ON THE KODAIRA DIMENSION

OSAMU FUJINO AND TARO FUJISAWA

Abstract. We discuss the behavior of the Kodaira dimension under smooth morphisms.

Contents

1. Introduction

We will discuss the behavior of the Kodaira dimension under smooth morphisms. Throughout this paper, we will work over $\mathbb{C}$, the field of complex numbers. One of the motivations of this paper is to understand [Pa]. In [Pa], Sung Gi Park established the following striking and unexpected theorem.

Theorem 1.1 (Park's logarithmic base change theorem, see [Pa, Theorem 1.2])**.** *Let X, Y, and Y' be smooth quasi-projective varieties and let E, D, and D' be simple normal crossing divisors on X, Y, and Y', respectively. Let $f\colon X \to Y$ and $g\colon Y' \to Y$ be projective surjective morphisms such that f and g are smooth over $Y \setminus D$, $f^{-1}(D) \subset E$ and $g^{-1}(D) \subset D'$, and that E and D' are relatively normal crossing over $Y \setminus D$. Let X' be the union of the irreducible components of $X \times_Y Y'$ dominating Y and $E' := g'^{-1}(E) \cup f'^{-1}(D')$. We consider the following commutative diagram:*

$$\begin{array}{ccccc} (X, E) & \xleftarrow{g'} & (X', E') & \xleftarrow{\mu} & (X'', E'') \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & \swarrow {\scriptstyle f''} & \\ (Y, D) & \xleftarrow[g]{} & (Y', D') & & \end{array}$$

Date: 2024/7/9, version 0.13.
2020 *Mathematics Subject Classification.* Primary 14D06; Secondary 14E30.
Key words and phrases. logarithmic base change theorem, Kodaira dimensions, logarithmic Kodaira dimensions, variation of mixed Hodge structure, Higgs sheaves, pseudo-effectivity, weak positivity, semipositivity.

where $\mu\colon X'' \to X'$ is a projective resolution of singularities such that μ is an isomorphism over $Y \setminus D$, E'' is a simple normal crossing divisor on X'' such that E'' coincides with E' over $Y \setminus D$ and that $(g \circ f'')^{-1}(D) \subset E''$. We put $\omega_{(X,E)} := \omega_X \otimes \mathscr{O}_X(E)$, $\omega_{(Y,D)} := \omega_Y \otimes \mathscr{O}_Y(D)$, $\omega_{(X,E)/(Y,D)} := \omega_{(X,E)} \otimes f^*\omega_{(Y,D)}^{\otimes -1}$, and so on. Then, for every positive integer N, there exists a generically isomorphic inclusion

$$\left(f_*\omega^{\otimes N}_{(X,E)/(Y,D)} \otimes g_*\omega^{\otimes N}_{(Y',D')/(Y,D)}\right)^{\vee\vee} \subset \left(h_*\omega^{\otimes N}_{(X'',E'')/(Y,D)}\right)^{\vee\vee} \tag{1.1}$$

where $h := g \circ f''$.

Precisely speaking, Park treated only the case where $f^{-1}(D) = E$ and $g^{-1}(D) = D'$ hold. However, we can easily see that [Pa, Proposition 2.5] implies the inclusion (1.1). As a direct and easy consequence of Theorem 1.1, he obtained the following very important result.

Corollary 1.2 (Park's logarithmic fiber product trick, see [Pa, Corollary 1.4])**.** *Let $f\colon X \to Y$ be a projective surjective morphism of smooth quasi-projective varieties. Let E and D be simple normal crossing divisors on X and Y, respectively. Assume that f is smooth over $Y \setminus D$, E is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. Let s be a positive integer and let $f^s\colon X^s := X \times_Y \cdots \times_Y X \to Y$ be the s-fold fiber product of $f\colon X \to Y$. We put $E^s := \sum_{i=1}^s p_i^*E$, where $p_i\colon X^s \to X$ is the i-th projection for every i. Let $\varphi\colon X^{(s)} \to X^s$ be a projective resolution of singularities of the union of the irreducible components of X^s dominating Y such that φ is an isomorphism over $Y \setminus D$. Let $E^{(s)}$ be a simple normal crossing divisor on $X^{(s)}$ such that $E^{(s)}$ coincides with E^s over $Y \setminus D$ and that $(f^{(s)})^{-1}(D) \subset E^{(s)}$, where $f^{(s)} := f^s \circ \varphi$. Then we have the following generically isomorphic injection*

$$\left(\bigotimes^s f_*\left(\omega^{\otimes N}_{(X,E)/(Y,D)}\right)\right)^{\vee\vee} \hookrightarrow \left(f^{(s)}_*\left(\omega^{\otimes N}_{(X^{(s)},E^{(s)})/(Y,D)}\right)\right)^{\vee\vee} \tag{1.2}$$

for every positive integer N. Note that $X^{(s)}$ is smooth, $f^{(s)}$ is smooth over $Y \setminus D$, $E^{(s)}$ is a simple normal crossing divisor on $X^{(s)}$ and is relatively normal crossing over $Y \setminus D$, and $(f^{(s)})^{-1}(D) \subset E^{(s)}$.

We make a small remark on Corollary 1.2.

Remark 1.3. If $f\colon X \to Y$ has connected fibers in Corollary 1.2, then $X^{(s)}$ is a smooth variety, that is, $X^{(s)}$ is connected, by construction. In general, $X^{(s)}$ may have some connected components.

In this paper, we will establish the following theorem, which is a slight generalization of [Pa, Section 3], as an application of Corollary 1.2 and the theory of variations of mixed Hodge structure. In [Pa], Theorem 1.4 was treated under the assumption that $f^{-1}(D) = E$ holds.

Theorem 1.4 (see [Pa, Section 3])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties and let E and D be simple normal crossing divisors on X and Y, respectively. Assume that f is smooth over $Y \setminus D$, E is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. Let $\mathscr{L}$ be a line bundle on Y such that there exists a nonzero homomorphism*

$$\mathscr{L}^{\otimes N} \to \left(f_*\omega^{\otimes N}_{(X,E)/(Y,D)}\right)^{\vee\vee}$$

for some positive integer N. Then there exists a pseudo-effective line bundle $\mathscr{P}$ on Y and a nonzero homomorphism

$$\mathscr{L}^{\otimes r} \otimes \mathscr{P} \to \left(\Omega^1_Y(\log D)\right)^{\otimes kr}$$

for some $r > 0$ and $k \geq 0$.

By using Theorem 1.4, we will prove the following results.

Theorem 1.5 (see [Pa, Theorem 1.5])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let E and D be simple normal crossing divisors on X and Y, respectively. Assume that $f^{-1}(D) \subset E$, f is smooth over $Y \setminus D$, and E is relatively normal crossing over $Y \setminus D$. We further assume that $\kappa(F, (K_X + E)|_F) \geq 0$ holds, where F is a sufficiently general fiber of $f\colon X \to Y$. Then $\kappa(Y, K_Y + D) = \dim Y$ holds if and only if $\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$.*

Remark 1.6. In Theorem 1.5, it is well known that

$$\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

holds under the assumption that $\kappa(Y, K_Y + D) = \dim Y$. This is due to Maehara (see [Ma] and [Fn1]). Hence the opposite implication is new and nontrivial.

Corollary 1.7 is an obvious consequence of Theorem 1.5.

Corollary 1.7. *Let $f\colon X \to Y$, D, and E be as in Theorem 1.5, that is, $f^{-1}(D) \subset E$, f is smooth over $Y \setminus D$, and E is relatively normal crossing over $Y \setminus D$. If $\kappa(X, K_X + E) = \dim X$, then $\kappa(Y, K_Y + D) = \dim Y$ and $\kappa(F, (K_X + E)|_F) = \dim F$ hold, where F is a general fiber of $f\colon X \to Y$.*

Theorem 1.8 (see [Pa, Theorem 1.7 (1)])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties and let E and D be simple normal crossing divisors on X and Y, respectively. Assume that f is smooth over $Y \setminus D$, E is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. In this situation, if $\kappa(X, K_X + E - \varepsilon f^*D) \geq 0$ holds for some positive rational number ε, then there exists some positive rational number δ such that $K_Y + (1 - \delta)D$ is pseudo-effective.*

If $f^{-1}(D) = E$ in Theorems 1.5 and 1.8, then they are nothing but [Pa, Theorem 1.5] and [Pa, Theorem 1.7], respectively. In Theorem 1.8, if $\kappa(X, K_X + E) \geq 0$, then we can prove that $K_Y + D$ is pseudo-effective without using Theorem 1.4. It will be treated in Theorem 4.5.

Let us consider a conjecture on the behavior of the (logarithmic) Kodaira dimension under smooth morphisms.

Conjecture 1.9 (see [Po, Conjecture 3.6] and [Pa, Conjecture 5.1])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let E and D be simple normal crossing divisors on X and Y, respectively. Assume that f is smooth over $Y \setminus D$, E is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. Then*

$$\kappa(X, K_X + E) = \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F)$$

holds, where F is a sufficiently general fiber of $f\colon X \to Y$.

For the classical Iitaka subadditivity conjecture and some related topics, see [Fn5]. For the details of relatively new conjectures on the behavior of the Kodaira dimension under morphisms of smooth complex varieties, see [Po] and the references therein.

We explain some related conjectures. Conjecture 1.10 is a special case of the generalized abundance conjecture, which is one of the most important conjectures in the theory of minimal models. For the details of Conjecture 1.10, see [Fn2, Section 4.1].

Conjecture 1.10 (Generalized abundance conjecture for projective smooth pairs)**.** *Let X be a smooth projective variety and let E be a simple normal crossing divisor on X. Then*

$$\kappa(X, K_X + E) = \kappa_\sigma(X, K_X + E)$$

holds, where κ_σ denotes Nakayama's numerical dimension.

Conjecture 1.10 is still widely open. Conjecture 1.10 contains Conjecture 1.11 as a special case. For the details of the nonvanishing conjecture, see, for example, [Fn2, Section 4.8] and [H1].

Conjecture 1.11 (Nonvanishing conjecture for projective smooth pairs)**.** *Let X be a smooth projective variety and let E be a simple normal crossing divisor on X. Assume that $K_X + E$ is pseudo-effective. Then*

$$\kappa(X, K_X + E) \geq 0$$

holds.

On Conjecture 1.9, we have a partial result, which is obviously a generalization of [Pa, Theorem 1.12].

Theorem 1.12 (Superadditivity, see [Pa, Theorem 1.12])**.** *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let E and D be simple normal crossing divisors on X and Y, respectively. Assume that f is smooth over $Y \setminus D$, E is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. We further assume that $\kappa(Y, K_Y + D) \geq 0$ and that the generalized abundance conjecture holds for sufficiently general fibers of the Iitaka fibration of Y with respect to $K_Y + D$. Then*

$$\kappa(X, K_X + E) \leq \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F)$$

holds, where F is a sufficiently general fiber of $f\colon X \to Y$.

We have already known that Conjecture 1.10 holds true when $X \setminus E$ is affine. More generally, Conjecture 1.10 holds true under the assumption that there exists a projective birational morphism $X \setminus E \to V$ onto an affine variety V. Hence we have:

Corollary 1.13. *Let $f\colon X \to Y$, E, and D be as in Conjecture 1.9. We assume that $Y \setminus D$ is affine. Then the following superadditivity*

$$\kappa(X, K_X + E) \leq \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F) \tag{1.3}$$

holds.

In Conjecture 1.9, we have already known that the subadditivity

$$\kappa(X, K_X + E) \geq \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F)$$

follows from the generalized abundance conjecture (see Conjecture 1.10). For the details, see [Fn3], [Fn4], and [H2]. Hence, if the generalized abundance conjecture holds true, then Conjecture 1.9 is also true. Roughly speaking, we have:

Theorem 1.14. *Let $f\colon X \to Y$, E, and D be as in Conjecture 1.9. We assume that Conjecture 1.10 holds true. Then we have*

$$\kappa(X, K_X + E) = \kappa(Y, K_Y + D) + \kappa(F, (K_X + E)|_F),$$

where F is a sufficiently general fiber of $f\colon X \to Y$, that is, Conjecture 1.9 holds true.

We look at the organization of this paper. In Section 2, we collect some basic definitions and results necessary for this paper for the reader's convenience. Subsection 2.1 collects some basic definitions. In Subsection 2.2, we recall various notions of positivities. In Subsection 2.3, we explain systems of Hodge bundles. In Section 3, we discuss Hodge theoretic weak positivity results. They seem to be more general than the usual ones slightly. In Section 4, we construct graded logarithmic Higgs sheaves and prove Theorem 1.4 following [Pa, Section 3.1] closely. Here we use variations of mixed Hodge structure. In Section 5, we prove results explained in Section 1. In Section 6, we discuss variations of mixed Hodge structure necessary for Section 4 for the sake of completeness. In the final section: Section 7, we prove a variant of semipositivity theorems of Fujita–Zucker–Kawamata necessary for Section 3. Although we could not find any easily accessible accounts of the topics in Sections 6 and 7, the results may be more or less known to the experts.

Acknowledgments. The first author was partially supported by JSPS KAKENHI Grant Numbers JP19H01787, JP20H00111, JP21H00974, JP21H04994, JP23K20787. The second author was partially supported by JSPS KAKENHI Grant Number JP20K03542.

2. Preliminaries

In this section, we will collect some basic definitions and properties necessary for this paper.

2.1. **Basic definitions.** We will work over $\mathbb{C}$, the field of complex numbers. A *variety* means an irreducible and reduced separated scheme of finite type over $\mathbb{C}$.

2.1.1 (κ and κ_σ)**.** Let X be a smooth projective variety. Then $\kappa(X, \bullet)$ and $\kappa_\sigma(X, \bullet)$ denote *the Iitaka dimension* and *Nakayama's numerical dimension* of $\bullet$, respectively, where $\bullet$ is a $\mathbb{Q}$-Cartier divisor or a line bundle on X. For the details of κ and κ_σ, see [U], [Mo], [N], and so on.

2.1.2 (Canonical bundles and log canonical bundles)**.** Let X be a smooth variety and let E be a simple normal crossing divisor on X. Then we put

$$\omega_X := \det \Omega^1_X$$

and

$$\omega_{(X,E)} := \omega_X \otimes \mathscr{O}_X(E) =: \omega_X(E).$$

We note that

$$\omega_{(X,E)} = \det \Omega^1_X(\log E).$$

Let $f\colon X \to Y$ be a surjective morphism of smooth varieties and let D be a simple normal crossing divisor on Y. Then we put

$$\omega_{(X,E)/(Y,D)} := \omega_{(X,E)} \otimes f^*\omega^{\otimes -1}_{(Y,D)} = \omega_X(E) \otimes (f^*\omega_Y(D))^{\otimes -1}.$$

Let K_X be a Cartier divisor with $\mathscr{O}_X(K_X) \simeq \omega_X$. Then it is obvious that $\omega_{(X,E)} \simeq \mathscr{O}_X(K_X + E)$ holds.

2.1.3 (Duals and double duals)**.** Let $\mathscr{F}$ be a coherent sheaf on a smooth variety X. Then we put

$$\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$$

and

$$\mathscr{F}^{\vee\vee} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^\vee, \mathscr{O}_X).$$

We further assume that $\mathscr{F}$ is torsion-free. Then $\widehat{\det}\mathscr{F}$ and $\widehat{S}^\alpha(\mathscr{F})$ denote $(\det \mathscr{F})^{\vee\vee}$ and $S^\alpha(\mathscr{F})^{\vee\vee}$, respectively, where $S^\alpha(\mathscr{F})$ is the α-symmetric product of $\mathscr{F}$.

2.1.4 (Sufficiently general fibers and general fibers)**.** Let $f\colon X \to Y$ be a surjective morphism between varieties. Then a *sufficiently general fiber* (resp. *general fiber*) F of $f\colon X \to Y$ means that $F = f^{-1}(y)$, where y is any closed point contained in a countable intersection of nonempty Zariski open sets (resp. a nonempty Zariski open set) of Y. A sufficiently general fiber is sometimes called a *very general fiber* in the literature.

2.2. **Weakly positive sheaves.** Let us recall the necessary definitions around various positivity. The following definition is well known and standard in the study of higher-dimensional algebraic varieties.

Definition 2.2.1. Let X be a projective variety and let $\mathscr{L}$ be a line bundle on X.

(i) $\mathscr{L}$ is *big* if $\mathscr{L}^{\otimes k} \simeq \mathscr{H} \otimes \mathscr{O}_X(B)$ for some positive integer k, an ample line bundle $\mathscr{H}$, and an effective Cartier divisor B on X.
(ii) $\mathscr{L}$ is *nef* if $\mathscr{L} \cdot C \geq 0$ holds for every curve C on X.
(iii) $\mathscr{L}$ is *pseudo-effective* if $\mathscr{L}^{\otimes m} \otimes \mathscr{H}$ is big for every positive integer m and every ample line bundle $\mathscr{H}$ on X.

We note that a nef line bundle is always pseudo-effective. Let $\mathscr{E}$ be a locally free sheaf of finite rank on a projective variety X.

(iv) $\mathscr{E}$ is *nef* if $\mathscr{O}_{\mathbb{P}_X(\mathscr{E})}(1)$ is a nef line bundle on $\mathbb{P}_X(\mathscr{E})$.

We note that a nef locally free sheaf is sometimes called a *semipositive* locally free sheaf.

We recall the definition of weakly positive sheaves, which was first introduced by Viehweg. For the basic properties of weakly positive sheaves, see [Fn5].

Definition 2.2.2 (Weakly positive sheaves)**.** Let X be a normal quasi-projective variety and let $\mathscr{A}$ be a torsion-free coherent sheaf on X. We say that $\mathscr{A}$ is *weakly positive* if, for every positive integer α and every ample line bundle $\mathscr{H}$ on X, there exists a positive integer β such that $\widehat{S}^{\alpha\beta}(\mathscr{A}) \otimes \mathscr{H}^{\otimes\beta}$ is generically generated by global sections, where $\widehat{S}^{\alpha\beta}(\mathscr{A})$ denotes the double dual of the $\alpha\beta$-symmetric product of $\mathscr{A}$.

It is well known that $\widehat{\det}\mathscr{A}$ is weakly positive when $\mathscr{A}$ is a torsion-free weakly positive coherent sheaf on a normal quasi-projective variety. Moreover, a line bundle on a normal projective variety is pseudo-effective if and only if it is weakly positive. In general, the weak positivity does not behave well under extensions.

Remark 2.2.3. In [EjFI], we constructed a short exact sequence

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$

of locally free sheaves on a smooth projective surface such that $\mathscr{E}'$ and $\mathscr{E}''$ are pseudo-effective line bundles but $\mathscr{E}$ is not weakly positive.

We make a small remark on [EjFI] for the reader's convenience. Professor Robert Lazarsfeld pointed out that the following example answers [EjFI, Question 3.2] negatively.

Example 2.2.4 (Gieseker)**.** There exists a rank two ample vector bundle E on $\mathbb{P}^2$ sitting in an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^2}(-1)^{\oplus 4} \to E \to 0,$$

where d is a sufficiently large positive integer d (see, for example, [L, Example 6.3.67]). Let $\pi\colon X \to \mathbb{P}^2$ be any generically finite surjective morphism from a smooth variety X. Then we see that $H^0(X, \pi^*E) = 0$ since $H^1(X, \pi^*\mathscr{O}_{\mathbb{P}^2}(-d)^{\oplus 2}) = 0$ by the Kawamata–Viehweg vanishing theorem. In particular, π^*E is not generically globally generated.

2.3. **Systems of Hodge bundles.** Let $V_0 = (\mathbb{V}_0, W_0, F_0)$ be a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure on a complex manifold X_0, where $\mathbb{V}_0$ is a local system of finite-dimensional $\mathbb{R}$-vector spaces on X_0, W_0 is an increasing filtration of $\mathbb{V}_0$ by local subsystems, and $F_0 = \{F_0^p\}$ is the Hodge filtration. Then we obtain a Higgs bundle (E_0, θ_0) on X_0 by setting

$$E_0 = \mathrm{Gr}^{\bullet}_{F_0} \mathscr{V}_0 = \bigoplus_p F_0^p / F_0^{p+1}$$

where $\mathscr{V}_0 = \mathbb{V}_0 \otimes \mathscr{O}_{X_0}$. Note that θ_0 is induced by the Griffiths transversality

$$\nabla\colon F_0^p \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} F_0^{p-1}.$$

More precisely, ∇ induces

$$\theta_0^p\colon F_0^p/F_0^{p+1} \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} \left(F_0^{p-1}/F_0^p\right)$$

for every p. Then

$$\theta_0 = \bigoplus_p \theta_0^p\colon E_0 \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} E_0.$$

The pair (E_0, θ_0) is usually called the *system of Hodge bundles* associated to $V_0 = (\mathbb{V}_0, W_0, F_0)$ and θ_0 is called the *Higgs field* of (E_0, θ_0).

We further assume that X_0 is a Zariski open subset of a complex manifold X such that $D = X \setminus X_0$ is a normal crossing divisor on X. We note that the local monodromy of $\mathbb{V}_0$ around D is quasi-unipotent because V_0 is admissible. Let ${}^{\ell}\mathscr{V}$ be the lower canonical extension of $\mathscr{V}_0$ on X, that is, the *Deligne extension* of $\mathscr{V}_0$ on X such that the eigenvalues of the residue of the connection are contained in $[0, 1)$. Let ${}^{\ell}F^p$ be the lower canonical extension of F_0^p, that is,

$${}^{\ell}F^p = j_*F_0^p \cap {}^{\ell}\mathscr{V},$$

where $j\colon X_0 \hookrightarrow X$ is the natural open immersion, for every p. Since V_0 is admissible, ${}^{\ell}F^p$ is a subbundle of ${}^{\ell}\mathscr{V}$ for every p, and we can extend (E_0, θ_0) to (E, θ) on X, where

$$E = \mathrm{Gr}^{\bullet}_F {}^{\ell}\mathscr{V} = \bigoplus_p {}^{\ell}F^p / {}^{\ell}F^{p+1}$$

and

$$\theta\colon E \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} E.$$

When all the local monodromies of V_0 around D are unipotent, we simply write $\mathscr{V}$ and F^p to denote ${}^{\ell}\mathscr{V}$ and ${}^{\ell}F^p$, respectively. We say that $\mathscr{V}$ (resp. F^p) is the canonical extension of $\mathscr{V}_0$ (resp. F_0^p).

Remark 2.3.1. Although we formulated systems of Hodge bundles for graded polarizable *admissible* variations of $\mathbb{R}$-mixed Hodge structure, we do not need the relative monodromy weight filtration in this paper. We will only use Hodge bundles and their extensions. For the details of the necessary conditions, see (7.4.1) and (7.4.2) in 7.4 below and Remark 7.5.

We will discuss a variant of semipositivity theorems of Fujita–Zucker–Kawamata in Section 7 below (see Theorem 7.6).

3. Hodge theoretic weak positivity theorem

In this section, we will prove Theorem 3.3. For related topics, see [Ft], [Zuc], [Kw], [Zuo], [FnFs1], [FnFsS], [B1], [PoW], [PoS1], [Fs2], [FnFs2], [B2], and so on. In this section, we will freely use the notation in Subsection 2.3. Let us start with the following theorem: Theorem 3.1. As far as we know, there is no rigorous published proof of Theorem 3.1. The reader can find an approach to Theorem 3.1 in Section 7 below, where we will give a Hodge theoretic semipositivity theorem which implies Theorem 3.1. The approach in Section 7 is traditional and classical (see [FnFs1], [FnFs2], [FnFsS], and [Fs2]).

Theorem 3.1. *Let X be a smooth projective variety and let $X_0 \subset X$ be a Zariski open subset such that $D = X \setminus X_0$ is a simple normal crossing divisor on X. Let V_0 be a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure over X_0. We assume that all the local monodromies of V_0 around D are unipotent. We consider the logarithmic Higgs field*

$$\theta\colon \operatorname{Gr}_F^{\bullet} \mathscr{V} \to \Omega^1_X(\log D) \otimes \operatorname{Gr}_F^{\bullet} \mathscr{V}.$$

If $\mathscr{E}$ is a subbundle of $\operatorname{Gr}_F^{\bullet} \mathscr{V}$ and $\mathscr{E}$ is contained in the kernel of θ, then $\mathscr{E}^{\vee}$ is a nef locally free sheaf on Y.

Proof. We consider the dual variation of $\mathbb{R}$-mixed Hodge structure (see the description in [Fs2, 2.5]). Then, by Theorem 7.6 below (see also [Fs2, Theorem 4.2]), we see that $\mathscr{E}^{\vee}$ is a nef locally free sheaf on Y. □

Remark 3.2. When V_0 is pure in Theorem 3.1, a slightly better result was proved in [FnFs2, Theorem 1.5].

The main result of this section is as follows.

Theorem 3.3 (Hodge theoretic weak positivity theorem)**.** *Let X be a smooth projective variety and let $X_0 \subset X$ be a Zariski open subset such that $D = X \setminus X_0$ is a simple normal crossing divisor on X. Let V_0 be a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure over X_0. If $\mathscr{A}$ is a coherent sheaf on X such that $\mathscr{A}$ is contained in the kernel of the logarithmic Higgs field*

$$\theta\colon \operatorname{Gr}_F^{\bullet} {}^{\ell}\mathscr{V} \to \Omega^1_X(\log D) \otimes \operatorname{Gr}_F^{\bullet} {}^{\ell}\mathscr{V},$$

then the dual coherent sheaf $\mathscr{A}^{\vee}$ is weakly positive.

Proof of Theorem 3.3. Let U be the largest Zariski open subset of X such that $\mathscr{A}|_U$ is locally free. Since $\mathscr{A}$ is torsion-free, we have $\operatorname{codim}_X(X \setminus U) \geq 2$. By Kawamata's unipotent reduction theorem, we have a finite surjective flat morphism $f\colon Y \to X$ from a smooth projective variety such that $f^{-1}D$ is a simple normal crossing divisor on Y, $f^{-1}V_0$ is a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure on $Y_0 := Y \setminus f^{-1}D$, and all the local momodromies of $f^{-1}V_0$ around $f^{-1}D$ are unipotent. By considering the canonical extension of the system of Hodge bundles associated to $f^{-1}V_0$, we have

$$\theta_Y\colon \operatorname{Gr}^{\bullet}_F \mathscr{V}_Y \to \Omega^1_Y(\log f^{-1}D) \otimes \operatorname{Gr}^{\bullet}_F \mathscr{V}_Y,$$

where $\mathscr{V}_Y$ is the canonical extension of $f^{-1}\mathbb{V}_0 \otimes \mathscr{O}_{Y_0}$. Then $f^*\mathscr{A}$ is contained in the kernel of θ_Y. If $(f^*\mathscr{A})^\vee$ is weakly positive, then it is obvious that $f^*(\mathscr{A}^\vee|_U)$ is weakly positive. Hence we see that $\mathscr{A}^\vee$ is also weakly positive. Therefore, by replacing $\mathscr{A}$ and V_0 with $f^*\mathscr{A}$ and $f^{-1}V_0$, respectively, we may assume that V_0 has unipotent monodromies around D. We apply the flattening theorem to $\mathscr{G}/\mathscr{A}$, where $\mathscr{G} := \operatorname{Gr}^{\bullet}_F \mathscr{V}$. Then we get a projective birational morphism $f\colon Y \to X$ from a smooth projective variety Y such that $f^*(\mathscr{G}/\mathscr{A})$/torsion is locally free and that $f^{-1}D$ is a simple normal crossing divisor on Y. By construction, we obtain a subbundle $\mathscr{E}$ of $f^*\mathscr{G}$ such that there exists a generically isomorphic injection $f^*\mathscr{A} \hookrightarrow \mathscr{E}$ on $f^{-1}(U)$. Thus $\mathscr{E}$ is contained in the kernel of $f^*\theta$, where

$$f^*\theta\colon \operatorname{Gr}^{\bullet}_F f^*\mathscr{V} \to \Omega^1_Y(\log f^{-1}D) \otimes_{\mathscr{O}_Y} \operatorname{Gr}^{\bullet}_F f^*\mathscr{V}.$$

Hence $\mathscr{E}^\vee$ is a nef locally free sheaf on Y by Theorem 3.1. In particular, $\mathscr{E}^\vee$ is weakly positive. Since we have a generically isomorphic injection $\mathscr{E}^\vee \hookrightarrow (f^*\mathscr{A})^\vee$ on $f^{-1}(U)$, $(f^*\mathscr{A})^\vee$ is weakly positive on $f^{-1}(U)$. This implies that $\mathscr{A}^\vee$ is weakly positive on U. Hence, $\mathscr{A}^\vee$ is a weakly positive reflexive sheaf on X since $\operatorname{codim}_X(X \setminus U) \geq 2$. This is what we wanted. □

4. Graded logarithmic Higgs sheaves

This section is a direct generalization of [Pa, Section 3.1]. The original idea goes back to Viehweg and Zuo (see [VZ1] and [VZ2]). Let us recall the definition of graded logarithmic Higgs sheaves following [Pa] (see also [VZ1], [VZ2], [PoS1], and so on).

Definition 4.1 (Graded logarithmic Higgs sheaves)**.** Let Y be a smooth variety and let D be a simple normal crossing divisor on Y. A graded $\mathcal{O}_Y$-module $\mathscr{F}_\bullet = \bigoplus_k \mathscr{F}_k$ is a *graded logarithmic Higgs sheaf* with poles along D if there exists a logarithmic Higgs structure

$$\phi\colon \mathscr{F}_\bullet \to \mathscr{F}_\bullet \otimes_{\mathscr{O}_Y} \Omega^1_Y(\log D)$$

such that $\phi = \bigoplus_k \phi_k$,

$$\phi_k\colon \mathscr{F}_k \to \mathscr{F}_{k+1} \otimes_{\mathscr{O}_Y} \Omega^1_Y(\log D)$$

for every k, and

$$\phi \wedge \phi\colon \mathscr{F}_\bullet \to \mathscr{F}_\bullet \otimes_{\mathscr{O}_Y} \Omega^2_Y(\log D)$$

is zero. We put

$$\mathscr{K}_k(\phi) := \ker\left(\phi_k\colon \mathscr{F}_k \to \mathscr{F}_{k+1} \otimes_{\mathscr{O}_Y} \Omega^1_Y(\log D)\right),$$

that is, $\mathscr{K}_k(\phi)$ is the kernel of the generalized Kodaira–Spencer map ϕ_k for every k.

Theorem 4.2 is a slight generalization of [Pa, Theorem 3.2]. One of the motivations of this paper is to understand [Pa, Theorem 3.2].

Theorem 4.2. *Let $f\colon X \to Y$ be a projective surjective morphism of smooth quasi-projective varieties. Let D (resp. E) be a simple normal crossing divisor on Y (resp. X). Assume that f is smooth over $Y \setminus D$, E is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. Let $\mathscr{L}$ be a line bundle on Y such that*

$$\kappa(X, \omega_{(X,E)/(Y,D)} \otimes f^*\mathscr{L}^{\otimes -1}) = \kappa(X, \omega_X(E) \otimes f^*(\omega_Y(D))^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}) \geq 0.$$

Then there exists a graded logarithmic Higgs sheaf $\mathscr{F}_\bullet$ with poles along D satisfying the following properties:

(i) $\mathscr{L} \subset \mathscr{F}_0$ *and* $\mathscr{F}_k = 0$ *for every* $k < 0$.
(ii) *There exists a positive integer* d *such that* $\mathscr{F}_k = 0$ *for every* $k > d$.
(iii) $\mathscr{F}_k$ *is a reflexive coherent sheaf on* Y *for every* k.
(iv) *Let* $\mathscr{A}$ *be a coherent subsheaf of* $\mathscr{F}_\bullet$ *contained in the kernel of* ϕ. *Then* $\mathscr{A}^\vee$ *is weakly positive. In particular,* $\mathscr{K}_k(\phi)^\vee$ *is weakly positive.*

Although the proof of Theorem 4.2 is essentially the same as that of [Pa, Theorem 3.2], we explain it in detail for the sake of completeness.

Proof of Theorem 4.2. We will closely follow the argument in [Pa]. Since it is sufficient to construct $\mathscr{F}_\bullet$ on the complement of a codimension two closed subset in Y, we will freely remove suitable codimension two closed subsets from Y throughout this proof. We put

$$\mathscr{N} := \omega_X(E) \otimes f^*\left(\omega_Y(D)\right)^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}.$$

Since $\kappa(X, \mathscr{N}) \geq 0$ by assumption, we can take a section s of $\mathscr{N}^{\otimes m}$ for some positive integer m. Let $X' \to X$ be the cyclic cover of X associated to s and let $Z \to X'$ be a suitable resolution of singularities. We put $\psi\colon Z \to X$ and $h := f \circ \psi\colon Z \to Y$. Hence we have the following commutative diagram:

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \downarrow{\scriptstyle \psi} & \searrow{\scriptstyle h} & \\ X' & \longrightarrow & X & \xrightarrow[f]{} & Y. \end{array}$$

Then there exists a natural inclusion $\psi^*\mathscr{N}^{\otimes -1} \hookrightarrow \mathscr{O}_Z$. Let E_Z be a simple normal crossing divisor on Z such that $\psi^{-1}(E) \subset E_Z$ and that $\psi^{-1}(E) = E_Z$ holds over the generic point of Y. After removing a suitable codimension two closed subset from Y and taking a birational modification of Z suitably, we may further assume that there exists a smooth divisor D' on Y such that $h\colon Z \to Y$ is smooth over $Y \setminus D'$, E_Z is a relatively normal crossing over $Y \setminus D'$, and $h^{-1}(D') \subset E_Z$. As usual, we put

$$\Omega^1_{X/Y}(\log E) := \operatorname{Coker}\left(f^*\Omega^1_Y(\log D) \to \Omega^1_X(\log E)\right)$$

and

$$\Omega^1_{Z/Y}(\log E_Z) := \operatorname{Coker}\left(h^*\Omega^1_Y(\log D') \to \Omega^1_Z(\log E_Z)\right).$$

Without loss of generality, we may assume that D is smooth and that $\Omega^1_{X/Y}(\log E)$ and $\Omega^1_{Z/Y}(\log E_Z)$ are both locally free sheaves. By construction, we see that $D \leq D'$ holds. We consider the Koszul filtration

$$\operatorname{Koz}^q\Omega^i_X(\log E) := \operatorname{Image}\left(f^*\Omega^q_Y(\log D) \otimes \Omega^{i-q}_X(\log E) \to \Omega^i_X(\log E)\right). \tag{4.1}$$

Then we have

$$\operatorname{Koz}^q/\operatorname{Koz}^{q+1}\Omega^i_X(\log E) \simeq f^*\Omega^q_Y(\log D) \otimes \Omega^{i-q}_{X/Y}(\log E)$$

and get the following short exact sequence:

$$0 \to f^*\Omega^1_Y(\log D) \otimes \Omega^{i-1}_{X/Y}(\log E) \to \operatorname{Koz}^0/\operatorname{Koz}^2\Omega^i_X(\log E) \to \Omega^i_{X/Y}(\log E) \to 0,$$

which is denoted by $\mathscr{C}^i_{X/Y}(\log E)$. Similarly, we can define $\mathrm{Koz}^q\Omega^i_Z(\log E_Z)$ and obtain $\mathscr{C}^i_{Z/Y}(\log E_Z)$. By construction, we have a natural map $\psi^*\mathscr{C}^i_{X/Y}(\log E) \to \mathscr{C}^i_{Z/Y}(\log E_Z)$ for every i. By tensoring with the natural injection $\psi^*\mathscr{N}^{\otimes -1} \hookrightarrow \mathscr{O}_Z$, we have

$$\psi^*\left(\mathscr{C}^i_{X/Y}(\log E) \otimes \mathscr{N}^{\otimes -1}\right) \to \mathscr{C}^i_{Z/Y}(\log E_Z).$$

Then, by using the edge homomorphism of the Leray spectral sequence, we obtain the natural homomorphism

$$R^{d-i}f_*\left(\Omega^i_{X/Y}(\log E) \otimes \mathscr{N}^{\otimes -1}\right) \to R^{d-i}h_*\left(\psi^*\left(\Omega^i_{X/Y}(\log E) \otimes \mathscr{N}^{\otimes -1}\right)\right),$$

where $d = \dim X - \dim Y$. Thus we get the following commutative diagram of the connecting homomorphisms.

$$\begin{array}{ccc} R^{d-i}f_*\left(\Omega^i_{X/Y}(\log E) \otimes \mathscr{N}^{\otimes -1}\right) & \longrightarrow & R^{d-i+1}f_*\left(\Omega^{i-1}_{X/Y}(\log E) \otimes \mathscr{N}^{\otimes -1}\right) \otimes \Omega^1_Y(\log D) \\ \downarrow{\scriptstyle \rho_{d-i}} & & \downarrow{\scriptstyle \rho_{d-i+1}\otimes\iota} \\ R^{d-i}h_*\left(\Omega^i_{Z/Y}(\log E_Z)\right) & \xrightarrow[\phi'_{d-i}]{} & R^{d-i+1}h_*\left(\Omega^{i-1}_{Z/Y}(\log E_Z)\right) \otimes \Omega^1_Y(\log D') \end{array}$$

We get a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure over $Y \setminus D'$ from $h\colon (Z, E_Z) \to (Y, D')$. By Theorem 6.1 below,

$$\bigoplus_{k=0}^{d} R^k h_*\left(\Omega^{d-k}_{Z/Y}(\log E_Z)\right)$$

is the lower canonical extension of the system of Hodge bundles associated to the above variation of $\mathbb{R}$-mixed Hodge structure. Then we put

$$\mathscr{F}_k := \left(\mathrm{Image}\left(\rho_k\colon R^k f_*\left(\Omega^{d-k}_{X/Y}(\log E) \otimes \mathscr{N}^{\otimes -1}\right) \to R^k h_*\left(\Omega^{d-k}_{Z/Y}(\log E_Z)\right)\right)\right)^{\vee\vee}$$

for $0 \le k \le d$. We put $\mathscr{F}_k = 0$ if $k < 0$ or $k > d = \dim X - \dim Y$. Then we see that $\mathscr{F}_\bullet$ is a graded logarithmic Higgs sheaf with poles along D. Since ρ_0 is the pushforward f_* of the inclusion

$$f^*\mathscr{L} = \omega_X(E) \otimes f^*\left(\omega_Y(D)\right)^{\otimes -1} \otimes \mathscr{N}^{\otimes -1} \to \psi_*\left(\omega_Z(E_Z) \otimes h^*\left(\omega_Y(D')\right)^{\otimes -1}\right).$$

This implies that $\mathscr{F}_0 = (\mathcal{L} \otimes f_*\mathscr{O}_X)^{\vee\vee}$. Hence $\mathscr{L} \subset \mathscr{F}_0$ holds. Therefore, (i), (ii), and (iii) hold. By taking a suitable compactification and applying Theorem 3.3, we obtain (iv). We finish the proof. □

Remark 4.3 (see [Pa, Remark 3.6])**.** In Theorem 4.2, we can replace the assumption

$$\kappa(X, \omega_X(E) \otimes f^*(\omega_Y(D))^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}) \ge 0$$

with the existence of a nonzero homomorphism

$$\mathscr{L}^{\otimes N} \to \left(f_*\omega^{\otimes N}_{(X,E)/(Y,D)}\right)^{\vee\vee} \tag{4.2}$$

for some positive integer N. Note that (4.2) implies the existence of a nonzero section of

$$\left(\omega_X(E) \otimes f^*(\omega_Y(D))^{\otimes -1} \otimes f^*\mathscr{L}^{\otimes -1}\right)^{\otimes N}$$

over the complement of some codimension two closed subset Σ in Y. Hence we can construct a desired graded logarithmic Higgs sheaf $\mathscr{F}_\bullet$ on $Y \setminus \Sigma$. By taking the reflexive hull of $\mathscr{F}_\bullet$, we can extend $\mathscr{F}_\bullet$ over Y.

For geometric applications, the following lemma is crucial.

Lemma 4.4 ([Pa, Lemma 3.7]). *Let Y be a smooth projective variety and let D be a simple normal crossing divisor on Y. Let $\mathscr{F}_\bullet$ be a graded logarithmic Higgs sheaf with poles along D satisfying* (i), (ii), (iii), *and* (iv) *in Theorem 4.2. Then we have a pseudo-effective line bundle $\mathscr{P}$ and a nonzero homomorphism*

$$\mathscr{L}^{\otimes r} \otimes \mathscr{P} \to \left(\Omega^1_Y(\log D)\right)^{\otimes kr}$$

for some $r > 0$ and $k \geq 0$.

Proof of Lemma 4.4. We have a sequence of homomorphisms

$$\phi_k \otimes \mathrm{id}\colon \mathscr{F}_k \otimes \left(\Omega^1_Y(\log D)\right)^{\otimes k} \to \mathscr{F}_{k+1} \otimes \left(\Omega^1_Y(\log D)\right)^{\otimes k+1}.$$

Note that $\mathscr{F}_k$ is zero for $k \gg 0$ and that $\mathscr{L} \subset \mathscr{F}_0$. Hence, the line bundle $\mathscr{L}$ is contained in the kernel of $\phi_k \otimes \mathrm{id}$ for some $k \geq 0$, that is,

$$\mathscr{L} \subset \mathscr{K}_k(\phi) \otimes \left(\Omega^1_Y(\log D)\right)^{\otimes k}.$$

This implies the existence of a nonzero homomorphism

$$\mathscr{K}_k(\phi)^\vee \to \left(\Omega^1_Y(\log D)\right)^{\otimes k} \otimes \mathscr{L}^{\otimes -1}.$$

Let $\mathscr{Q}$ be the image of the above homomorphism and let r be the rank of $\mathscr{Q}$. By considering the split surjection

$$\mathscr{Q}^{\otimes r} \to \det \mathscr{Q}$$

outside some suitable codimension two closed subset of Y, we have a nonzero homomorphism

$$\widehat{\det}\mathscr{Q} \to \left(\left(\Omega^1_Y(\log D)\right)^{\otimes k} \otimes \mathscr{L}^{\otimes -1}\right)^{\otimes r}.$$

Since $\mathscr{P} := \widehat{\det}\mathscr{Q}$ is a pseudo-effective line bundle by Theorem 4.2 (iv), we obtain a desired nonzero homomorphism. We finish the proof. ☐

Let us prove Theorem 1.4, which is one of the main results of this paper.

Proof of Theorem 1.4. By Remark 4.3, we can construct a graded logarithmic Higgs sheaf with poles along D satisfying (i), (ii), (iii), and (iv) in Theorem 4.2. Then, by Lemma 4.4, we obtain a desired pseudo-effective line bundle and a nonzero homomorphism. We finish the proof. ☐

As a direct consequence of Lemma 4.4, we have:

Theorem 4.5 (see [Pa, Theorem 1.7 (2)]). *Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties and let E and D be simple normal crossing divisors on X and Y, respectively. Assume that f is smooth over $Y \setminus D$, E is relatively normal crossing over $Y \setminus D$, and $f^{-1}(D) \subset E$. In this situation, if $\kappa(X, K_X + E) \geq 0$ holds, then $K_Y + D$ is pseudo-effective.*

Proof of Theorem 4.5. The proof of [Pa, Theorem 1.7 (2)] works. We put $\mathscr{L} := \omega^{\otimes -1}_{(Y,D)}$. By assumption, we have

$$\kappa(X, \omega_{(X,E)/(Y,D)} \otimes f^*\mathscr{L}^{\otimes -1}) = \kappa(X, K_X + E) \geq 0.$$

By Lemma 4.4, this implies that there exists a pseudo-effective line bundle $\mathscr{P}$ and a nonzero homomorphism

$$\omega_{(Y,D)}^{\otimes -r} \otimes \mathscr{P} \to \left(\Omega_Y^1(\log D)\right)^{\otimes kr}$$

for some $r > 0$ and $k \geq 0$. Thus $K_Y + D$ is pseudo-effective by [Pa, Theorem 3.9], which is due to [CP, Theorem 7.6]. We finish the proof. □

In Section 5, we will use Theorem 4.5 in the proofs of Corollary 1.13 and Theorem 1.14.

5. Proofs

In this section, we will prove results in Section 1. Let us start with the proof of Theorem 1.1.

Proof of Theorem 1.1. In [Pa, Section 2], this theorem is proved under the extra assumption that $f^{-1}(D) = E$ and $g^{-1}(D) = D'$ hold. However, we can easily see that [Pa, Proposition 2.5] implies the desired inclusion (1.1). Moreover, by the proof in [Pa, Section 2], it is easy to see that the inclusion (1.1) is an isomorphism over some nonempty Zariski open subset of Y. □

Proof of Corollary 1.2. This corollary is an easy consequence of Theorem 1.1. All we have to do is to apply Theorem 1.1 repeatedly. Note that the inclusion (1.2) is an isomorphism over some nonempty Zariski open subset of Y. □

We have already proved Theorem 1.4 in Section 4. Thus, let us prove Theorem 1.5.

Proof of Theorem 1.5. If $\kappa(Y, K_Y + D) = \dim Y$, that is, $K_Y + D$ is big, then

$$\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

holds by Maehara's theorem (see [Ma] and [Fn1]). On the other hand, if the equality

$$\kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

holds, then there exists a positive integer N and an ample Cartier divisor A on Y such that

$$f^*\mathscr{O}_Y(A) \subset \omega_{(X,E)}^{\otimes N}$$

by [Mo, Proposition 1.14]. We put $\mathscr{L} := \mathscr{O}_Y(A) \otimes \omega_{(Y,D)}^{\otimes -N}$. Hence we have

$$\mathscr{L}^{\otimes N} \subset \left(\bigotimes^N f_*\omega_{(X,E)/(Y,D)}^{\otimes N}\right)^{\vee\vee} \subset \left(f_*^{(N)}\omega_{(X^{(N)},E^{(N)})/(Y,D)}^{\otimes N}\right)^{\vee\vee}.$$

Here we used Corollary 1.2 with $s = N$. By Theorem 1.4, there exist a pseudo-effective line bundle $\mathscr{P}$ on Y and a nonzero homomorphism

$$\mathscr{L}^{\otimes r} \otimes \mathscr{P} \to \left(\Omega_Y^1(\log D)\right)^{\otimes kr}$$

for some $r > 0$ and $k \geq 0$. Hence we have a nonzero homomorphism

$$\mathscr{O}_Y(A)^{\otimes r} \otimes \mathscr{P} \to \left(\Omega_Y^1(\log D)\right)^{\otimes kr} \otimes \omega_{(Y,D)}^{\otimes Nr}.$$

Then, by [Pa, Theorem 3.9], which is due to [CP, Theorem 7.6], $K_Y + D$ is pseudo-effective. Since we have $\omega_{(Y,D)} \subset (\Omega_Y^1(\log D))^{\otimes \dim Y}$ by definition, we have a nonzero homomorphism

$$\mathscr{O}_Y(A)^{\otimes r} \otimes \mathscr{P} \to \left(\Omega_Y^1(\log D)\right)^{\otimes N'}$$

for some positive integer N'. Therefore, by [Pa, Theorem 3.8], which is due to [CP, Theorem 1.3], $K_Y + D$ is the sum of an ample divisor and a pseudo-effective divisor, so it is big as desired. We finish the proof. □

Proof of Corollary 1.7. By the easy addition formula, we have

$$\dim X = \kappa(X, K_X + E) = \kappa(F, (K_X + E)|_F) + \dim Y$$

and

$$\kappa(F, (K_X + E)|_F) = \dim F,$$

where F is a general fiber of $f\colon X \to Y$. By Theorem 1.5, we obtain $\kappa(Y, K_Y + D) = \dim Y$. We finish the proof. □

Proof of Theorem 1.8. We take a sufficiently large and divisible positive integer N such that

$$f^*\mathscr{O}_Y(D) \subset \omega_{(X,E)}^{\otimes N}.$$

As in the proof of Theorem 1.5 above, by Theorem 1.4, the proof of [Pa, Theorem 1.7 (1)] works. Then we obtain a positive rational number δ such that $K_X + (1-\delta)D$ is pseudo-effective. We finish the proof. □

For the proof of Theorem 1.12, we prepare the following lemma. Note that we need Corollary 1.2 in the proof of Lemma 5.1.

Lemma 5.1. *In Conjecture 1.9, we assume $\kappa_\sigma(Y, K_Y + D) = 0$. Then we have*

$$\kappa(X, K_X + E) \leq \kappa(F, (K_X + E)|_F),$$

where F is a sufficiently general fiber of $f\colon X \to Y$.

Proof of Lemma 5.1. Note that $K_Y + D$ is pseudo-effective by the assumption $\kappa_\sigma(Y, K_Y + D) = 0$. As in the proofs of Theorems 1.5 and 1.8, the proof of [Pa, Proposition 5.2] works. We finish the proof. □

Let us prove Theorem 1.12.

Proof of Theorem 1.12. Let $\mu\colon Y' \to Y$ be a projective birational morphism from a smooth variety Y' such that $D' := \mu^{-1}(D)$ is a simple normal crossing divisor on Y'. We replace (Y, D) with (Y', D') and take suitable birational modifications. Then we may assume that the Iitaka fibration $\Phi := \Phi_{|m(K_Y+D)|}\colon Y \to Z$ is a morphism onto a normal projective variety Z with connected fibers, where m is a sufficiently large positive integer. Let $(G, D|_G)$ (resp. $(H, E|_H)$) be a sufficiently general fiber of Φ (resp. $\Phi \circ f$), that is, $G = \Phi^{-1}(z)$ and $H = (\Phi \circ f)^{-1}(z)$, where z is a sufficiently general point of Z. Note that G and H are smooth projective varieties and $D|_G$ and $E|_H$ are simple normal crossing divisors. By construction, we see that $f|_H\colon H \to G$ satisfies that $f|_H$ is smooth over $G \setminus (D|_G)$, $E|_H$ is relatively normal crossing over $G \setminus (D|_G)$, and $(f|_H)^{-1}(D|_G) \subset E|_H$. By assumption,

$$\kappa_\sigma(G, K_G + D|_G) = \kappa(G, K_G + D|_G) = 0.$$

By Lemma 5.1,

$$\kappa(H, K_H + E|_H) \leq \kappa(F, (K_X + E)|_F)$$

holds. Therefore, by applying the easy addition formula to $\Phi\circ f\colon X\to Z$,

$$\begin{aligned}\kappa(X, K_X+E) &\leq \kappa(H, K_H+E|_H)+\dim Z\\ &\leq \kappa(F, (K_X+E)|_F)+\kappa(Y, K_Y+D).\end{aligned}$$

We finish the proof. □

We need Gongyo's theorem for the proof of Corollary 1.13.

Theorem 5.2 (see [Fn3, Proposition 4.1]). *Let X be a smooth projective variety and let E be a simple normal crossing divisor on X. Assume that there exists a projective birational morphism $\varphi\colon X\setminus E\to V$ onto an affine variety V. Then $\kappa_\sigma(X, K_X+E)=\kappa(X, K_X+E)$ holds, that is, the generalized abundance conjecture holds for (X, E). In particular, if K_X+E is pseudo-effective, then $\kappa(X, K_X+E)\geq 0$.*

Proof of Theorem 5.2. This theorem is an easy application of the minimal model program. For the details, see the proof of [Fn3, Proposition 4.1]. □

Proof of Corollary 1.13. If $\kappa(X, K_X+E)=-\infty$, then (1.3) is obvious. Hence we may assume that $\kappa(X, K_X+E)\geq 0$. By Theorems 4.5 and 5.2, we have $\kappa(Y, K_Y+D)\geq 0$. We can apply Theorem 5.2 to a sufficiently general fiber of the Iitaka fibration of Y with respect to K_Y+D. Therefore, by Theorem 1.12, we obtain the desired inequality (1.3). □

Finally, we prove Theorem 1.14.

Proof of Theorem 1.14. The subadditivity

$$\kappa(X, K_X+E)\geq \kappa(Y, K_Y+D)+\kappa(F, (K_X+E)|_F)$$

follows from $\kappa_\sigma(X, K_X+E)=\kappa(X, K_X+E)$ (see [Fn3] and [Fn4]) or $\kappa_\sigma(F, (K_X+E)|_F)=\kappa(F, (K_X+E)|_F)$ (see [H2]). Therefore, from now on, we will prove the superadditivity

$$\kappa(X, K_X+E)\leq \kappa(Y, K_Y+D)+\kappa(F, (K_X+E)|_F). \tag{5.1}$$

If $\kappa(X, K_X+E)=-\infty$, then (5.1) is obviously true. Hence we may assume that $\kappa(X, K_X+E)\geq 0$ holds. By Theorem 4.5, K_Y+D is pseudo-effective. By Conjecture 1.11, which is a special case of Conjecture 1.10, we have $\kappa(Y, K_Y+D)\geq 0$. Thus, by Theorem 1.12 and Conjecture 1.10, we obtain the desired superadditivity (5.1). We finish the proof. □

6. On variations of mixed Hodge structure

In this section, for the sake of completeness, we will explain the following theorem, which is more or less well known to the experts (see [StZ], [El], and so on). Theorem 6.1 has already been used in the proof of Theorem 4.2 and is one of the main ingredients of Theorem 4.2.

Theorem 6.1. *Let $f\colon X\to Y$ be a proper surjective morphism from a Kähler manifold X to a complex manifold Y with $d=\dim X-\dim Y$ and let Σ_X and Σ_Y be reduced simple normal crossing divisors on X and Y, respectively. We set $Y_0=Y\setminus\Sigma_Y, X_0=f^{-1}(Y_0), f_0=f|_{X_0}\colon X_0\to Y_0$ and $U=X\setminus\Sigma_X$ and assume that f_0 is a smooth morphism, $\Sigma_X\cap X_0$ is relatively normal crossing over Y_0, and $\operatorname{Supp} f^*\Sigma_Y\subset\operatorname{Supp}\Sigma_X$ holds. Then the local system $R^i(f_0|_U)_*\mathbb{R}_U$ underlies a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure on Y_0 for every i such that the Hodge filtration F on $\mathscr{V}=\mathscr{O}_{Y_0}\otimes R^i(f_0|_U)_*\mathbb{R}_U$ extends to a filtration F on the lower canonical extension ${}^{\ell}\mathscr{V}$ satisfying the following conditions:*

(6.1.1) $\mathrm{Gr}_F^p \mathrm{Gr}_m^W({}^\ell\mathscr{V})$ *is locally free of finite rank for every* m, p*, and*

(6.1.2) $\mathrm{Gr}_F^p({}^\ell\mathscr{V})$ *coincides with* $R^{i-p}f_*\Omega^p_{X/Y}(\log \Sigma_X)$ *for every* p*, after removing some suitable codimension two closed subset from* Y*.*

In the above statement, if $\Sigma_X \cap X_0 = 0$*, then* $R^i(f_0|_U)_*\mathbb{R}_U = R^i(f_0)_*\mathbb{R}_{X_0}$ *underlies a polarizable variation of* $\mathbb{R}$*-Hodge structure on* Y_0 *for every* i*.*

In this section, we adopt the same approach as in Sections 3, 4, and 7 in [FnFs3]. Before starting the proof of the theorem above, we recall several facts concerning on the Koszul complex.

6.2. In the situation above, we set $E = (f^*\Sigma_Y)_{\mathrm{red}}$ and $D = \Sigma_X - E$. Then D and E are reduced simple normal crossing divisors on X with no common irreducible components. The open immersions $X \setminus D \hookrightarrow X$ and $U \hookrightarrow X_0$ are denoted by j and j_0 respectively. The situation is summarized in the commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & X \setminus D & & \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle j} & & \\ X_0 & \longrightarrow & X & \longleftarrow & E \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ Y_0 & \longrightarrow & Y & \longleftarrow & \Sigma_Y \end{array}$$

where the left two squares are Cartesian.

We denote by $D = \sum_{i=1}^{l} D_i$ the irreducible decomposition of D and set

$$D^{(m)} = \coprod_{1 \le i_1 < \cdots < i_m \le l} D_{i_1} \cap \cdots \cap D_{i_m}$$

for $m \in \mathbb{Z}_{\geq 0}$. (For the case of $m = 0$, we set $D^{(0)} = X$ by definition.) The natural morphism from $D^{(m)}$ to X is denoted by a_m.

In order to define the desired weight filtration on

$$R^i(f_0|_U)_*\mathbb{R}_U \simeq \mathbb{R} \otimes R^i(f_0|_U)_*\mathbb{Q}_U \simeq \mathbb{R} \otimes R^i(f_0)_*(R(j_0)_*\mathbb{Q}_U)$$

we replace $R(j_0)_*\mathbb{Q}_U$ by a Koszul complex as follows. For the detail, see [Fs1, Sections 1 and 2] (cf. [I], [St], [FnFs3, Section 7]).

The divisor D on X defines a log structure $\mathcal{M}$ by $\mathcal{M} := \mathscr{O}_X \cap j_*\mathscr{O}^*_{X\setminus D}$ on X. A morphism of abelian sheaves $\mathscr{O}_X \to \mathcal{M}^{\mathrm{gp}}$ is defined as the composite of the exponential map $\mathscr{O}_X \ni a \mapsto e^{2\pi\sqrt{-1}a} \in \mathscr{O}^*_X$ and the inclusion $\mathscr{O}^*_X \hookrightarrow \mathcal{M}^{\mathrm{gp}}$. From the morphism $\mathbf{e} \otimes \mathrm{id}\colon \mathscr{O}_X \simeq \mathscr{O}_X \otimes \mathbb{Q} \to \mathcal{M}^{\mathrm{gp}} \otimes \mathbb{Q}$, $1 \in \Gamma(X, \mathbb{Q})$ which is a global section of the kernel of $\mathbf{e} \otimes \mathrm{id}$, and a subsheaf $\mathscr{O}^*_X \otimes \mathbb{Q} \subset \mathcal{M}^{\mathrm{gp}} \otimes \mathbb{Q}$, we obtain a complex of $\mathbb{Q}$-sheaves on X

$$\mathrm{Kos}(\mathcal{M}) := \mathrm{Kos}(\mathbf{e} \otimes \mathrm{id}; \infty; 1)$$

equipped with a finite increasing filtration $W := W(\mathscr{O}^*_X \otimes \mathbb{Q})$ as in [Fs1, Definition 1.8] (see also [FnFs3, Section 7]). By replacing $\mathcal{M}^{\mathrm{gp}}$ by $\mathscr{O}^*_{X\setminus D}$, we obtain a complex of $\mathbb{Q}$-sheaves on $X \setminus D$, denoted by $\mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$, by the same way as above. Moreover, we have a morphism of complexes of $\mathbb{Q}$-sheaves

$$\psi\colon \mathrm{Kos}(\mathcal{M}) \to \Omega_X(\log D)$$

as in [Fs1, (2.4)], which preserves the filtration W on the both sides.

The following two lemmas are more or less the same as Lemmas 7.6 and 7.7 in [FnFs3].

Lemma 6.3. *There exists a quasi-isomorphism of complexes*

$$(a_m)_*\mathbb{Q}_{D^{(m)}}[-m] \to \mathrm{Gr}_m^W \mathrm{Kos}(\mathcal{M}) \tag{6.1}$$

for all $m \in \mathbb{Z}_{\geq 0}$.

Lemma 6.4. *The quasi-isomorphism* (6.1) *makes the diagram*

$$\begin{array}{ccc} (a_m)_*\mathbb{Q}_{D^{(m)}}[-m] & \longrightarrow & \mathrm{Gr}_m^W \mathrm{Kos}(\mathcal{M}) \\ {\scriptstyle (2\pi\sqrt{-1})^{-m}\iota[-m]}\downarrow & & \downarrow{\scriptstyle \mathrm{Gr}_m^W \psi} \\ (a_m)_*\Omega_{D^{(m)}}[-m] & \longrightarrow & \mathrm{Gr}_m^W \Omega_X(\log D) \end{array}$$

commutative, where the bottom arrow is the inverse of the usual residue isomorphism and ι is the composite of the natural morphisms $\mathbb{Q}_{D^{(m)}} \hookrightarrow \mathbb{C}_{D^{(m)}}$ and $\mathbb{C}_{D^{(m)}} \to \Omega_{D^{(m)}}$. Consequently, the morphism ψ induces a filtered quasi-isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} \mathrm{Kos}(\mathcal{M}) \to \Omega_X(\log D)$$

with respect to the filtration W on the both sides.

6.5. Since $\mathrm{Kos}(\mathcal{M})|_{X\setminus D} = \mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$, we obtain a morphism of the complexes of $\mathbb{Q}$-sheaves $\mathrm{Kos}(\mathcal{M}) \to Rj_* \mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$ such that the diagram in the derived category

$$\begin{array}{ccc} \mathrm{Kos}(\mathcal{M}) & \longrightarrow & Rj_* \mathrm{Kos}(\mathscr{O}^*_{X\setminus D}) \\ {\scriptstyle \psi}\downarrow & & \downarrow \\ \Omega_X(\log D) & \longrightarrow & Rj_*\Omega_{X\setminus D} \end{array} \tag{6.2}$$

is commutative, where the right vertical arrow is induced from the morphism $\mathrm{Kos}(\mathscr{O}^*_{X\setminus D}) \to \Omega_{X\setminus D}$ defined by the same way as ψ.

Lemma 6.6. *We have the natural isomorphism*

$$\mathrm{Kos}(\mathcal{M}) \xrightarrow{\simeq} Rj_*\mathbb{Q}_{X\setminus D}$$

in the derived category. By restricting it to X_0, we obtain the isomorphism

$$\mathrm{Kos}(\mathcal{M})|_{X_0} \xrightarrow{\simeq} R(j_0)_*\mathbb{Q}_U$$

in the derived category.

Proof. It is sufficient to prove that the morphism

$$\mathbb{C} \otimes_{\mathbb{Q}} \mathrm{Kos}(\mathcal{M}) \to \mathbb{C} \otimes_{\mathbb{Q}} Rj_* \mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$$

is an isomorphism. Since we have the canonical quasi-isomorphism $\mathbb{Q}_{X\setminus D} \to \mathrm{Kos}(\mathscr{O}^*_{X\setminus D})$, the right vertical arrow in (6.2) induces an isomorphism $\mathbb{C} \otimes Rj_* \mathrm{Kos}(\mathscr{O}^*_{X\setminus D}) \simeq Rj_*\Omega_{X\setminus D}$ in the derived category. Hence Lemma 6.4 implies the conclusion because the bottom arrow in (6.2) is known to be isomorphisms in the derived category. □

Now, we prove Theorem 6.1.

Proof of Theorem 6.1. We set $E \cap D^{(m)} = (a_m)^*E$, which is a simple normal crossing divisor on $D^{(m)}$ for every $m \in \mathbb{Z}_{\geq 0}$.

First, we assume that the following conditions are satisfied:

(6.6.1) Σ_Y is a smooth hypersurface in Y, and

(6.6.2) $\Omega^1_{D^{(m)}/Y}(\log E \cap D^{(m)})$ is locally free of finite rank for all $m \in \mathbb{Z}_{\geq 0}$.

On the log de Rham complex $\Omega_X(\log \Sigma_X)$, we have the filtration $W(D)$, which induces a finite increasing filtration $W(D)$ on the relative log de Rham complex $\Omega_{X/Y}(\log \Sigma_X)$. A morphism of complexes $\overline{\psi}\colon \operatorname{Kos}(\mathcal{M}) \to \Omega_{X/Y}(\log \Sigma_X)$ is obtained by composing the three morphisms, $\psi\colon \operatorname{Kos}(\mathcal{M}) \to \Omega_X(\log D)$, the inclusion $\Omega_X(\log D) \to \Omega_X(\log \Sigma_X)$ and $\Omega_X(\log \Sigma_X) \to \Omega_{X/Y}(\log \Sigma_X)$. We set

$$\begin{aligned} K &= ((K_{\mathbb{R}}, W), (K_{\mathscr{O}}, W, F), \alpha) \\ &= (\mathbb{R} \otimes (Rf_* \operatorname{Kos}(\mathcal{M}), W)|_{Y_0}, (Rf_*\Omega_{X/Y}(\log \Sigma_X), W(D), F), \operatorname{id} \otimes (Rf_*\overline{\psi})|_{Y_0}), \end{aligned}$$

which is a triple as in [FnFs3, 3.7] on Y. Since

$$(\operatorname{Gr}_m^{W(D)} \Omega_{X/Y}(\log \Sigma_X), F) \simeq (a_m)_* \left(\Omega_{D^{(m)}/Y}(\log E \cap D^{(m)})[-m], F[-m]\right)$$

as filtered complexes by Lemmas 6.3 and 6.4, we have

$$\begin{aligned} &(\mathbb{R} \otimes \operatorname{Gr}_m^W \operatorname{Kos}(\mathcal{M}), (\operatorname{Gr}_m^{W(D)} \Omega_{X/Y}(\log \Sigma_X), F), \operatorname{id} \otimes \operatorname{Gr}_m^W \overline{\psi}) \\ &\qquad \simeq (a_m)_* \left(\mathbb{R}_{D^{(m)}}, (\Omega_{D^{(m)}/Y}(\log E \cap D^{(m)}), F[-m]), (2\pi\sqrt{-1})^{-m}\overline{\iota}\right)[-m] \end{aligned} \tag{6.4}$$

for every m, where $\overline{\iota}$ denote the composite of the canonical morphisms

$$\mathbb{R}_{D^{(m)}} \hookrightarrow \mathbb{C}_{D^{(m)}} \to \Omega_{D^{(m)}} \hookrightarrow \Omega_{D^{(m)}}(\log E \cap D^{(m)}) \to \Omega_{D^{(m)}/Y}(\log E \cap D^{(m)}).$$

Thus we can easily check that $K|_{Y_0}$ satisfies the conditions (3.7.1)–(3.7.3) in [FnFs3, 3.7] because $fa_m\colon D^{(m)} \to Y$ is smooth over Y_0 for every m. By Lemma 3.3 together with (3.7.6) of [FnFs3], we obtain a polarizable variation of $\mathbb{R}$-Hodge structure

$$\left(\mathbb{R} \otimes \operatorname{Gr}_m^W R^i f_* \operatorname{Kos}(\mathcal{M}), (\operatorname{Gr}_m^{W(D)} R^i f_*\Omega_{X/Y}(\log \Sigma_X), F), \operatorname{id} \otimes \operatorname{Gr}_m^W R^i f_*\overline{\psi}\right)\Big|_{Y_0}$$

of weight $m+i$ for every i, m. Using the Koszul filtration (4.1) in the proof of Theorem 4.2, we can check the Griffiths transversality for $(R^i f_*\Omega_{X/Y}(\log \Sigma_X), F)|_{Y_0}$ by the same way as in [KtO] (cf. [FnFs1, Lemma 4.5]). Thus the triple

$$\left((\mathbb{R} \otimes R^i f_* \operatorname{Kos}(\mathcal{M}), W[i]), (R^i f_*\Omega_{X/Y}(\log \Sigma_X), W(D)[i], F), \operatorname{id} \otimes R^i f_*\overline{\psi}\right)|_{Y_0}$$

is a graded polarizable variation of $\mathbb{R}$-mixed Hodge structure on Y_0 by [FnFs3, (3.7.5)], and all the local monodromies of $R^i f_* \operatorname{Kos}(\mathcal{M})|_{Y_0}$ along Σ_Y are quasi-unipotent by [FnFs3, (3.7.4)]. Since $R^i f_* \operatorname{Kos}(\mathcal{M})|_{Y_0} \simeq R^i(f_0|_U)_*\mathbb{Q}_U$ by Lemma 6.6, the local system $R^i(f_0|_U)_*\mathbb{R}_U$ is of quasi-unipotent local monodromy along Σ_Y and underlies a graded polarizable variation of $\mathbb{R}$-mixed Hodge structure on Y_0.

Moreover, K satisfies all the assumptions in Theorem 3.9 of [FnFs3] by (6.4) and by [FnFs3, Lemma 4.3]. Therefore, there exist isomorphisms

$$\begin{aligned} R^i f_*\Omega_{X/Y}(\log \Sigma_X) &\simeq {}^{\ell}R^i f_*\Omega_{X/Y}(\log \Sigma_X)|_{Y_0} \\ W(D)_m R^i f_*\Omega_{X/Y}(\log \Sigma_X) &\simeq {}^{\ell}W(D)_m R^i f_*\Omega_{X/Y}(\log \Sigma_X)|_{Y_0} \end{aligned}$$

whose restriction to Y_0 coincide with the identities and the natural isomorphisms

$$\begin{aligned} F^p R^i f_*\Omega_{X/Y}(\log \Sigma_X) &\simeq R^i f_* F^p\Omega_{X/Y}(\log \Sigma_X) \\ \operatorname{Gr}_F^p R^i f_*\Omega_{X/Y}(\log \Sigma_X) &\simeq R^{i-p} f_*\Omega^p_{X/Y}(\log \Sigma_X) \end{aligned}$$

by (3.9.1) and by (3.9.3) of [FnFs3, Theorem 3.9], and $\operatorname{Gr}_F^p \operatorname{Gr}_m^{W(D)} R^i f_*\Omega_{X/Y}(\log \Sigma_X)$ is locally free of finite rank for every $i, m, p \in \mathbb{Z}$ by (3.9.4) of [FnFs3, Theorem 3.9]. Thus the filtration F on $R^i f_*\Omega_{X/Y}(\log \Sigma_X)$ satisfies (6.1.1) and (6.1.2).

For the general case, by Lemma 4.6 of [FnFs3], there exists a closed subspace $\Sigma'_Y \subset \Sigma_Y$ with $\operatorname{codim}_Y \Sigma'_Y \geq 2$ such that $f\colon X \to Y$ restricted over $Y \setminus \Sigma'_Y$ satisfies the conditions

(6.6.1) and (6.6.2). Therefore the filtration F on ${}^{\ell}\mathscr{V}|_{Y\setminus\Sigma'_Y}$ is obtained by the argument above. Moreover, the filtration F on $\mathrm{Gr}_m^{W(D)}\mathscr{V}$ extends to $\mathrm{Gr}_m^{W(D)}({}^{\ell}\mathscr{V})$ by Schmid's nilpotent orbit theorem. Applying Lemma 1.11.2 of [Ks], we obtain an extension of F on ${}^{\ell}\mathscr{V}$ satisfying (6.1.1) and (6.1.2).

In order to prove the admissibility, we may assume $(Y,\Sigma_Y)=(\Delta,\{0\})$ by the definition of admissibility (cf. [Ks, 1,9]). Pulling back the variation by the morphism

$$\Delta \ni t \mapsto t^m \in \Delta \tag{6.5}$$

changes the logarithm of the unipotent part of the monodromy automorphism to its multiple by m. Therefore the existence of the relative monodromy weight filtration can be checked after the pull-back by the morphism (6.5). Moreover, Lemma 1.9.1 of [Ks] enables us to check the extendability of the Hodge filtration after the pull-back by the morphism (6.5). Thus we may assume that $f\colon X\to\Delta$ satisfies the following three conditions:

- X is a Kähler manifold,
- $f^{-1}(0)_{\mathrm{red}}$ is a simple normal crossing divisor on X, and
- the local system $R^i(f_0|_U)_*\mathbb{R}_U$, which underlies the variation of mixed Hodge structure in question, is of unipotent monodromy automorphism around the origin.

Then we obtain the conclusion by [El, Théorème I.1.10 and Proposition I.3.10] (cf. [StZ, §5], [PeS, Theorem 14.51], [BE, Theorem 8.2.13], and so on). □

Remark 6.7. As in [FnFs3, Section 7], we can use Koszul complexes when we construct a cohomological $\mathbb{Q}$-mixed Hodge complex for the proof of the above admissibility.

7. Semipositivity theorems

In this section, we give a variant of semipositivity theorems of Fujita–Zucker–Kawamata (cf. e.g. [B1], [B2], [FnFs1], [FnFs2], [FnFsS], [Kw], [Zuc], [Zuo]), which is necessary for the proof of Theorem 3.1. The arguments here are essentially the same as, but simpler than that in [Fs2].

7.1. Apparently, Theorem 3.1 is a direct consequence of [B1, Theorem 4.5]. However, Theorem 4.5 of [B1] is not correct because there exists a counter example (see [Fs2, Example 4.6]). In this section, we formulate another semipositivity theorem, which is slightly different from [B1, Theorem 4.5], and give the proof of it by adapting the idea in [B1] to our formulation. For more detail, see [Fs2].

7.2. Let X be a smooth complex variety. A filtered vector bundle is a pair $(\mathscr{V},F)$ consisting of a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of finite rank and a finite decreasing filtration F on $\mathscr{V}$, such that $\mathrm{Gr}_F^p\mathscr{V}=F^p\mathscr{V}/F^{p+1}\mathscr{V}$ is locally free for every $p\in\mathbb{Z}$. The notion of a morphism of filtered vector bundles is defined in the trivial way. For a filtered vector bundle $(\mathscr{V},F)$, we set $\mathrm{Gr}_F^{\bullet}\mathscr{V}=\bigoplus_p\mathrm{Gr}_F^p\mathscr{V}$. Similarly, a bifiltered vector bundle is a triple $(\mathscr{V},W,F)$ consisting of a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of finite rank, a finite increasing filtration W, and a finite decreasing filtration F such that $\mathrm{Gr}_F^p\mathrm{Gr}_m^W\mathscr{V}\simeq\mathrm{Gr}_m^W\mathrm{Gr}_F^p\mathscr{V}$ is locally free for every $m,p\in\mathbb{Z}$.

It is said that a filtered vector bundle $(\mathscr{V},F)$ on X underlies a (polarizable) variation of $\mathbb{R}$-Hodge structure of weight w, if there exist a polarizable variation of $\mathbb{R}$-Hodge structure $(\mathbb{V},F)$ of weight w and an isomorphism $(\mathscr{V},F)\simeq(\mathscr{O}_X\otimes_{\mathbb{R}}\mathbb{V},F)$ as filtered vector bundles.

7.3. Let X be a smooth complex variety and D a simple normal crossing divisor on X. We set $X_0=X\setminus D$. A variation of $\mathbb{R}$-Hodge structure $(\mathbb{V},F)$ of weight w on X_0 is said to be unipotent, if the monodromy automorphism around each irreducible component of D is unipotent. Let $(\mathscr{V}, F)$ be a filtered vector bundle on X such that $(\mathscr{V}, F)|_{X_0}$ underlies a unipotent variation of $\mathbb{R}$-Hodge structure $(\mathbb{V}, F)$ of weight w on X_0. We call $(\mathscr{V}, F)$ the canonical extension of $(\mathbb{V}, F)$ if there exists an integrable log connection

$$\nabla\colon \mathscr{V} \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \mathscr{V}$$

such that the residue morphism of ∇ along each irreducible component of D is nilpotent and that $\nabla|_{X_0} \simeq d \otimes \mathrm{id}$ under the isomorphisms $\mathscr{V}|_{X_0} \simeq \mathscr{O}_X \otimes_{\mathbb{R}} \mathbb{V}$ and $(\Omega^1_X(\log D) \otimes_{\mathscr{O}_{X_0}} \mathscr{V})|_{X_0} \simeq \Omega^1_{X_0} \otimes_{\mathbb{R}} \mathbb{V}$. We note that the integrable log connection ∇ above satisfies the condition

$$\nabla(F^p\mathscr{V}) \subset \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} F^{p-1}\mathscr{V}$$

for every $p \in \mathbb{Z}$ because of the Griffiths transversality for $(\mathbb{V}, F)$ on X_0. Then a morphism

$$\theta^p\colon \operatorname{Gr}^p_F \mathscr{V} \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^{p-1}_F \mathscr{V}$$

is induced for every p. Thus the morphism defined by

$$\theta = \bigoplus_p \theta^p\colon \operatorname{Gr}^\bullet_F \mathscr{V} \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^\bullet_F \mathscr{V}$$

is called the Higgs field of $(\mathscr{V}, F)$ as in Section 2.

In the situation above, the filtered vector bundle $(\mathscr{V}, F)$ is called the canonical extension of its restriction $(\mathscr{V}, F)|_{X_0}$ when the variation of $\mathbb{R}$-Hodge structure $(\mathbb{V}, F)$ is not explicitly specified.

7.4. From now, we study a bifiltered vector bundle $(\mathscr{V}, W, F)$ on X satisfying the following two conditions:

(7.4.1) $(\operatorname{Gr}^W_m \mathscr{V}, F)|_{X_0}$ underlies a unipotent polarizable variation of $\mathbb{R}$-Hodge structure of a certain weight on X_0 for every $m \in \mathbb{Z}$.

(7.4.2) $(\operatorname{Gr}^W_m \mathscr{V}, F)$ is the canonical extension of $(\operatorname{Gr}^W_m \mathscr{V}, F)|_{X_0}$ for every $m \in \mathbb{Z}$.

For a bifiltered vector bundle $(\mathscr{V}, W, F)$ satisfying (7.4.1) and (7.4.2), the Higgs field of $(\operatorname{Gr}^W_m \mathscr{V}, F)$ is denoted by θ_m. The composite

$$\operatorname{Gr}^\bullet_F \operatorname{Gr}^W_m \mathscr{V} \xrightarrow{\theta_m} \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^\bullet_F \operatorname{Gr}^W_m \mathscr{V} \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^W_m \operatorname{Gr}^\bullet_F \mathscr{V} \tag{7.1}$$

is denoted by $\tilde{\theta}_m$ for every m, where the second arrow is induced from the canonical isomorphism $\operatorname{Gr}^p_F \operatorname{Gr}^W_m \mathscr{V} \xrightarrow{\simeq} \operatorname{Gr}^W_m \operatorname{Gr}^p_F \mathscr{V}$ for all p. We often omit the subscript m for θ_m and for $\tilde{\theta}_m$ if there is no danger of confusion.

Remark 7.5. If a bifiltered vector bundle $(\mathscr{V}, W, F)$ is the canonical extension of a graded polarizable admissible variation of $\mathbb{R}$-mixed Hodge structure, then it satisfies the conditions (7.4.1) and (7.4.2).

Theorem 7.6. *Let X be a smooth projective variety, D a simple normal crossing divisor on X, and $(\mathscr{V}, W, F)$ a bifiltered vector bundle on X satisfying* (7.4.1) *and* (7.4.2)*. Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module equipped with a surjection $\operatorname{Gr}^\bullet_F \mathscr{V} \to \mathscr{F}$. The filtration W on $\mathscr{V}$ induces a filtration W on $\operatorname{Gr}^\bullet_F \mathscr{V}$, and then the filtration W on $\mathscr{F}$ is induced via the surjection $\operatorname{Gr}^\bullet_F \mathscr{V} \to \mathscr{F}$. Then $\mathscr{F}$ is semipositive if the composite*

$$\operatorname{Gr}^\bullet_F \operatorname{Gr}^W_m \mathscr{V} \xrightarrow{\tilde{\theta}_m} \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^W_m \operatorname{Gr}^\bullet_F \mathscr{V} \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^W_m \mathscr{F} \tag{7.2}$$

is the zero morphism for every $m \in \mathbb{Z}$, where the second morphism is induced from the surjection $\operatorname{Gr}^\bullet_F \mathscr{V} \to \mathscr{F}$.

The rest of this section is devoted to the proof of this theorem.

7.7. Let us assume $\dim X \geq 2$. We take an irreducible component of D denoted by Y, and set $E = D - Y$. Then Y is a smooth projective variety and $E|_Y$ is a simple normal crossing divisor on Y. We set $Y_0 = Y \setminus E|_Y$.

We consider the restriction $\mathscr{V}_Y = \mathscr{O}_Y \otimes_{\mathscr{O}_X} \mathscr{V}$ with the induced filtrations W and F. The residue $\mathrm{Res}_Y(\nabla_m)$ of the associated integrable log connection ∇_m on $\mathrm{Gr}_m^W \mathscr{V}$ is a nilpotent endomorphism of $\mathrm{Gr}_m^W \mathscr{V}_Y \simeq \mathscr{O}_Y \otimes_{\mathscr{O}_X} \mathrm{Gr}_m^W \mathscr{V}$ by the assumption (7.4.2). Then the monodromy weight filtration for $\mathrm{Res}_Y(\nabla_m)$ on $\mathrm{Gr}_m^W \mathscr{V}_Y$ is denoted by $W(Y)$ for every m. Via the surjection $\pi_m \colon W_m\mathscr{V}_Y \to \mathrm{Gr}_m^W \mathscr{V}_Y$, we obtain a sequence of coherent $\mathscr{O}_Y$-modules

$$W_{m-1}\mathscr{V}_Y \subset \cdots \subset \pi_m^{-1}(W(Y)_{l-1} \mathrm{Gr}_m^W \mathscr{V}_Y) \subset \pi_m^{-1}(W(Y)_l \mathrm{Gr}_m^W \mathscr{V}_Y) \subset \cdots \subset W_m\mathscr{V}_Y$$

of finite length for all m. By renumbering the $\mathscr{O}_Y$-submodules $\pi_m^{-1}(W(Y)_l \mathrm{Gr}_m^W \mathscr{V}_Y)$ of $\mathscr{V}_Y$, we obtain a finite increasing filtration M on $\mathscr{V}_Y$ satisfying the following two conditions:

(7.7.1) There exists a strictly increasing map $\chi \colon \mathbb{Z} \to \mathbb{Z}$ such that $M_{\chi(m)}\mathscr{V}_Y = W_m\mathscr{V}_Y$ for every m.

(7.7.2) For every m, the filtration on $\mathrm{Gr}_m^W \mathscr{V}_Y$ induced from M coincides with $W(Y)$ up to a shift.

Below, we will see that $(\mathscr{V}_Y, M, F)$ is a bifiltered vector bundle on Y satisfying the conditions (7.4.1) and (7.4.2).

7.8. First, we consider the pure case, that is, the case where $W_m\mathscr{V} = V, W_{m-1}\mathscr{V} = 0$ for some m. Then $(\mathscr{V}, F)$ is a filtered vector bundle on X such that $(\mathscr{V}, F)|_{X_0}$ underlies a unipotent polarizable variation of $\mathbb{R}$-Hodge structure $(\mathbb{V}, F)$ of weight w on X_0, and that $(\mathscr{V}, F)$ is the canonical extension of $(\mathbb{V}, F)$. This case was already studied in Section 5 of [FnFs1]. Here we briefly recall some results, which will be needed later.

The associated integrable log connection on $\mathscr{V}$ is denoted by ∇. As mentioned in 7.7 above, the nilpotent endomorphism $\mathrm{Res}_Y(\nabla)$ gives us the monodromy weight filtration $W(Y)$ on $\mathscr{V}_Y$. In this case, we set $M = W(Y)$ on $\mathscr{V}_Y$.

The canonical morphism $\Omega_X^1(\log E) \to \Omega_X^1(\log D)$ induces a morphism of $\mathscr{O}_Y$-modules $\mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log E) \to \mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)$, which factors through the surjection $\mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log E) \to \Omega_Y^1(\log E|_Y)$. Thus we obtain a morphism of $\mathscr{O}_Y$-modules $\Omega_Y^1(\log E|_Y) \to \mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)$, which fits in the short exact sequence

$$0 \longrightarrow \Omega_Y^1(\log E|_Y) \longrightarrow \mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D) \longrightarrow \mathscr{O}_Y \longrightarrow 0 \tag{7.3}$$

as in 5.14 of [FnFs1]. By restricting ∇ to Y, we obtain a $\mathbb{C}$-morphism $\mathscr{V}_Y \to \Omega_X^1(\log D) \otimes_{\mathscr{O}_X} \mathscr{V}_Y$ denote by $\nabla|_Y$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & \mathscr{V}_Y & = & \mathscr{V}_Y & & \\ & & & & \downarrow{\scriptstyle \nabla|_Y} & & \downarrow{\scriptstyle \mathrm{Res}_Y(\nabla)} & & \\ 0 & \longrightarrow & \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \mathscr{V}_Y & \longrightarrow & \Omega_X^1(\log D) \otimes_{\mathscr{O}_X} \mathscr{V}_Y & \longrightarrow & \mathscr{V}_Y & \longrightarrow & 0 \end{array} \tag{7.4}$$

by the definition of $\mathrm{Res}_Y(\nabla)$, where the bottom row, which is induced from (7.3), is exact. From the local description in 7.9 below, $\nabla|_Y$ preserves the filtration $W(Y)$, that is,

$$(\nabla|_Y)(W(Y)_l\mathscr{V}_Y) \subset \Omega_X^1(\log D) \otimes_{\mathscr{O}_X} W(Y)_l\mathscr{V}_Y$$

for every l. Therefore the morphism

$$\mathrm{Gr}_l^{W(Y)} \nabla|_Y \colon \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y$$

is induced. Because the composite of $\mathrm{Gr}_l^{W(Y)} \nabla|_Y$ and the morphism

$$\Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y \to \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y$$

induced by the morphism in the bottom row of (7.4) is the zero morphism by the commutativity of (7.4) and by the inclusion $\mathrm{Res}_Y(\nabla)(W(Y)_l\mathscr{V}_Y) \subset W(Y)_{l-2}\mathscr{V}$, a morphism

$$\nabla_Y^{(l)} \colon \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y \to \Omega^1_Y(\log E|_Y) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y$$

is induced from $\mathrm{Gr}_l^{W(Y)} \nabla|_Y$. In other words, we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y & = & \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y \\
\Big\downarrow{\scriptstyle \nabla_Y^{(l)}} & & \Big\downarrow{\scriptstyle \mathrm{Gr}_l^{W(Y)} \nabla|_Y} \\
\Omega^1_Y(\log E|_Y) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y & \longrightarrow & \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y
\end{array}$$

for every l. Since $\nabla_Y^{(l)}$ is an integrable log connection on $\mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y$ by the local description 7.9 below, we obtain a $\mathbb{C}$-local system $\ker(\nabla_Y^{(l)})|_{Y_0}$ such that

$$\mathscr{O}_{Y_0} \otimes \ker(\nabla_Y^{(l)})|_{Y_0} \simeq \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y|_{Y_0}$$

for every l.

7.9. In this paragraph, we give a local description of the objects appeared in 7.8. Let us assume that $X = \Delta^{n+1}$ with the coordinate $(y, z_1, \dots, z_n)$, on which the divisors Y and E are defined by y and $z_1 \cdots z_k$ respectively for some k with $1 \le k \le n$. By the assumption that $(\mathscr{V}, F)|_{X_0}$ underlies a unipotent polarizable variation of $\mathbb{R}$-Hodge structure $(\mathbb{V}, F)$ of weight w on X_0, there exist a finite dimensional $\mathbb{R}$-vector space V, the mutually commuting nilpotent endomorphisms $N, N_1, \dots, N_k$ of V, and an isomorphism

$$\varpi \colon \mathscr{O}_X \otimes_{\mathbb{R}} V \xrightarrow{\simeq} \mathscr{V}$$

such that the pull-back $\varpi^*\nabla$ on $\mathscr{O}_X \otimes_{\mathbb{R}} V$ is given by

$$(\varpi^*\nabla)(f \otimes v) = df \otimes v + (2\pi\sqrt{-1})^{-1} f\Big(\frac{dy}{y} \otimes N(v) + \sum_{i=1}^{k} \frac{dz_i}{z_i} \otimes N_i(v)\Big) \tag{7.5}$$

for $f \in \mathscr{O}_X$ and $v \in V$. Moreover, once we define a (multi-valued) morphism of $\mathscr{O}_{X_0}$-modules $\rho \colon \mathscr{O}_{X_0} \otimes_{\mathbb{R}} V \to \mathscr{O}_{X_0} \otimes_{\mathbb{R}} V$ by

$$\rho = \exp\Big(-(2\pi\sqrt{-1})^{-1}(\log y(\mathrm{id} \otimes N) + \sum_{i=1}^{k} \log z_i(\mathrm{id} \otimes N_i))\Big), \tag{7.6}$$

then we have $(\varpi|_{X_0})^{-1}\mathbb{V} = \rho(V)$. We denote the isomorphism $\mathscr{O}_Y \otimes_{\mathbb{R}} V \xrightarrow{\simeq} \mathscr{V}_Y$ induced from ϖ by ϖ_Y. Since $\mathrm{Res}_Y(\nabla)$ is identified with $(2\pi\sqrt{-1})^{-1} \mathrm{id} \otimes N$ under the isomorphism ϖ_Y, we have

$$\varpi_Y^{-1}(W(Y)_l\mathscr{V}_Y) = \mathscr{O}_Y \otimes_{\mathbb{R}} W(N)_l V,$$

for every l, where $W(N)$ denotes the monodromy weight filtration for N on V. Therefore ϖ_Y induces an isomorphism

$$\varpi_Y^{(l)} \colon \mathscr{O}_Y \otimes_{\mathbb{R}} \mathrm{Gr}_l^{W(N)} V \xrightarrow{\simeq} \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y$$

for every l. Because N_i commutes with N for all $i = 1, \dots, k$, the filtration $W(N)$ is preserved by N_i for all $i = 1, \dots, k$. Therefore $\varpi^*\nabla$ preserves the filtration $W(N)$ by (7.5). Thus $\nabla|_Y$ preserves the filtration $W(Y)$ on $\mathscr{V}_Y$. For the morphism $\nabla_Y^{(l)}$ defined in 7.8, we have

$$((\varpi_Y^{(l)})^*\nabla_Y^{(l)})(f \otimes v) = df \otimes v + (2\pi\sqrt{-1})^{-1} f \sum_{i=1}^{k} \frac{dz_i}{z_i} \otimes N_i(v) \tag{7.7}$$

for $f \in \mathscr{O}_Y$ and $v \in \mathrm{Gr}_l^{W(N)} V$. On the other hand, the morphism ρ in (7.6) induces a morphism $\rho_Y \colon \mathscr{O}_{Y_0} \otimes_{\mathbb{C}} V \to \mathscr{O}_{Y_0} \otimes_{\mathbb{C}} V$, which preserves the filtration $W(N)$ on V. Because of the inclusion $N(W(N)_l V) \subset W(N)_{l-2} V$, we have

$$\mathrm{Gr}_l^{W(N)} \rho_Y = \exp\Big(-(2\pi\sqrt{-1})^{-1} \sum_{i=1}^{k} \log z_i (\mathrm{id} \otimes N_i)\Big)$$

for every l. Therefore the equality

$$\ker(\nabla_Y^{(l)})|_{Y_0} = (\varpi_Y^{(l)}|_{Y_0})((\mathrm{Gr}_l^{W(N)} \rho_Y)(\mathbb{C} \otimes_{\mathbb{R}} V))$$

holds for every l by (7.7). Then

$$(\varpi_Y^{(l)}|_{Y_0})((\mathrm{Gr}_l^{W(N)} \rho_Y)(V)) \subset (\varpi_Y^{(l)}|_{Y_0})((\mathrm{Gr}_l^{W(N)} \rho_Y)(\mathbb{C} \otimes_{\mathbb{R}} V)) = \ker(\nabla_Y^{(l)})|_{Y_0} \tag{7.8}$$

gives us an $\mathbb{R}$-local subsystem of $\ker(\nabla_Y^{(l)})|_{Y_0}$.

Changing the coordinate function z_i to az_i for some $i \in \{1, \dots, k\}$ with $a \in \Gamma(X, \mathscr{O}_X^*)$ yields the commutative diagram

$$\begin{array}{ccc} \mathscr{O}_Y \otimes_{\mathbb{R}} \mathrm{Gr}_l^{W(N)} V & \xrightarrow{\simeq} & \mathscr{O}_Y \otimes_{\mathbb{R}} \mathrm{Gr}_l^{W(N)} V \\ {\scriptstyle \varpi_Y^{(l)}} \downarrow & & \downarrow {\scriptstyle \varpi_Y'^{(l)}} \\ \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y & = & \mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y \end{array} \tag{7.9}$$

where the vertical arrows $\varpi_Y^{(l)}$ and $\varpi_Y'^{(l)}$ are the isomorphisms given by the coordinates $(y, z_1, \dots, z_i, \dots, z_n)$ and $(y, z_1, \dots, az_i, \dots, z_n)$ respectively, and the top horizontal arrow is the isomorphism

$$\exp(-(2\pi\sqrt{-1})^{-1} (\log a) \,\mathrm{id} \otimes N_i) \colon \mathscr{O}_Y \otimes_{\mathbb{R}} \mathrm{Gr}_l^{W(N)} V \to \mathscr{O}_Y \otimes_{\mathbb{R}} \mathrm{Gr}_l^{W(N)} V$$

by choosing a brunch of $\log a$.

7.10. We return to the situation in 7.8. By Lemma 5.10 and Corollary 5.13 of [FnFs1], the triple $(\mathscr{V}_Y, W(Y), F)$ is a bifiltered vector bundle on Y. Next task is to define an $\mathbb{R}$-local system $\mathbb{V}^{(l)}$ on Y_0 with $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{V}^{(l)} \simeq \ker(\nabla_Y^{(l)})|_{Y_0}$ for every l. Such $\mathbb{V}^{(l)}$ can be constructed by gluing as follows: In the local situation in 7.9, the $\mathbb{R}$-local subsystem (7.8) of $\ker(\nabla_Y^{(l)})|_{Y_0}$ is invariant under the change of the coordinate function z_i to az_i with $a \in \mathscr{O}_X^*$ by the commutativity of the diagram (7.9) and the equality

$$\mathrm{Gr}_l^{W(N)} \rho'_Y = \exp(-(2\pi\sqrt{-1})^{-1} (\log a) \,\mathrm{id} \otimes N_i) \cdot \mathrm{Gr}_l^{W(N)} \rho_Y$$

by the suitable choice of a brunch of $\log a$, where ρ'_Y denotes the morphism defined by the same way as ρ_Y from the coordinate $(y, z_1, \dots, az_i, \dots, z_n)$. Thus we obtain an $\mathbb{R}$-local system $\mathbb{V}^{(l)}$ on Y_0 with the desired property $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{V}^{(l)} \simeq \ker(\nabla_Y^{(l)})|_{Y_0}$ for every l. Then Corollary 5.13 and Proposition 5.19 of [FnFs1] imply that $(\mathbb{V}^{(l)}, (\mathrm{Gr}_l^{W(Y)} \mathscr{V}_Y, F)|_{Y_0})$ is a polarizable variation of $\mathbb{R}$-Hodge structure of weight $w + l$ on Y_0. Combined with the fact that the residue morphism of $\nabla_Y^{(l)}$ along every irreducible component of $E|_Y$ is nilpotent by the local description 7.9, the bifiltered vector bundle $(\mathscr{V}_Y, W(Y), F)$ satisfies the conditions (7.4.1) and (7.4.2).

7.11. Next, we discuss the general case in 7.7. For every $k \in \mathbb{Z}$, there exists the unique $m(k) \in \mathbb{Z}$ such that $\chi(m(k)-1) < k \leq \chi(m(k))$ by (7.7.1). Then we have

$$W_{m(k)-1}\mathscr{V}_Y = M_{\chi(m(k)-1)}\mathscr{V}_Y \subset M_{k-1}\mathscr{V}_Y \subset M_k\mathscr{V}_Y \subset M_{\chi(m(k))}\mathscr{V}_Y = W_{m(k)}\mathscr{V}_Y \tag{7.10}$$

for every k. By (7.10), the inclusion $M_k\mathscr{V} \hookrightarrow W_{m(k)}\mathscr{V}$ induces an isomorphism

$$\operatorname{Gr}_k^M \mathscr{V}_Y \xrightarrow{\simeq} \operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y, \tag{7.11}$$

under which the filtration F on the both sides are identified. Because there exists an integer l such that $(\operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y, F) \simeq (\operatorname{Gr}_l^{W(Y)} \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y, F)$ as filtered vector bundles by (7.7.2), the triple $(\mathscr{V}_Y, M, F)$ is a bifiltered vector bundle satisfying (7.4.1) and (7.4.2) for $(Y, E|_Y)$.

Next, we relate the Higgs field of $(\operatorname{Gr}_k^M \mathscr{V}_Y, F)$ to that of $(\operatorname{Gr}_{m(k)}^W \mathscr{V}, F)$. We fix $k \in \mathbb{Z}$ and denote the integrable log connections associated to $(\operatorname{Gr}_{m(k)}^W \mathscr{V}, F)$, $(\operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y, F)$, and $(\operatorname{Gr}_k^M \mathscr{V}_Y, F)$ by $\nabla, \overline{\nabla}$ and ∇_Y respectively. Similarly, the Higgs fields of $(\operatorname{Gr}_{m(k)}^W \mathscr{V}, F)$ and $(\operatorname{Gr}_k^M \mathscr{V}_Y, F)$ are simply denoted by θ and θ_Y respectively. By definition, the diagram

$$\begin{array}{ccc}
\operatorname{Gr}_k^M \mathscr{V}_Y & \xrightarrow{\nabla_Y} & \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_k^M \mathscr{V}_Y \\
\simeq\downarrow & & \downarrow\simeq \\
\operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y & \xrightarrow[\overline{\nabla}]{} & \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y
\end{array}$$

is commutative, where the left vertical arrow is the isomorphism (7.11) and the right is the isomorphism induced by (7.11).

On the other hand, the filtration $W(Y)$ on $\operatorname{Gr}_{m(k)}^W \mathscr{V}_Y$ is preserved by $\nabla|_Y$, and hence so is M on $\operatorname{Gr}_{m(k)}^W \mathscr{V}_Y$ by (7.7.2). Moreover, the diagram

$$\begin{array}{ccc}
\operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y & \xrightarrow{\overline{\nabla}} & \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y \\
\| & & \downarrow \\
\operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y & \xrightarrow[\operatorname{Gr}_k^M(\nabla|_Y)]{} & (\mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y
\end{array}$$

is commutative, where the right vertical arrow is induced from the injection $\Omega_Y^1(\log E|_Y) \hookrightarrow \mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)$ above. Combining these two commutative diagram, we obtain the commutative diagram

$$\begin{array}{ccc}
\operatorname{Gr}_F^p \operatorname{Gr}_k^M \mathscr{V}_Y & \xrightarrow{\theta_Y} & \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_F^{p-1} \operatorname{Gr}_k^M \mathscr{V}_Y \\
\simeq\downarrow & & \downarrow \\
\operatorname{Gr}_F^p \operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y & \xrightarrow{\operatorname{Gr}_F^p \operatorname{Gr}_k^M(\nabla|_Y)} & (\mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_F^{p-1} \operatorname{Gr}_k^M \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y \\
\simeq\downarrow & & \downarrow \\
\operatorname{Gr}_k^M \operatorname{Gr}_F^p \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y & \xrightarrow[\operatorname{Gr}_k^M(\theta|_Y)]{} & (\mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)) \otimes_{\mathscr{O}_Y} \operatorname{Gr}_k^M \operatorname{Gr}_F^{p-1} \operatorname{Gr}_{m(k)}^W \mathscr{V}_Y
\end{array}$$

for every p, where the lower vertical arrows are induced from the exchanging isomorphism between $\mathrm{Gr}_k^M \mathrm{Gr}_F^p$ and $\mathrm{Gr}_F^p \mathrm{Gr}_k^M$ and between $\mathrm{Gr}_k^M \mathrm{Gr}_F^{p-1}$ and $\mathrm{Gr}_F^{p-1} \mathrm{Gr}_k^M$ respectively. Then, by Lemma 7.12 and Corollary 7.13 below, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_F^\bullet \mathrm{Gr}_k^M \mathscr{V}_Y & \xrightarrow{\tilde{\theta}_Y} & \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_k^M \mathrm{Gr}_F^\bullet \mathscr{V}_Y \\ \downarrow & & \downarrow \\ \mathrm{Gr}_k^M \mathrm{Gr}_F^\bullet \mathrm{Gr}_{m(k)}^W \mathscr{V}_Y & \xrightarrow[\mathrm{Gr}_k^M \tilde{\theta}|_Y]{} & (\mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_k^M \mathrm{Gr}_{m(k)}^W \mathrm{Gr}_F^\bullet \mathscr{V}_Y, \end{array} \tag{7.12}$$

where the top horizontal arrow is the composite (7.1) for θ_Y and the bottom horizontal arrow is obtained by taking Gr_k^M on the restriction to Y of the morphism $\tilde{\theta}$.

Lemma 7.12. *Let V be an object of an abelian category equipped with a finite decreasing filtrations F and two increasing filtrations W, M. Under the canonical isomorphism*

$$\mathrm{Gr}_F^p \mathrm{Gr}_m^W V \simeq \mathrm{Gr}_m^W \mathrm{Gr}_F^p V$$

for every m, p, the subobject $M_k \mathrm{Gr}_F^p \mathrm{Gr}_m^W V$ on the left hand side is isomorphic to the subobject $M_k \mathrm{Gr}_m^W \mathrm{Gr}_F^p V$ if $W_{m-1}V \subset M_kV \subset W_mV$.

Proof. Under the canonical isomorphisms

$$\mathrm{Gr}_F^p \mathrm{Gr}_m^W V \simeq \frac{F^pV \cap W_mV}{F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V} \simeq \mathrm{Gr}_m^W \mathrm{Gr}_F^p V \tag{7.13}$$

we compare $M_k \mathrm{Gr}_F^p \mathrm{Gr}_m^W V$ and $M_k \mathrm{Gr}_m^W \mathrm{Gr}_F^p V$. We have

$$\begin{aligned} &M_k \mathrm{Gr}_m^W V \cap F^p \mathrm{Gr}_m^W V \\ &\qquad = \frac{(M_kV \cap W_mV + W_{m-1}V) \cap (F^pV \cap W_mV + W_{m-1}V) + W_{m-1}V}{W_{m-1}V} \\ &\qquad = \frac{(M_kV \cap W_mV + W_{m-1}V) \cap F^pV + W_{m-1}V}{W_{m-1}V} \end{aligned}$$

on $\mathrm{Gr}_m^W V = W_mV/W_{m-1}V$ by definition. Therefore, under the first isomorphism in (7.13),

$$\begin{aligned} &M_k \mathrm{Gr}_F^p \mathrm{Gr}_m^W V \\ &\qquad \simeq \frac{(M_kV \cap W_mV + W_{m-1}V) \cap F^pV + F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V}{F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V} \\ &\qquad = \frac{M_kV \cap F^pV + F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V}{F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V} \end{aligned}$$

by the assumption $W_{m-1}V \subset M_kV \subset W_mV$. On the other hand, we have

$$\begin{aligned} &M_k \mathrm{Gr}_F^p V \cap W_m \mathrm{Gr}_F^p V \\ &\qquad = \frac{(M_kV \cap F^pV + F^{p+1}V) \cap (W_mV \cap F^pV + F^{p+1}V) + F^{p+1}V}{F^{p+1}V} \\ &\qquad = \frac{(M_kV \cap F^pV + F^{p+1}V) \cap W_mV + F^{p+1}V}{F^{p+1}V} \end{aligned}$$

and then

$$M_k \mathrm{Gr}_m^W \mathrm{Gr}_F^p V \simeq \frac{(M_kV \cap F^pV + F^{p+1}V) \cap W_mV + F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V}{F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V}$$

under the second isomorphism in (7.13). By the assumption $M_kV \subset W_mV$, the equality

$$(M_kV \cap F^pV + F^{p+1}V) \cap W_mV = M_kV \cap F^pV + F^{p+1}V \cap W_mV$$

can be easily checked. Thus we obtain

$$M_k \operatorname{Gr}_m^W \operatorname{Gr}_F^p V \simeq \frac{M_kV \cap F^pV + F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V}{F^{p+1}V \cap W_mV + F^pV \cap W_{m-1}V}$$

which implies the conclusion. □

Corollary 7.13. *In the same situation as in Lemma 7.12, we have the commutative diagram*

$$\begin{array}{ccc}
\operatorname{Gr}_F^p \operatorname{Gr}_k^M V & \xrightarrow{\simeq} & \operatorname{Gr}_k^M \operatorname{Gr}_F^p V \\
\downarrow \simeq & & \\
\operatorname{Gr}_F^p \operatorname{Gr}_k^M \operatorname{Gr}_m^W V & & \downarrow \simeq \\
\downarrow \simeq & & \\
\operatorname{Gr}_k^M \operatorname{Gr}_F^p \operatorname{Gr}_m^W V & \xrightarrow[\simeq]{} & \operatorname{Gr}_k^M \operatorname{Gr}_m^W \operatorname{Gr}_F^p V
\end{array}$$

for the canonical isomorphisms if $W_{m-1}V \subset M_{k-1}V \subset M_kV \subset W_mV$.

Proof. As in the proof of Lemma 7.12, all objects in the diagram above are the quotient of $M_kV \cap F^pV$. Thus the conclusion is easily obtained. □

Now we are ready to prove Theorem 7.6.

Proof of Theorem 7.6. What we have to show is the inequality $\deg \mathscr{L} \geq 0$ for every quotient line bundle $\mathscr{L}$ of $f^*\mathscr{F}$ and for a morphism $f\colon C \to X$ from a smooth projective curve C. We prove it by induction on $\dim X$.

First we consider the case of $\dim X = 1$. Then we may assume that f is surjective. Therefore $f^*(\mathscr{V}, W, F)$ satisfies the conditions (7.4.1) and (7.4.2) for $f^{-1}(X_0) \subset C$. Thus we may assume that $X = C$ and $f = \mathrm{id}$. Take the smallest integer m such that the induced morphism $W_m\mathscr{F} \to \mathscr{L}$ is not the zero morphism. Then its image $\mathscr{M}$ admits a surjective morphism $\operatorname{Gr}_m^W \mathscr{F} \to \mathscr{M}$ by definition. On the other hand, $\mathscr{M}$ is an invertible subsheaf of $\mathscr{L}$ because $\dim X = 1$. Then it is sufficient to prove $\deg \mathscr{M} \geq 0$. By replacing $(\mathscr{V}, F)$ by $(\operatorname{Gr}_m^W \mathscr{V}, F)$, we may assume that $(\mathscr{V}, W, F)$ is pure. Then we can apply the argument in the proof of Lemma 4.7 of [B1] to the dual $\mathscr{F}^\vee$, which is a subbundle of $(\operatorname{Gr}_F^\bullet \mathscr{V})^\vee \simeq \operatorname{Gr}_F^\bullet \mathscr{V}^\vee$ contained in the kernel of the Higgs field θ, and obtain the desired inequality $\deg \mathscr{M} \geq 0$ (see also Remark 3.2).

Next, we assume $\dim X \geq 2$. If $f(C) \cap X_0 \neq \emptyset$, then $f^*(\mathscr{V}, W, F)$ satisfies the conditions (7.4.1) and (7.4.2) for $f^{-1}(X_0) \subset C$. In such a case, the same argument as above implies the desired conclusion. Therefore we may assume $f(C) \subset D$. Then there exists an irreducible component Y of D with $f(C) \subset Y$. As constructed in the paragraph 7.7, we have a bifiltered vector bundle $(\mathscr{V}_Y, M, F)$ on Y satisfying the conditions (7.4.1) and (7.4.2) for $(Y, E|_Y)$. The locally free $\mathscr{O}_Y$-module $\mathscr{F}_Y = \mathscr{O}_Y \otimes_{\mathscr{O}_X} \mathscr{F}$ is equipped with the induced surjection $\mathrm{Gr}_F^\bullet \mathscr{V}_Y \to \mathscr{F}_Y$. Then we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_F^\bullet \mathrm{Gr}_k^M \mathscr{V}_Y & \longrightarrow & \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_k^M \mathscr{F}_Y \\ \downarrow & & \downarrow \\ \mathrm{Gr}_k^M \mathrm{Gr}_F^\bullet \mathrm{Gr}_{m(k)}^W \mathscr{V}_Y & \longrightarrow & (\mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_k^M \mathrm{Gr}_{m(k)}^W \mathscr{F}_Y \end{array}$$

by the diagram (7.12). Therefore the composite of $\tilde{\theta}_Y$ and

$$\Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_k^M \mathrm{Gr}_F^\bullet \mathscr{V}_Y \to \Omega_Y^1(\log E|_Y) \otimes_{\mathscr{O}_Y} \mathrm{Gr}_k^M \mathscr{F}$$

as in (7.2), which is the top horizontal arrow in the diagram above, is the zero morphism because the morphism $\Omega_Y^1(\log E|_Y) \to \mathscr{O}_Y \otimes_{\mathscr{O}_X} \Omega_X^1(\log D)$ is injective and because the bottom horizontal arrow is the zero morphism by the assumption for $\mathscr{F}$. Therefore $\mathscr{F}_Y$ is semipositive by the induction hypothesis. Thus $f^*\mathscr{F} = f^*\mathscr{F}_Y$ satisfies the positivity property as desired. □

References

[BE] P. Brosnan, F. El Zein, Variations of mixed hodge structure, in *Hodge Theory*, Mathematical Notes **49** (2014), 333–409. Princeton University Press.

[B1] Y. Brunebarbe, Symmetric differentials and variations of Hodge structures, J. Reine Angew. Math. **743** (2018), 133–161.

[B2] Y. Brunebarbe, Semi-positivity from Higgs bundles, preprint (2017). arXiv:1707.08495 [math.AG]

[C] F. Campana, Kodaira additivity, birational isotriviality, and specialness, Mosc. Math. J. **23** (2023), no. 3, 319–330.

[CP] F. Campana, M. Păun, Foliations with positive slopes and birational stability of orbifold cotangent bundles, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 1–49.

[EjFI] S. Ejiri, O. Fujino, M. Iwai, Positivity of extensions of vector bundles, Math. Z. **306** (2024), no. 3, Paper no. 47.

[El] F. El Zein, Théorie de Hodge des cycles évanescents, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 107–184.

[Fn1] O. Fujino, Notes on the weak positivity theorems, *Algebraic varieties and automorphism groups*, 73–118, Adv. Stud. Pure Math., **75**, Math. Soc. Japan, Tokyo, 2017.

[Fn2] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[Fn3] O. Fujino, On subadditivity of the logarithmic Kodaira dimension, J. Math. Soc. Japan **69** (2017), no. 4, 1565–1581.

[Fn4] O. Fujino, Corrigendum to "On subadditivity of the logarithmic Kodaira dimension", J. Math. Soc. Japan **72** (2020), no. 4, 1181–1187.

[Fn5] O. Fujino, *Iitaka conjecture—an introduction*, SpringerBriefs Math. Springer, Singapore, 2020.

[FnFs1] O. Fujino, T. Fujisawa, Variations of mixed Hodge structure and semipositivity theorems, Publ. Res. Inst. Math. Sci. **50** (2014), no. 4, 589–661.

[FnFs2] O. Fujino, T. Fujisawa, On semipositivity theorems, Math. Res. Lett. **26** (2019), no. 5, 1359–1382.

[FnFs3] O. Fujino, T. Fujisawa, Variation of mixed Hodge structure and its applications, preprint (2023). arXiv:2304.00672 [math.AG]

[FnFsS] O. Fujino, T. Fujisawa, M. Saito, Some remarks on the semipositivity theorems, Publ. Res. Inst. Math. Sci. **50** (2014), no. 1, 85–112.

[Fs1] T. Fujisawa, Mixed Hodge structures on log smooth degenerations, Tohoku Math. J. (2) **60** (2008), no. 1, 71–100.

[Fs2] T. Fujisawa, A remark on semipositivity theorems, preprint (2017). arXiv:1710.01008 [math.AG]

[Ft] T. Fujita, On Kähler fiber spaces over curves, J. Math. Soc. Japan **30** (1978), no. 4, 779–794.

[H1] K. Hashizume, On the non-vanishing conjecture and existence of log minimal models, Publ. Res. Inst. Math. Sci. **54** (2018), no. 1, 89–104.

[H2] K. Hashizume, Log Iitaka conjecture for abundant log canonical fibrations, Proc. Japan Acad. Ser. A Math. Sci. **96** (2020), no. 10, 87–92.

[I] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Math., vol. **239**. Springer-Verlag, Berlin–New York, 1971.

[Ks] M. Kashiwara, A study of variation of mixed Hodge structure, Publ. Res. Inst. Math. Sci. **22** (1986), no. 5, 991–1024.

[KtO] N. Katz, T. Oda, On the differentiation of De Rham cohomology classes with respect to parameters, J. Math. Kyoto Univ. **8** (1968), 199–213.

[Kw] Y. Kawamata, Characterization of abelian varieties, Compositio Math. **43** (1981), no. 2, 253–276.

[L] R. Lazarsfeld, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], **49**. Springer-Verlag, Berlin, 2004.

[Ma] K. Maehara, The weak 1-positivity of direct image sheaves, J. Reine Angew. Math. **364** (1986), 112–129.

[Mo] S. Mori, Classification of higher-dimensional varieties, *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, 269–331, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[N] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, 2004.

[Pa] S. G. Park, Logarithmic base change theorem and smooth descent of positivity of log canonical divisor, preprint (2022). arXiv:2210.02825 [math.AG]

[PeS] C. A. M. Peters, J. H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], **52**. Springer-Verlag, Berlin, 2008.

[Po] M. Popa, Conjectures on the Kodaira dimension, to appear in the London Math. Soc. Lecture Note Ser. volume in honor of V. Shokurov's 70th birthday.

[PoS1] M. Popa, C. Schnell, Viehweg's hyperbolicity conjecture for families with maximal variation, Invent. Math. **208** (2017), no. 3, 677–713.

[PoS2] M. Popa, C. Schnell, On the behavior of the Kodaira dimension under smooth morphisms, Algebr. Geom. **10** (2023), no. 5, 607–619.

[PoW] M. Popa, L. Wu, Weak positivity for Hodge modules, Math. Res. Lett. **23** (2016), no. 4, 1139–1155.

[Sc] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Invent. Math. **22** (1973), 211–319.

[St] J. Steenbrink, Logarithmic embeddings of varieties with normal crossings and mixed Hodge structures, Math. Ann. **301** (1995), no. 1, 105–118.

[StZ] J. Steenbrink, S. Zucker, Variation of mixed Hodge structure. I, Invent. Math. **80** (1985), no. 3, 489–542.

[U] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Notes written in collaboration with P. Cherenack, Lecture Notes in Mathematics, Vol. **439**. Springer-Verlag, Berlin-New York, 1975.

[VZ1] E. Viehweg, K. Zuo, On the isotriviality of families of projective manifolds over curves, J. Algebraic Geom. **10** (2001), no. 4, 781–799.

[VZ2] E. Viehweg, K. Zuo, Base spaces of non-isotrivial families of smooth minimal models, *Complex geometry (Göttingen, 2000)*, 279–328, Springer, Berlin, 2002.

[Zuc] S. Zucker, Remarks on a theorem of Fujita, J. Math. Soc. Japan **34** (1982), no. 1, 48–54.

[Zuo] K. Zuo, On the negativity of kernels of Kodaira–Spencer maps on Hodge bundles and applications, Kodaira's issue, Asian J. Math. **4** (2000), no. 1, 279–301.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: `fujino@math.kyoto-u.ac.jp`

Department of Mathematics, School of Engineering, Tokyo Denki University, Tokyo, Japan
Email address: fujisawa@mail.dendai.ac.jp