$$D^{(m)} = \coprod_{1 \le i_1 < \cdots < i_m \le l} D_{i_1} \cap \cdots \cap D_{i_m}$$

for $m \in \mathbb{Z}_{\ge 0}$. (For the case of $m = 0$, we set $D^{(0)} = X$ by definition.) The natural morphism from $D^{(m)}$ to $X$ is denoted by $a_m$.

In order to define the desired weight filtration on

$$R^i(f_0|_U)_*\mathbb{R}_U \simeq \mathbb{R} \otimes R^i(f_0|_U)_*\mathbb{Q}_U \simeq \mathbb{R} \otimes R^i(f_0)_*(R(j_0)_*\mathbb{Q}_U)$$

we replace $R(j_0)_*\mathbb{Q}_U$ by a Koszul complex as follows. For the detail, see [Fs1, Sections 1 and 2] (cf. [I], [St], [FnFs3, Section 7]).

The divisor $D$ on $X$ defines a log structure $\mathcal{M}$ by $\mathcal{M} := \mathscr{O}_X \cap j_*\mathscr{O}^*_{X\setminus D}$ on $X$. A morphism of abelian sheaves $\mathscr{O}_X \to \mathcal{M}^{\mathrm{gp}}$ is defined as the composite of the exponential map $\mathscr{O}_X \ni a \mapsto e^{2\pi\sqrt{-1}a} \in \mathscr{O}^*_X$ and the inclusion $\mathscr{O}^*_X \hookrightarrow \mathcal{M}^{\mathrm{gp}}$. From the morphism $\mathbf{e}\otimes\mathrm{id}\colon \mathscr{O}_X \simeq \mathscr{O}_X \otimes \mathbb{Q} \to \mathcal{M}^{\mathrm{gp}} \otimes \mathbb{Q}$, $1 \in \Gamma(X, \mathbb{Q})$ which is a global section of the kernel of $\mathbf{e}\otimes\mathrm{id}$, and a subsheaf $\mathscr{O}^*_X \otimes \mathbb{Q} \subset \mathcal{M}^{\mathrm{gp}} \otimes \mathbb{Q}$, we obtain a complex of $\mathbb{Q}$-sheaves on $X$

$$\mathrm{Kos}(\mathcal{M}) := \mathrm{Kos}(\mathbf{e}\otimes\mathrm{id}; \infty; 1)$$

equipped with a finite increasing filtration $W := W(\mathscr{O}^*_X \otimes \mathbb{Q})$ as in [Fs1, Definition 1.8] (see also [FnFs3, Section 7]). By replacing $\mathcal{M}^{\mathrm{gp}}$ by $\mathscr{O}^*_{X\setminus D}$, we obtain a complex of $\mathbb{Q}$-sheaves on $X \setminus D$, denoted by $\mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$, by the same way as above. Moreover, we have a morphism of complexes of $\mathbb{Q}$-sheaves

$$\psi\colon\ \mathrm{Kos}(\mathcal{M}) \to \Omega_X(\log D)$$

as in [Fs1, (2.4)], which preserves the filtration $W$ on the both sides.

The following two lemmas are more or less the same as Lemmas 7.6 and 7.7 in [FnFs3].

**Lemma 6.3.** *There exists a quasi-isomorphism of complexes*

$$(a_m)_*\mathbb{Q}_{D(m)}[-m] \to \mathrm{Gr}^W_m \mathrm{Kos}(\mathcal{M}) \tag{6.1}$$

*for all* $m \in \mathbb{Z}_{\geq 0}$.

**Lemma 6.4.** *The quasi-isomorphism* (6.1) *makes the diagram*

$$\begin{array}{ccc} (a_m)_*\mathbb{Q}_{D(m)}[-m] & \longrightarrow & \mathrm{Gr}^W_m \mathrm{Kos}(\mathcal{M}) \\ {\scriptstyle (2\pi\sqrt{-1})^{-m}\iota[-m]}\downarrow & & \downarrow{\scriptstyle \mathrm{Gr}^W_m \psi} \\ (a_m)_*\Omega_{D(m)}[-m] & \longrightarrow & \mathrm{Gr}^W_m \Omega_X(\log D) \end{array}$$

*commutative, where the bottom arrow is the inverse of the usual residue isomorphism and* $\iota$ *is the composite of the natural morphisms* $\mathbb{Q}_{D(m)} \hookrightarrow \mathbb{C}_{D(m)}$ *and* $\mathbb{C}_{D(m)} \to \Omega_{D(m)}$. *Consequently, the morphism* $\psi$ *induces a filtered quasi-isomorphism*

$$\mathbb{C} \otimes_{\mathbb{Q}} \mathrm{Kos}(\mathcal{M}) \to \Omega_X(\log D)$$

*with respect to the filtration* $W$ *on the both sides.*

**6.5.** Since $\mathrm{Kos}(\mathcal{M})|_{X\setminus D} = \mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$, we obtain a morphism of the complexes of $\mathbb{Q}$-sheaves $\mathrm{Kos}(\mathcal{M}) \to Rj_* \mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$ such that the diagram in the derived category

$$\begin{array}{ccc} \mathrm{Kos}(\mathcal{M}) & \longrightarrow & Rj_* \mathrm{Kos}(\mathscr{O}^*_{X\setminus D}) \\ {\scriptstyle \psi}\downarrow & & \downarrow \\ \Omega_X(\log D) & \longrightarrow & Rj_*\Omega_{X\setminus D} \end{array} \tag{6.2}$$

is commutative, where the right vertical arrow is induced from the morphism $\mathrm{Kos}(\mathscr{O}^*_{X\setminus D}) \to \Omega_{X\setminus D}$ defined by the same way as $\psi$.

**Lemma 6.6.** *We have the natural isomorphism*

$$\mathrm{Kos}(\mathcal{M}) \xrightarrow{\simeq} Rj_*\mathbb{Q}_{X\setminus D}$$

*in the derived category. By restricting it to* $X_0$, *we obtain the isomorphism*

$$\mathrm{Kos}(\mathcal{M})|_{X_0} \xrightarrow{\simeq} R(j_0)_*\mathbb{Q}_U$$

*in the derived category.*

*Proof.* It is sufficient to prove that the morphism

$$\mathbb{C} \otimes_{\mathbb{Q}} \operatorname{Kos}(\mathcal{M}) \to \mathbb{C} \otimes_{\mathbb{Q}} Rj_* \operatorname{Kos}(\mathscr{O}^*_{X \setminus D})$$

is an isomorphism. Since we have the canonical quasi-isomorphism $\mathbb{Q}_{X\setminus D} \to \operatorname{Kos}(\mathscr{O}^*_{X\setminus D})$, the right vertical arrow in (6.2) induces an isomorphism $\mathbb{C} \otimes Rj_* \operatorname{Kos}(\mathscr{O}^*_{X\setminus D}) \simeq Rj_*\Omega_{X\setminus D}$ in the derived category. Hence Lemma 6.4 implies the conclusion because the bottom arrow in (6.2) is known to be isomorphisms in the derived category. □

Now, we prove Theorem 6.1.

*Proof of Theorem 6.1.* We set $E \cap D^{(m)} = (a_m)^*E$, which is a simple normal crossing divisor on $D^{(m)}$ for every $m \in \mathbb{Z}_{\geq 0}$.

First, we assume that the following conditions are satisfied:

(6.6.1) $\Sigma_Y$ is a smooth hypersurface in $Y$, and

(6.6.2) $\Omega^1_{D^{(m)}/Y}(\log E \cap D^{(m)})$ is locally free of finite rank for all $m \in \mathbb{Z}_{\geq 0}$.

On the log de Rham complex $\Omega_X(\log \Sigma_X)$, we have the filtration $W(D)$, which induces a finite increasing filtration $W(D)$ on the relative log de Rham complex $\Omega_{X/Y}(\log \Sigma_X)$. A morphism of complexes $\overline{\psi}\colon \operatorname{Kos}(\mathcal{M}) \to \Omega_{X/Y}(\log \Sigma_X)$ is obtained by composing the three morphisms, $\psi\colon \operatorname{Kos}(\mathcal{M}) \to \Omega_X(\log D)$, the inclusion $\Omega_X(\log D) \to \Omega_X(\log \Sigma_X)$ and $\Omega_X(\log \Sigma_X) \to \Omega_{X/Y}(\log \Sigma_X)$. We set

$$\begin{aligned} K &= ((K_{\mathbb{R}}, W), (K_{\mathscr{O}}, W, F), \alpha) \\ &= (\mathbb{R} \otimes (Rf_* \operatorname{Kos}(\mathcal{M}), W)|_{Y_0}, (Rf_*\Omega_{X/Y}(\log \Sigma_X), W(D), F), \operatorname{id} \otimes (Rf_*\overline{\psi})|_{Y_0}), \end{aligned}$$

which is a triple as in [FnFs3, 3.7] on $Y$. Since

$$(\operatorname{Gr}^{W(D)}_m \Omega_{X/Y}(\log \Sigma_X), F) \simeq (a_m)_* \left(\Omega_{D^{(m)}/Y}(\log E \cap D^{(m)})[-m], F[-m]\right)$$

as filtered complexes by Lemmas 6.3 and 6.4, we have

$$\begin{aligned} &(\mathbb{R} \otimes \operatorname{Gr}^W_m \operatorname{Kos}(\mathcal{M}), (\operatorname{Gr}^{W(D)}_m \Omega_{X/Y}(\log \Sigma_X), F), \operatorname{id} \otimes \operatorname{Gr}^W_m \overline{\psi}) \\ &\qquad \simeq (a_m)_* \left(\mathbb{R}_{D^{(m)}}, (\Omega_{D^{(m)}/Y}(\log E \cap D^{(m)}), F[-m]), (2\pi\sqrt{-1})^{-m}\overline{\iota}\right)[-m] \end{aligned} \tag{6.4}$$

for every $m$, where $\overline{\iota}$ denote the composite of the canonical morphisms

$$\mathbb{R}_{D^{(m)}} \hookrightarrow \mathbb{C}_{D^{(m)}} \to \Omega_{D^{(m)}} \hookrightarrow \Omega_{D^{(m)}}(\log E \cap D^{(m)}) \to \Omega_{D^{(m)}/Y}(\log E \cap D^{(m)}).$$

Thus we can easily check that $K|_{Y_0}$ satisfies the conditions (3.7.1)–(3.7.3) in [FnFs3, 3.7] because $fa_m\colon D^{(m)} \to Y$ is smooth over $Y_0$ for every $m$. By Lemma 3.3 together with (3.7.6) of [FnFs3], we obtain a polarizable variation of $\mathbb{R}$-Hodge structure

$$\left.\left(\mathbb{R} \otimes \operatorname{Gr}^W_m R^if_* \operatorname{Kos}(\mathcal{M}), (\operatorname{Gr}^{W(D)}_m R^if_*\Omega_{X/Y}(\log \Sigma_X), F), \operatorname{id} \otimes \operatorname{Gr}^W_m R^if_*\overline{\psi}\right)\right|_{Y_0}$$

of weight $m + i$ for every $i, m$. Using the Koszul filtration (4.1) in the proof of Theorem 4.2, we can check the Griffiths transversality for $(\operatorname{Gr}^{W(D)}_m R^if_*\Omega_{X/Y}(\log \Sigma_X), F)|_{Y_0}$ by the same way as in [KtO] (cf. [FnFs1, Lemma 4.5]). Thus the triple

$$\left((\mathbb{R} \otimes R^if_* \operatorname{Kos}(\mathcal{M}), W[i]), (R^if_*\Omega_{X/Y}(\log \Sigma_X), W(D)[i], F), \operatorname{id} \otimes R^if_*\overline{\psi}\right)|_{Y_0}$$

is a graded polarizable variation of $\mathbb{R}$-Hodge structure on $Y_0$ by [FnFs3, (3.7.5)], and all the local monodromies of $R^if_* \operatorname{Kos}(\mathcal{M})|_{Y_0}$ along $\Sigma_Y$ are quasi-unipotent by [FnFs3, (3.7.4)]. Since $R^if_* \operatorname{Kos}(\mathcal{M})|_{Y_0} \simeq R^i(f_0|_U)_*\mathbb{Q}_U$ by Lemma 6.6, the local system $R^i(f_0|_U)_*\mathbb{R}_U$ is of quasi-unipotent local monodromy along $\Sigma_Y$ and underlies a graded polarizable variation of $\mathbb{R}$-Hodge structure on $Y_0$.

Moreover, $K$ satisfies all the assumptions in Theorem 3.9 of [FnFs3] by (6.4) and by [FnFs3, Lemma 4.3]. Therefore, there exist isomorphisms

$$R^i f_* \Omega_{X/Y}(\log \Sigma_X) \simeq {}^{\ell} R^i f_* \Omega_{X/Y}(\log \Sigma_X)|_{Y_0}$$
$$W(D)_m R^i f_* \Omega_{X/Y}(\log \Sigma_X) \simeq {}^{\ell} W(D)_m R^i f_* \Omega_{X/Y}(\log \Sigma_X)|_{Y_0}$$

whose restriction to $Y_0$ coincide with the identities and the natural isomorphisms

$$F^p R^i f_* \Omega_{X/Y}(\log \Sigma_X) \simeq R^i f_* F^p \Omega_{X/Y}(\log \Sigma_X)$$
$$\mathrm{Gr}_F^p R^i f_* \Omega_{X/Y}(\log \Sigma_X) \simeq R^{i-p} f_* \Omega^p_{X/Y}(\log \Sigma_X)$$

by (3.9.1) and by (3.9.3) of [FnFs3, Theorem 3.9], and $\mathrm{Gr}_F^p \mathrm{Gr}_m^{W(D)} R^i f_* \Omega_{X/Y}(\log \Sigma_X)$ is locally free of finite rank for every $i, m, p \in \mathbb{Z}$ by (3.9.4) of [FnFs3, Theorem 3.9]. Thus the filtration $F$ on $R^i f_* \Omega_{X/Y}(\log \Sigma_X)$ satisfies (6.1.1) and (6.1.2).

For the general case, by Lemma 4.6 of [FnFs3], there exists a closed subspace $\Sigma'_Y \subset \Sigma_Y$ with $\mathrm{codim}_Y \Sigma'_Y \geq 2$ such that $f \colon X \to Y$ restricted over $Y \setminus \Sigma'_Y$ satisfies the conditions (6.6.1) and (6.6.2). Therefore the filtration $F$ on ${}^{\ell}\mathscr{V}|_{Y \setminus \Sigma'_Y}$ is obtained by the argument above. Moreover, the filtration $F$ on $\mathrm{Gr}_m^{W(D)} \mathscr{V}$ extends to $\mathrm{Gr}_m^{W(D)}({}^{\ell}\mathscr{V})$ by Schmid's nilpotent orbit theorem. Applying Lemma 1.11.2 of [Ks], we obtain an extension of $F$ on ${}^{\ell}\mathscr{V}$ satisfying (6.1.1) and (6.1.2).

In order to prove the admissibility, we may assume $(Y, \Sigma_Y) = (\Delta, \{0\})$ by the definition of admissibility (cf. [Ks, 1,9]). Pulling back the variation by the morphism

$$\Delta \ni t \mapsto t^m \in \Delta \tag{6.5}$$

changes the logarithm of the unipotent part of the monodromy automorphism to its multiple by $m$. Therefore the existence of the relative monodromy weight filtration can be checked after the pull-back by the morphism (6.5). Moreover, Lemma 1.9.1 of [Ks] enables us to check the extendability of the Hodge filtration after the pull-back by the morphism (6.5). Thus we may assume that $f \colon X \to \Delta$ satisfies the following three conditions:

- $X$ is a Kähler manifold,
- $f^{-1}(0)_{\mathrm{red}}$ is a simple normal crossing divisor on $X$, and
- the local system $R^i(f_0|_U)_* \mathbb{R}_U$, which underlies the variation of mixed Hodge structure in question, is of unipotent monodromy automorphism around the origin.

Then we obtain the conclusion by [El, Théorème I.1.10 and Proposition I.3.10] (cf. [StZ, §5], [PS, Theorem 14.51], [BE, Theorem 8.2.13], and so on). □

**Remark 6.7.** As in [FnFs3, Section 7], we can use Koszul complexes when we construct a cohomological $\mathbb{Q}$-mixed Hodge complex for the proof of the above admissibility.

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