

ON INVERSION OF ADJUNCTION FOR LOG CANONICITY

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ABSTRACT. This is a short note on inversion of adjunction for log canonicity.

1. INTRODUCTION

The following theorem is Kawakita's inversion of adjunction on log canonicity (see [K, Theorem]). Although [K, Theorem] is formulated and proved only for algebraic varieties, his clever and mysterious proof in [K] works in the complex analytic setting. Here we will prove it as an application of the minimal model theory for projective morphisms of complex analytic spaces established in [F] following the argument in [H] with some suitable modifications. Our proof is more geometric than Kawakita's.

Theorem 1.1 (Inversion of adjunction for log canonicity, see [K, Theorem]). *Let X be a normal complex variety and let $S + B$ be an effective $\mathbb{R}$ -divisor on X such that $K_X + S + B$ is $\mathbb{R}$ -Cartier, S is reduced, and S and B have no common irreducible components. Let $\nu: S^\nu \rightarrow S$ be the normalization with $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$, where B_{S^ν} denotes Shokurov's different. Then $(X, S + B)$ is log canonical in a neighborhood of S if and only if (S^ν, B_{S^ν}) is log canonical.*

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We note that X is not necessarily an algebraic variety in Theorem 1.1. It is only a complex analytic space. In this note, we will freely use [F] and [BS].

2. PROOF OF THEOREM 1.1

Let us prove Theorem 1.1 following the argument in [H], where the inversion of adjunction on log canonicity was established for log canonical centers of arbitrary dimension. Our proof given below uses Zariski's subspace theorem as in [K].

Proof of Theorem 1.1. In this proof, we will closely follow the argument in [H] with some suitable modifications. If $(X, S + B)$ is log canonical in a neighborhood of S , then it is easy to see that (S^ν, B_{S^ν}) is log canonical by adjunction. Therefore, it is sufficient to prove that $(X, S + B)$ is log canonical near S under the assumption that (S^ν, B_{S^ν}) is log canonical. Without loss of generality, we may assume that S is irreducible. We take an arbitrary point $P \in S$. We can replace X with a relatively compact Stein open

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This note will be contained in [F].

neighborhood of P since the statement is local. From now on, we will freely shrink X around P suitably throughout the proof without mentioning it explicitly.

Step 1. In this step, we will see that we can reduce the problem to the case where $K_X + S + B$ is $\mathbb{Q}$ -Cartier.

The argument here is more or less well known to the experts and is standard in the theory of minimal models. Hence we will only give a sketch of the proof. As usual, we can write

$$K_X + S + B = \sum_{p=1}^q r_p (K_X + S + B_p)$$

such that $K_X + S + B_p$ is $\mathbb{Q}$ -Cartier, $0 < r_p < 1$ for every p with $\sum_{p=1}^q r_p = 1$, and (S^ν, B_p^ν) is log canonical for every p , where $K_{S^\nu} + B_p^\nu = \nu^*(K_X + S + B_p)$. Note that if $(X, S + B_p)$ is log canonical near S for every p then $(X, S + B)$ is log canonical in a suitable neighborhood of S . Therefore, we can replace $(X, S + B)$ with $(X, S + B_p)$ and assume that $K_X + S + B$ is $\mathbb{Q}$ -Cartier. This is what we wanted.

Step 2. In this step, we will make a good partial resolution of singularities of the pair $(X, S + B)$ by using the minimal model program established in [F].

Let W be a Stein compact subset of X such that $\Gamma(W, \mathcal{O}_X)$ is noetherian and that W contains some open neighborhood of P . By [F, Theorem 1.21], we can take a projective bimeromorphic morphism $\mu: Y \rightarrow X$ with $K_Y + \Delta_Y = \mu^*(K_X + S + B)$ such that

- (i) Y is $\mathbb{Q}$ -factorial over W ,
- (ii) Δ_Y is effective and $\Delta_Y = \sum_j d_j \Delta_j$ is the irreducible decomposition,
- (iii) the pair

$$\left(Y, \Delta'_Y := \sum_{d_j \leq 1} d_j \Delta_j + \sum_{d_j > 1} \Delta_j \right)$$

is divisorial log terminal, and

- (iv) every μ -exceptional divisor appears in $(\Delta'_Y)^{=1}$.

Note that $\mu: Y \rightarrow X$ is sometimes called a dlt blow-up of $(X, S + B)$ in the literature (see [F, Theorem 1.21]). We write $\Delta'_Y = T + \Gamma$, where T is the strict transform of S and $\Gamma := \Delta'_Y - T$, and put

$$\Sigma := \Delta_Y - T - \Gamma = \Delta_Y - \Delta'_Y.$$

We take an effective Cartier divisor E on Y such that $-E$ is μ -ample and $K_Y + T + \Gamma - E$ is μ -nef over W . We note that we can choose E such that E and T have no common components. Then we run the $(K_Y + T + \Gamma)$ -minimal model program with scaling of $-E$ over X around W . We obtain a sequence of flips and divisorial contractions:

$$(Y, T + \Gamma) =: (Y_0, T_0 + \Gamma_0) \xrightarrow{\phi_0} (Y_1, T_1 + \Gamma_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{i-1}} (Y_i, T_i + \Gamma_i) \xrightarrow{\phi_i} \cdots$$

Note that each step exists only after shrinking X around W suitably. Let $\mu_i: Y_i \rightarrow X$ be the induced morphism. For any divisor G on Y , we let G_i denote the pushforward of G on Y_i . We put $\lambda_{-1} := 1$. By construction, there exists a non-increasing sequence of rational numbers $\lambda_i \geq \lambda_{i+1}$ with $i \geq 0$ that is either

- finite with $\lambda_{N-1} > \lambda_N = 0$, or
- infinite with $\lim_{i \rightarrow \infty} \lambda_i = 0$

such that $K_{Y_i} + T_i + \Gamma_i - \lambda E_i$ is nef over W for all $\lambda_{i-1} \geq \lambda \geq \lambda_i$. Without loss of generality, we may assume that each ϕ_i is a flip for every $i \geq i_0$ or that $i_0 = N$, that is, the minimal model program stops at $i_0 = N$. For any positive rational number t , there is an effective $\mathbb{Q}$ -divisor Θ_t on Y such that $\Theta_t \sim_{\mathbb{Q}} \Gamma - tE$ and $(Y, T + \Theta_t)$ is purely log terminal with $\lfloor T + \Theta_t \rfloor = T$. In this case, we see that if $t < \lambda_{i-1}$ then $(Y_i, T_i + \Theta_{t,i})$ is purely log terminal. In particular, $(Y_i, \Theta_{t,i})$ is kawamata log terminal.

Step 3. In this step, we will check that $T_i \cap \Sigma_i = \emptyset$ holds for every i .

We note that T_i is normal since $(Y_i, T_i + \Gamma_i)$ is a divisorial log terminal pair. Therefore, $\mu_i: T_i \rightarrow S$ factors through $\nu: S^\nu \rightarrow S$. By construction, we have $K_{Y_i} + T_i + \Gamma_i + \Sigma_i = \mu_i^*(K_X + S + B)$. Hence

$$(2.1) \quad K_{T_i} + \text{Diff}_{T_i}(\Gamma_i + \Sigma_i) := (K_{Y_i} + T_i + \Gamma_i + \Sigma_i)|_{T_i} = (\mu'_i)^*(K_{S^\nu} + B_{S^\nu})$$

holds, where $\mu'_i: T_i \rightarrow S^\nu$. Assume that $T_i \cap \Sigma_i$ is not empty. Then we see that $(T_i, \text{Diff}_{T_i}(\Gamma_i + \Sigma_i))$ is not log canonical. By (2.1), this is a contradiction since (S^ν, B_{S^ν}) is log canonical by assumption. This implies that $T_i \cap \Sigma_i = \emptyset$ holds for every i . In particular, we have

$$K_{T_i} + \text{Diff}_{T_i}(\Gamma_i + \Sigma_i) = (K_{Y_i} + T_i + \Gamma_i + \Sigma_i)|_{T_i} = (K_{Y_i} + T_i + \Gamma_i)|_{T_i} =: K_{T_i} + \text{Diff}_{T_i}(\Gamma_i).$$

Step 4. In this step, we will show that $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$ is an isomorphism for every i . Moreover, we will prove that if ϕ_i is a flip then ϕ_i is an isomorphism on some open neighborhood of T_i .

First, we assume that ϕ_i is a flip. We consider the following flipping diagram

$$\begin{array}{ccc} (Y_i, T_i + \Gamma_i) & \xrightarrow{\phi_i} & (Y_{i+1}, T_{i+1} + \Gamma_{i+1}) \\ & \searrow \varphi_i \quad \swarrow \varphi_i^+ & \\ & Z_i & \end{array}$$

and we let W_i denote the normalization of $\varphi_i(T_i)$. Let C be any flipping curve. If C is contained in T_i , then we obtain

$$(2.2) \quad (K_{Y_i} + T_i + \Gamma_i) \cdot C = (K_{Y_i} + T_i + \Gamma_i + \Sigma_i) \cdot C = 0$$

since $T_i \cap \Sigma_i = \emptyset$ by Step 3. This is absurd. Hence this implies that the natural map $T_i \rightarrow W_i$ is an isomorphism. By the same argument, we see that the natural map $T_{i+1} \rightarrow W_i$ is also an isomorphism. This means that $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$ is an isomorphism when ϕ_i is a flip. By the above argument, we see that T_{i+1} (resp. T_i) does not contain any flipped (resp. flipping) curves. Note that if $T_i \cdot C > 0$ holds for some flipping curve C then $-T_{i+1}$ is φ_i^+ -ample. Hence T_i is disjoint from the flipping locus. This implies that ϕ_i is an isomorphism near T_i when ϕ_i is a flip.

Next, we assume that ϕ_i is a divisorial contraction. In this case, $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$ is obviously a projective bimeromorphic morphism between normal complex varieties. Let C be any curve contracted by ϕ_i . Assume that C is contained in T_i . Then, by the same computation as in (2.2), we get a contradiction. This means that $\phi_i|_{T_i}: T_i \rightarrow T_{i+1}$ does not contract any curves. Thus, $\phi_i|_{T_i}: T_i \dashrightarrow T_{i+1}$ is an isomorphism.

We get the desired statement.

Step 5. In this step, we will prove that the natural restriction map

$$(\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0}) \rightarrow (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

is surjective over some open neighborhood of P for every positive integer $m \geq a/\lambda_{i_0-1}$ such that $m\Sigma$ is an integral divisor, where a is the smallest positive integer such that aE_{i_0} is Cartier.

By definition, aE_{i_0} is Cartier. By Step 4, $Y_{i_0} \dashrightarrow Y_i$ is an isomorphism on some open neighborhood of T_{i_0} . Therefore, aE_i is Cartier on some open neighborhood of T_i for every $i \geq i_0$. Since $(Y_i, T_i + \Gamma_i)$ is divisorial log terminal and T_i is a $\mathbb{Q}$ -Cartier integral divisor, we have the following short exact sequence:

$$(2.3) \quad 0 \rightarrow \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i - T_i) \rightarrow \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) \rightarrow \mathcal{O}_{T_i}(-aE_i) \rightarrow 0$$

for every $i_0 \geq i$ and every m such that $m\Sigma_i$ is integral. Here, we used the fact that $T_i \cap \Sigma_i = \emptyset$ (see Step 3). Let U be an open neighborhood of P contained in W . For every positive integer $m \geq a$ such that $m\Sigma$ is an integral divisor, there exists i such that $\lambda_{i-1} \geq a/m \geq \lambda_i$. Since

$$-m\Sigma_i - aE_i - T_i \sim_{\mathbb{Q}, \mu_i} K_{Y_i} + \Theta_{\frac{a}{m}, i} + (m-1) \left(K_{Y_i} + T_i + \Gamma_i - \frac{a}{m} E_i \right),$$

$(Y_i, \Theta_{\frac{a}{m}, i})$ is kawamata log terminal, $K_{Y_i} + T_i + \Gamma_i - \frac{a}{m} E_i$ is nef over U , we obtain that

$$(2.4) \quad R^1(\mu_i)_* \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i - T_i) = 0$$

on U by the Kawamata–Viehweg vanishing theorem for projective bimeromorphic morphisms of complex analytic spaces. If further $m \geq a/\lambda_{i_0-1}$, then $i \geq i_0$. Hence the natural restriction map

$$(\mu_i)_* \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) \rightarrow (\mu_i)_* \mathcal{O}_{T_i}(-m\Sigma_i - aE_i) = (\mu_i)_* \mathcal{O}_{T_i}(-aE_i)$$

is surjective on U by (2.3) and (2.4). Note that

$$(\mu_i)_* \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) = (\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0})$$

and

$$(\mu_i)_* \mathcal{O}_{T_i}(-aE_i) = (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

hold because $Y_{i_0} \dashrightarrow Y_i$ is an isomorphism in codimension one and $Y_{i_0} \dashrightarrow Y_i$ is an isomorphism on some open neighborhood of T_{i_0} by Step 4, respectively. Thus, the natural restriction map

$$(2.5) \quad (\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0}) \rightarrow (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

is surjective on U for every positive integer $m \geq a/\lambda_{i_0-1}$ such that $m\Sigma$ is an integral divisor. This is what we wanted.

Step 6. In this final step, we will get a contradiction by assuming that $(X, S+B)$ is not log canonical at P . Here, we will use Zariski's subspace theorem as in [K].

The assumption implies that $P \in \mu(\Sigma)$. Note that the non-log canonical locus of $(X, S+B)$ is $\mu(\Sigma)$ set theoretically. By construction, $(Y_i, T_i + \Gamma_i)$ is divisorial log terminal. Therefore, the non-log canonical locus of $(Y_i, T_i + \Gamma_i + \Sigma_i)$ is nothing but the support of Σ_i . Therefore, $\mu(\Sigma) = \mu_i(\Sigma_i)$ holds set theoretically for every i . Hence we have $P \in \mu_{i_0}(\Sigma_{i_0})$.

Claim. Let $\mathcal{O}_{X,P}$ be the localization of $\mathcal{O}_X$ at P and let m_P denote the maximal ideal of $\mathcal{O}_{X,P}$. For every positive integer n , there exists a divisible positive integer $\nu(n)$ such that

$$(\mu_{i_0})_* \mathcal{O}_{Y_0}(-\nu(n)\Sigma_{i_0} - aE_{i_0})_P \subset m_P^n \subset \mathcal{O}_{X,P}$$

holds, where $(\mu_{i_0})_* \mathcal{O}_{Y_0}(-\nu(n)\Sigma_{i_0} - aE_{i_0})_P$ is the localization of $(\mu_{i_0})_* \mathcal{O}_{Y_0}(-\nu(n)\Sigma_{i_0} - aE_{i_0})$ at P .

Proof of Claim. We take $Q \in \Sigma_{i_0}$ such that $\mu_{i_0}(Q) = P$. We consider $\mathcal{O}_{X,P} \hookrightarrow \mathcal{O}_{Y_0,Q}$, where $\mathcal{O}_{Y_0,Q}$ is the localization of $\mathcal{O}_{Y_0}$ at Q . It is well known that $\mathcal{O}_{X,P}$ is excellent. Therefore, $\mathcal{O}_{X,P}$ is analytically irreducible since X is normal. Since $\mu_{i_0}: Y_{i_0} \rightarrow X$ is a projective bimeromorphic morphism, the quotient field of $\mathcal{O}_{Y_0,Q}$ coincides with the one of $\mathcal{O}_{X,P}$. We note that the natural map $\mathcal{O}_{X,P} \rightarrow \mathcal{O}_{Y_0,Q}/m_Q$ is surjective, where m_Q is the maximal ideal of $\mathcal{O}_{Y_0,Q}$. Hence we can use Zariski's subspace theorem (see [A, (10.13). (A form of ZST)]). Thus we get a large and divisible positive integer $\nu(n)$ with the desired property. $\square$

We consider the localization of the following restriction map $\mathcal{O}_X \simeq (\mu_{i_0})_* \mathcal{O}_{Y_0} \rightarrow (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}$ at P . We put $A = \mathcal{O}_{X,P}$, $M = ((\mu_{i_0})_* \mathcal{O}_{T_{i_0}})_P$, and $N = ((\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0}))_P$. Then, by the surjection (2.5) in Step 5 and Claim, we obtain that $N = (0)$ by Lemma 2.1 below. This is a contradiction.

Hence, we obtain that $(X, S + B)$ is log canonical at P . Since P is an arbitrary point of S , $(X, S + B)$ is log canonical in a neighborhood of S . We finish the proof of Theorem 1.1. $\square$

We used the following easy lemma on commutative algebra in the above proof of Theorem 1.1.

Lemma 2.1. *Let $(A, \mathfrak{m})$ be a noetherian local ring, let M be a finitely generated A -module, and let $\varphi: A \rightarrow M$ be a homomorphism of A -modules. Let $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$ be a chain of ideals of A such that there exists $\nu(n)$ satisfying $I_{\nu(n)} \subset \mathfrak{m}^n$ for every positive integer n . Let N be an A -submodule of M . Assume that $\varphi(I_k) = N$ holds for every positive integer k . Then we have $N = (0)$.*

Proof. Let b be any element of N . Then we can take $a \in I_{\nu(n)} \subset \mathfrak{m}^n$ such that $\varphi(a) = b$. This implies that $b = \varphi(a) \in \mathfrak{m}^n M$. Hence $b \in \mathfrak{m}^n M$ holds for every positive integer n . Thus we obtain $b \in \bigcap_n \mathfrak{m}^n M = (0)$. Therefore, $b = 0$ holds, that is, $N = (0)$. $\square$

We close this short note with a remark.

Remark 2.2. If $(X, S + B)$ is algebraic in Theorem 1.1, then we do not need [F]. It is sufficient to use the minimal model program at the level of [BCHM], the well-known relative Kawamata–Viehweg vanishing theorem, and Zariski's subspace theorem (see, for example, [A, (10.6)]). Our proof given here is longer than Kawakita's one (see [K]). However, it looks more accessible for the experts of the minimal model program since the argument is more or less standard.

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