On Kawamata's theorem

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Abstract. We give an alternate proof of the main theorem of Kawamata's paper: Pluricanonical systems on minimal algebraic varieties. Our proof also works for varieties in class $\mathcal{C}$. We note that our proof is completely different from Kawamata's.

2010 Mathematics Subject Classification. Primary 14C20; Secondary 14E30.
Keywords. Base point free theorem, Abundance conjecture, Canonical bundle formula.

1. Introduction

One of the main purposes of this paper is to cut a chain of troubles caused by [Ka, Theorem 4.3]. We give an alternate proof of the following famous theorem, which we call *Kawamata's theorem* in this paper. This theorem is indispensable for the abundance conjecture.

Theorem 1.1 (cf. [KMM, Theorem 6-1-11]). *Let (X, B) be a klt pair and $\pi : X \to S$ a proper surjective morphism of normal varieties. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier divisor on X,*

(b) *$H - (K_X + B)$ is π-nef and π-abundant, and*

(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.*

Then H is π-semi-ample.

It was first proved in [Ka] on the assumption that S is a point. Kawamata's proof heavily depends on a very technical generalization of Kollár's injectivity theorem on *generalized normal crossing varieties* (see [Ka, Section 4]). Once we adopt this difficult injectivity theorem, the X-method works and the proof is essentially the same as the one of the Kawamata–Shokurov base point free theorem. Unfortunately, there is an ambiguity in the proof of [Ka, Theorem 4.3] (see [F2, Remark 3.10.3] and 5.1 below). Thus, our proof is the first rigorous proof of Kawamata's theorem. It is completely different from Kawamata's. His proof relies on the theory of mixed Hodge structures for reducible varieties. Our proof grew out from the theory of variation of Hodge structures, especially, Deligne's canonical extensions

*The author was partially supported by The Sumitomo Foundation, The Inamori Foundation, and by the Grant-in-Aid for Young Scientists (A) ♯20684001 from JSPS.

of Hodge bundles. We note that our method saves Kawamata's theorem but does not recover the results in [Ka, Section 4]. They are completely generalized in [F4, Chapter 2] for *embedded simple normal crossing pairs*. However, [F4] does not recover [Ka, Theorem 4.3]. Compare the arguments in [F4, Chapter 2] with Kawamata's ones. The reader can find a slight generalization of Kawamata's theorem and some other applications of our methods in [F3], [F5], and [FG].

We summarize the contents of this paper. In Section 2, we will give an alternate proof of Kawamata's theorem. By using Ambro's formula, we will reduce Kawamata's theorem to a reformulated version of the Kawamata–Shokurov base point free theorem. Section 3 is an appendix, where we will quickly review Ambro's formula for the reader's convenience. In Section 4, we will prove Kawamata's theorem for varieties in class $\mathcal{C}$, which is [N2, Theorem 5.5]. We separate this section from Section 2 in order not to make needless confusion. In the final section, Section 5, we will make some comments on topics related to Kawamata's theorem for the coming generation.

This paper was first circulated as "A remark on the base point free theorem" on 28 August, 2005 (arXiv:math/0508554v1).

Acknowledgments. The author would like to thank Professors Yujiro Kawamata and Noboru Nakayama for answering his questions. He thanks Professors Daisuke Matsushita and Shigefumi Mori for encouraging him during the preparation of this paper.

We will work over an algebraically closed field k of characteristic zero throughout this paper. We adopt the language of *b-divisors* and use the standard notation of the log minimal model program. See, for example, [C].

2. Proof of Kawamata's theorem

The following theorem is a reformulation of the Kawamata–Shokurov base point free theorem. The original proof works without any changes (cf. [KMM, Theorem 3-1-1]).

Theorem 2.1 (Base point free theorem). *Let (X, B) be a sub klt pair, let $\pi : X \to S$ be a proper surjective morphism of normal varieties, and D a π-nef Cartier divisor on X. Assume the following conditions*:

(1) *$rD - (K_X + B)$ is nef and big over S for some positive integer r, and*

(2) *$\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B) \urcorner + j\overline{D}) \subseteq \pi_*\mathcal{O}_X(jD)$ for every positive integer j, where $\mathbf{A}(X, B)$ is the discrepancy $\mathbb{Q}$-b-divisor and $\overline{D}$ is the Cartier closure of D (see* [C, Example 2.3.12 (1) (3)]*).*

Then mD is π-generated for $m \gg 0$, that is, there exists a positive integer m_0 such that for every $m \geq m_0$ the natural homomorphism $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective.

Before the proof of Theorem 1.1, let us recall the definition of *abundant* divisors, which are called *good* divisors in [Ka]. See [KMM, §6-1].

Definition 2.2 (Abundant divisor). Let X be a complete normal variety and D a $\mathbb{Q}$-Cartier nef divisor on X. We define the *numerical Iitaka dimension* to be

$$\nu(X, D) = \max\{e; D^e \not\equiv 0\}.$$

This means that $D^{e'} \cdot S = 0$ for any e'-dimensional subvarieties S of X with $e' > e$ and there exists an e-dimensional subvariety T of X such that $D^e \cdot T > 0$. Then it is easy to see that $\kappa(X, D) \le \nu(X, D)$, where $\kappa(X, D)$ denotes *Iitaka's D-dimension.* A nef $\mathbb{Q}$-divisor D is said to be *abundant* if the equality $\kappa(X, D) = \nu(X, D)$ holds. Let $\pi : X \to S$ be a proper surjective morphism of normal varieties and D a $\mathbb{Q}$-Cartier divisor on X. Then D is said to be π-abundant if $D|_{X_\eta}$ is abundant, where X_η is the generic fiber of π.

Proof of Theorem 1.1. If $H - (K_X + B)$ is π-big, then the statement follows from the original Kawamata–Shokurov base point free theorem. Thus, from now on, we assume that $H - (K_X + B)$ is not π-big. Then there exists a diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & Z \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\varphi} \\ X & \xrightarrow[\ \pi\ ]{} & S \end{array}$$

which satisfies the following conditions (see [KMM, Proposition 6-1-3 and Remark 6-1-4] or [N1, Lemma 6]):

(i) μ, f and φ are projective morphisms,

(ii) Y and Z are non-singular varieties,

(iii) μ is a birational morphism and f is a surjective morphism having connected fibers,

(iv) there exists a φ-nef and φ-big $\mathbb{Q}$-divisor M_0 on Z such that

$$\mu^*(H - (K_X + B)) \sim_{\mathbb{Q}} f^* M_0,$$

and

(v) there is a φ-nef $\mathbb{Q}$-divisor D on Z such that

$$\mu^* H \sim_{\mathbb{Q}} f^* D.$$

Note that $f : Y \to Z$ is the Iitaka fibration with respect to $H - (K_X + B)$ over S. We put $K_Y + B_Y = \mu^*(K_X + B)$ and $H_Y = \mu^* H$. We note that (Y, B_Y) is not necessarily klt but *sub* klt. Thus, we have $H_Y - (K_Y + B_Y) \sim_{\mathbb{Q}} f^* M_0$ (resp. $H_Y \sim_{\mathbb{Q}} f^* D$), where M_0 (resp. D) is a φ-nef and φ-big (resp. φ-nef) $\mathbb{Q}$-divisor as we saw in (iv) and (v). Furthermore, we can assume that D and H are Cartier divisors and $H_Y \sim f^* D$ by replacing D and H by sufficiently divisible multiples. If necessary, we modify Y and Z birationally and can assume the following conditions:

(1) $K_Y + B_Y \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M)$, where B_Z is the *discriminant* $\mathbb{Q}$-divisor of (Y, B_Y) on Z and M is the *moduli* $\mathbb{Q}$-divisor on Z,

(2) (Z, B_Z) is a sub klt pair,

(3) M is a φ-nef $\mathbb{Q}$-divisor on Z,

(4) $\varphi_*\mathcal{O}_Z(\ulcorner \mathbf{A}(Z, B_Z)\urcorner + j\overline{D}) \subseteq \varphi_*\mathcal{O}_Z(jD)$ for every positive integer j, and

(5) $D - (K_Z + B_Z)$ is φ-nef and φ-big.

Indeed, let $P \subset Z$ be a prime divisor. Let a_P be the largest real number t such that $(Y, B_Y + tf^*P)$ is sub lc over the generic point of P. It is obvious that $a_P = 1$ for all but finitely many prime divisors P of Z. We note that a_P is a positive rational number for any P. The *discriminant* $\mathbb{Q}$*-divisor* on Z is defined by the following formula

$$B_Z = \sum_P (1 - a_P)P.$$

We note that $\llcorner B_Z \lrcorner \leq 0$. By properties (iv) and (v), we can write

$$K_Y + B_Y \sim_{\mathbb{Q}} f^*(M_1)$$

for a $\mathbb{Q}$-Cartier divisor M_1 on Z. We define $M = M_1 - (K_Z + B_Z)$ and call it the *moduli* $\mathbb{Q}$*-divisor* on Z, where B_Z is the discriminant $\mathbb{Q}$-divisor defined above. Note that M is called the *log-semistable part* in [FM, Section 4]. So, the condition (1) obviously holds by the definitions of the discriminant $\mathbb{Q}$-divisor B_Z and the moduli $\mathbb{Q}$-divisor M. If we take birational modifications of Y and Z suitably, we have that M is φ-nef and (Z, B_Z) is sub klt. Thus we obtain (2) and (3). For the details, see [A1, Theorems 0.2 and 2.7] or Theorem 3.2 below. We note the following lemma (cf. [A1, Lemma 6.2]), which we need to apply [A1, Theorems 0.2 and 2.7] or Theorem 3.2 to $f : Y \to Z$ (see the condition (2) in 3.1).

Lemma 2.3. *We have* $\operatorname{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) = 1$.

Proof of Lemma 2.3. Since $\mathcal{O}_Z \simeq f_*\mathcal{O}_Y \subseteq f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner)$, we know

$$\operatorname{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) \geq 1.$$

Without loss of generality, we can shrink S and assume that S is affine. Let A be a φ-very ample divisor such that $f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) \otimes \mathcal{O}_Z(A)$ is φ-generated. Since M_0 is a φ-big $\mathbb{Q}$-divisor on Z, we have $\mathcal{O}_Z(A) \subset \mathcal{O}_Z(mM_0)$ for a sufficiently divisible positive integer m. We note that

$$\pi_*\mu_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner + \overline{f^*(mM_0)}) \simeq \pi_*\mu_*\mathcal{O}_Y(f^*(mM_0)),$$

where $\overline{f^*(mM_0)}$ is the Cartier closure of $f^*(mM_0)$ (see [C, Example 2.3.12 (1)]). It is because $\mu^*(H - (K_X + B)) = H_Y - (K_Y + B_Y) \sim_{\mathbb{Q}} f^*M_0$. Therefore,

$$\begin{aligned} &\varphi_*(f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) \otimes \mathcal{O}_Z(A)) \\ &\subseteq \varphi_*(f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) \otimes \mathcal{O}_Z(mM_0)) \\ &\simeq \varphi_*\mathcal{O}_Z(mM_0). \end{aligned}$$

So, we see that $\operatorname{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) \leq 1$. This completes the proof. □

We know the following lemma by Lemma 9.2.2 and Proposition 9.2.3 in [A2] (see also Theorem 3.2 (a) below).

Lemma 2.4. *We have*

$$\mathcal{O}_Z(\ulcorner \mathbf{A}(Z, B_Z)\urcorner + j\overline{D}) \subseteq f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner + j\overline{H_Y})$$

for every integer j.

Pushing forward by φ, we obtain that

$$\begin{aligned}\varphi_*\mathcal{O}_Z(\ulcorner \mathbf{A}(Z, B_Z)\urcorner + j\overline{D}) &\subseteq \varphi_*f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner + j\overline{H_Y})\\ &\simeq \pi_*\mu_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner + j\overline{H_Y})\\ &\simeq \pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + j\overline{H})\\ &\simeq \pi_*\mathcal{O}_X(jH)\\ &\simeq \pi_*\mu_*\mathcal{O}_Y(jH_Y)\\ &\simeq \varphi_*f_*\mathcal{O}_Y(jH_Y)\\ &\simeq \varphi_*\mathcal{O}_Z(jD)\end{aligned}$$

for every integer j. Thus, we have (4). The relation $H_Y - (K_Y + B_Y) \sim_{\mathbb{Q}} f^*(D - (K_Z + B_Z + M))$ implies that $D - (K_Z + B_Z + M)$ is φ-nef and φ-big. By (3), M is φ-nef. Therefore, $D - (K_Z + B_Z) = D - (K_Z + B_Z + M) + M$ is φ-nef and φ-big. This is condition (5). Apply Theorem 2.1 to D on (Z, B_Z). Then we obtain that D is φ-semi-ample. This implies that H is π-semi-ample. This completes the proof. □

Theorem 1.1 has the following obvious corollaries.

Corollary 2.5. *Let (X, B) be a klt pair and $\pi : X \to S$ a proper surjective morphism of normal varieties. Assume that $K_X + B$ is π-nef and π-abundant. Then $K_X + B$ is π-semi-ample.*

Corollary 2.6. *Let X be a complete normal variety such that $K_X \sim_{\mathbb{Q}} 0$. Assume that X has only klt singularities. Let H be a nef and abundant $\mathbb{Q}$-Cartier divisor on X. Then H is semi-ample.*

We close this section with a useful remark.

Remark 2.7 (cf. [F5, Remark 3.5]). Let $\pi : X \to S$ be a proper surjective morphism of normal varieties and D a π-nef and π-abundant Cartier divisor on X. Then we can easily check that

$$\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(mD)$$

is finitely generated if and only if D is π-semi-ample. See, for example, [F5, Lemma 3.10].

Let B be an effective $\mathbb{Q}$-divisor on X such that (X, B) is klt. By [BCHM], we know that

$$\bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\llcorner m(K_X + B) \lrcorner)$$

is finitely generated.

Assume that $K_X + B$ is π-nef. By the above observation, we obtain that $K_X + B$ is π-semi-ample if and only if $K_X + B$ is π-nef and π-abundant. Therefore, we do not need Theorem 1.1 to obtain Corollaries 2.5 and 2.6.

3. Appendix: Quick review of Ambro's formula

In this appendix, we quickly review Ambro's formula. For the details, see the original paper [A1] or Kollár's survey article [Ko].

3.1. Let $f : X \to Y$ be a proper surjective morphism of normal varieties and $p : Y \to S$ a proper morphism onto a variety S. Assume the following conditions:

(1) $K_X + B$ is $\mathbb{Q}$-Cartier and (X, B) is sub klt over the generic point of Y,

(2) rank $f_* \mathcal{O}_X(\ulcorner \mathbf{A}(X, B) \urcorner) = 1$, and

(3) $K_X + B \sim_{\mathbb{Q}, f} 0$.

By (3), we can write $K_X + B \sim_{\mathbb{Q}} f^* D$ for some $\mathbb{Q}$-Cartier divisor D on Y. Let B_Y be the *discriminant* $\mathbb{Q}$*-divisor* on Y. For the definition, see the proof of Theorem 1.1. We put $M_Y = D - (K_Y + B_Y)$ and call M_Y the *moduli* $\mathbb{Q}$*-divisor* on Y. Then we have $K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y)$. Let $\sigma : Y' \to Y$ be a proper birational morphism from a normal variety Y'. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\ \mu\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[\ \sigma\ ]{} & Y' \end{array}$$

such that

(i) μ is a birational morphism from a normal variety X';

(ii) if we put $K_{X'} + B' = \mu^*(K_X + B)$, then we can write $K_{X'} + B' \sim_{\mathbb{Q}} f'^*(K_{Y'} + B_{Y'} + M_{Y'})$, where $B_{Y'}$ is the discriminant $\mathbb{Q}$-divisor on Y' associated to $f' : X' \to Y'$.

Ambro's theorem [A1, Theorems 0.2 and 2.7] says

Theorem 3.2. *If we choose Y' appropriately, then we have the following properties for every proper birational morphism $\nu : Y'' \to Y'$ from a normal variety Y''.*

(a) *$K_{Y'} + B_{Y'}$ is $\mathbb{Q}$-Cartier and $\nu^*(K_{Y'} + B_{Y'}) = K_{Y''} + B_{Y''}$. In particular, $\mathbf{A}(Y', B_{Y'})_{Y''} = -B_{Y''}$.*

(b) *The moduli $\mathbb{Q}$-divisor $M_{Y'}$ is nef over S and $\nu^*(M_{Y'}) = M_{Y''}$.*

We note that the nefness of the moduli $\mathbb{Q}$-divisor can be proved by using Fujita–Kawamata's semi-positivity theorem. It is a consequence of the theory of variation of Hodge structures. For details, see, for example, [M, Section 5], [F1, Section 5], or [Ko].

4. Kawamata's theorem for varieties in class $\mathcal{C}$

In this section, we treat Nakayama's theorem: [N2, Theorem 5.5], which is Kawamata's theorem for varieties in class $\mathcal{C}$. First, let us recall the definition of the varieties in class $\mathcal{C}$.

Definition 4.1 (Class $\mathcal{C}$). A compact complex variety in class $\mathcal{C}$ is a variety which is dominated by a compact Kähler manifold. It is known that X is in class $\mathcal{C}$ if and only if X is bimeromorphically equivalent to a compact Kähler manifold.

Next, we recall the definitions of the Kähler cone and the nef line bundles on a compact Kähler manifold.

Definition 4.2 (Kähler cone). Let Y be a d-dimensional compact Kähler manifold. We define the Kähler cone $\mathrm{KC}(Y)$ of Y to be the set

$$\{[\omega] \in H^{1,1}(Y, \mathbb{R}); \omega \text{ is a Kähler form on } Y\},$$

where $H^{1,1}(Y, \mathbb{R}) := H^2(Y, \mathbb{R}) \cap H^{1,1}(Y, \mathbb{C})$. Then $\mathrm{KC}(Y)$ is an open convex cone in $H^{1,1}(Y, \mathbb{R})$. $\overline{\mathrm{KC}}(Y)$ is the closure of $\mathrm{KC}(Y)$ in $H^{1,1}(Y, \mathbb{R})$.

Definition 4.3 (cf. [N2, Definition 2.4]). Let L be a line bundle on a compact Kähler manifold Y. L is said to be *nef* if the real first Chern class $c_1(L)$ is contained in $\overline{\mathrm{KC}}(Y)$.

Remark 4.4. For a new numerical characterization of the Kähler cone of a compact Kähler manifold, see [DP, Main Theorem 0.1]. A nef line bundle on a compact Kähler manifold can be characterized numerically by [DP, Corollaries 0.3 and 0.4].

Finally, we recall the definitions of the quasi-nef line bundles, the homological Kodaira dimension, and the big and abundant line bundles, which were introduced in [N2].

Definition 4.5 (cf. [N2, Definition 2.6]). Let X be a compact complex variety in class $\mathcal{C}$. A line bundle L on X is called *quasi-nef* if there exists a bimeromorphic morphism $\mu : Y \to X$ from a compact Kähler manifold Y such that μ^*L is nef.

Definition 4.6 (cf. [N2, Definition 2.9]). Let L be a quasi-nef line bundle on a complex variety X in class $\mathcal{C}$. Take a bimeromorphic morphism $\mu : Y \to X$ from a compact Kähler manifold Y such that μ^*L is nef. Then we define

$$\kappa_{\mathrm{hom}}(L) := \max\{l \geq 0; 0 \neq c_1(\mu^*L)^l \in H^{l,l}(Y, \mathbb{R})\}$$

and call it the *homological Kodaira dimension* of L. It is well-defined, because it is independent of the choice of Y.

Definition 4.7 (cf. [N2, Definition 2.11]). Let L be a line bundle on a compact complex variety X in class $\mathcal{C}$. L is said to be *big* if $\kappa(X, L) = \dim X$. If L is quasi-nef and $\kappa(X, L) = \kappa_{\mathrm{hom}}(L)$, then L is called *abundant*.

Now, we state the main theorem of this section. It is nothing but [N2, Theorem 5.5]. The reader can find some applications of Theorem 4.8 in [COP].

Theorem 4.8 (cf. [N2, Theorem 5.5]). *Let X be a compact normal complex variety in class $\mathcal{C}$, B an effective $\mathbb{Q}$-divisor on X, and H a $\mathbb{Q}$-Cartier divisor on X. Then H is semi-ample under the following conditions*:

(1) *(X, B) is klt,*

(2) *H is quasi-nef,*

(3) *$H - (K_X + B)$ is quasi-nef and abundant, and*

(4) *$\kappa_{\mathrm{hom}}(aH - (K_X + B)) = \kappa_{\mathrm{hom}}(H - (K_X + B))$ and $\kappa(X, aH - (K_X + B)) \geq 0$ for some $a \in \mathbb{Q}$ with $a > 1$.*

Sketch of the proof. First, we recall Nakamaya's result.

Lemma 4.9 ([N2, Proposition 2.14 and Corollary 2.16]). *There exists the following diagram*

$$X \xleftarrow{\ \mu\ } Y \xrightarrow{\ f\ } Z,$$

where

(a) *Y is a compact Kähler manifold and μ is a bimeromorphic morphism,*

(b) *Z is a smooth projective variety,*

(c) *f is surjective and has connected fibers,*

(d) *there exists a nef and big $\mathbb{Q}$-divisor M_0 on Z such that*

$$\mu^*(H - (K_X + B)) \sim_{\mathbb{Q}} f^*M_0,$$

and

(e) *there is a nef $\mathbb{Q}$-divisor D on Z such that*

$$\mu^*H \sim_{\mathbb{Q}} f^*D.$$

We note that Z is a smooth projective *variety.*

Let $f : Y \to Z$ be the proper surjective morphism from a compact Kähler manifold Y to a normal projective variety Z obtained in Lemma 4.9. Let B_Y be a $\mathbb{Q}$-divisor on Y such that $K_Y + B_Y = \mu^*(K_X + B)$. Then we have the following properties:

(1) $K_Y + B_Y$ is $\mathbb{Q}$-Cartier and (Y_z, B_z) is sub klt for general $z \in Z$, where $Y_z = f^{-1}(z)$ and $B_z = B_Y|_{Y_z}$,

(2) $\operatorname{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) = 1$, and

(3) $K_Y + B_Y \sim_{\mathbb{Q},f} 0$.

We note that (1) is obvious by the definition of B_Y, (2) follows from the proof of Lemma 2.3, and (3) is also obvious by Lemma 4.9. Under these conditions (1), (2), and (3), Ambro's theorem (see [A1, Theorems 0.2 and 2.7] or Theorem 3.2) holds if we use [N3, 3.7. Theorem (4)] in the proof of Ambro's theorem. Note that it is not difficult to modify the arguments in [A1] for our setting. More explicitly, let $\sigma : Z' \to Z$ be a proper birational morphism from a normal projective variety Z'. If we choose Z' appropriately, then we have the following properties for every proper birational morphism $\nu : Z'' \to Z'$ from a normal projective variety Z''.

(a) $K_{Z'} + B_{Z'}$ is $\mathbb{Q}$-Cartier and $\nu^*(K_{Z'} + B_{Z'}) = K_{Z''} + B_{Z''}$, where $B_{Z'}$ and $B_{Z''}$ are the discriminant $\mathbb{Q}$-divisors. In particular, $\mathbf{A}(Z', B_{Z'})_{Z''} = -B_{Z''}$.

(b) The moduli $\mathbb{Q}$-divisor $M_{Z'}$ is nef and $\nu^*(M_{Z'}) = M_{Z''}$.

For the details and the notation, see Section 3.

By applying Ambro's theorem to $f : Y \to Z$, the proof of Theorem 1.1 works without any modifications. We note that Z is a *projective* variety. Thus, we obtain the semi-ampleness of H. □

5. Comments for the coming generation

The results in [Ka] had already been used in various papers. We think that almost all the papers only used the main results of [Ka], that is, Theorems 1.1 and 6.1 in [Ka]. Therefore, by this paper, almost all the troubles caused by [Ka, Theorem 4.3] were removed. However, some authors used arguments in [Ka]. We give some comments for the coming generation.

5.1. As we pointed out in [F2, Remark 3.10.3], the proof of [Ka, Theorem 4.3] is not completed (see also [KMM, Theorem 6-1-6]). We recall the trouble in [Ka] here for the reader's convenience.

We use the same notation as in the proof of Theorem 4.3 in [Ka]. By [Ka, Theorem 3.2], ${}'E_1^{p,q} \to {}''E_1^{p,q}$ are zero for all p and q. It does not directly say that

$$H^i(X, \mathcal{O}_X(-\ulcorner L\urcorner)) \to H^i(D, \mathcal{O}_D(-\ulcorner L\urcorner))$$

are zero for all i. So, the proofs of Theorems 4.4, 4.5, 5.1, and 6.1 in [Ka] do not work. It is because everything depends on Theorem 4.3 in [Ka]. Thus, we have no rigorous proofs for [KMM, Theorems 6-1-8, 6-1-9]. In [Ka], there seem to be no troubles except the proof of Theorem 4.3.

If someone corrects the proof of [Ka, Theorem 4.3], then the following comments are unnecessary.

5.2. In [N1], Nakayama obtained the relative version of Kawamata's theorem. The proof given there heavily depends on Kawamata's original proof. So, it does not work by the trouble in [Ka, Theorem 4.3]. Of course, [N1, Theorem 5] is true by our main theorem: Theorem 1.1.

5.3. Section 5 in [N2] contains the same trouble. It is because it depends on Kawamata's paper [Ka]. In Section 4, we give a rigorous proof of [N2, Theorem 5.5].

5.4. In [Fk], Fukuda obtained a slight generalization of Kawamata's theorem. See [Fk, Proposition 3.3]. In the final step of the proof of [Fk, Proposition 3.3], Fukuda used [Ka, Theorem 5.1]. So, Fukuda's original proof also has some troubles by [Ka, Theorem 4.3]. Fortunately, we can prove a slight generalization of [Fk, Proposition 3.3] in [F3, Section 6].

6. References

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