Log canonical inversion of adjunction

By Osamu FUJINO

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Abstract: This is a short note on the log canonical inversion of adjunction.

Key words: inversion of adjunction; adjunction; log canonical singularities; complex analytic spaces.

1. Introduction The following theorem is Kawakita's inversion of adjunction on log canonicity (see [8, Theorem]). Although [8, Theorem] is formulated and proved only for algebraic varieties, his clever and mysterious proof in [8] works in the complex analytic setting. Here we will prove it as an application of the minimal model theory for projective morphisms of complex analytic spaces established in [6] following the argument in [7] with some suitable modifications. Our proof is more geometric than Kawakita's.

Theorem 1.1 (Log canonical inversion of adjunction, see [8, Theorem]). *Let X be a normal complex variety and let $S+B$ be an effective $\mathbb{R}$-divisor on X such that K_X+S+B is $\mathbb{R}$-Cartier, S is reduced, and S and B have no common irreducible components. Let $\nu\colon S^\nu \to S$ be the normalization with $K_{S^\nu}+B_{S^\nu}=\nu^*(K_X+S+B)$, where B_{S^ν} denotes Shokurov's different. Then $(X, S+B)$ is log canonical in a neighborhood of S if and only if (S^ν, B_{S^ν}) is log canonical.*

We note that X is not necessarily an algebraic variety in Theorem 1.1. It is only a complex analytic space. In this note, we will freely use [6] and [2]. We assume that the reader is familiar with the basic definitions and results of the minimal model theory for algebraic varieties (see, for example, [9], [3], [4], [5], and so on).

2. Quick review of the analytic MMP In this section, we quickly explain the minimal model theory for projective morphisms between complex analytic spaces established in [6].

2.1 (Singularities of pairs). As in the algebraic case, we can define *kawamata log terminal pairs*, *log canonical pairs*, *purely log terminal pairs*, *divisorial log terminal pairs*, and so on, for complex analytic spaces. For the details, see [6, Section 3].

One of the main contributions of [6] is to find out a suitable complex analytic formulation in order to make the original proof of [3] work with only some minor modifications.

2020 Mathematics Subject Classification. Primary 14E30; Secondary 14N30, 32S05.

2.2. Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces. A compact subset of an analytic space is said to be *Stein compact* if it admits a fundamental system of Stein open neighborhoods. It is well known that if W is a Stein compact semianalytic subset of Y then $\Gamma(W, \mathcal{O}_Y)$ is noetherian. From now on, we fix a Stein compact subset W of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Then we can formulate and prove the cone and contraction theorem over some open neighborhood of W as in the usual algebraic case. This is essentially due to Nakayama (see [10]). We say that X is *$\mathbb{Q}$-factorial over W* if every prime divisor defined on an open neighborhood of $\pi^{-1}(W)$ is $\mathbb{Q}$-Cartier at any point $x \in \pi^{-1}(W)$. Then, in [6], we show that we can translate almost all the results in [3] into the above analytic setting suitably (see [6, Section 1]).

Hence we have the minimal model program with ample scaling as in the algebraic case. In Section 4, we will use it in the proof of Theorem 1.1.

2.3 (Minimal model program with ample scaling). Let (X, Δ) be a divisorial log terminal pair such that X is $\mathbb{Q}$-factorial over W and let $C \geq 0$ be a π-ample $\mathbb{R}$-divisor on X such that $(X, \Delta+C)$ is log canonical and that $K_X+\Delta+C$ is nef over W. Then we can run the $(K_X+\Delta)$-*minimal model program with scaling of C over Y around W* from $(X_0, \Delta_0) := (X, \Delta)$ as in the algebraic case. We put $C_0 := C$. Thus we get a sequence of flips and divisorial contractions

$$(X_0, \Delta_0) \overset{\phi_0}{\dashrightarrow} (X_1, \Delta_1) \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} (X_i, \Delta_i) \overset{\phi_i}{\dashrightarrow} (X_{i+1}, \Delta_{i+1}) \overset{\phi_{i+1}}{\dashrightarrow} \cdots$$

over Y with $\Delta_i := (\phi_{i-1})_*\Delta_{i-1}$ and $C_i := (\phi_{i-1})_*C_{i-1}$ for every $i \geq 1$. We note that each step ϕ_i exists only after shrinking Y around W suitably. We also note that

$$\lambda_i := \inf\{\mu \in \mathbb{R}_{\geq 0} \mid K_{X_i}+\Delta_i+\mu C_i \text{ is nef over } W\}$$

and that each step ϕ_i is induced by a $(K_{X_i}+\Delta_i)$-negative extremal ray R_i such that $(K_{X_i}+\Delta_i+\lambda_i C_i)\cdot R_i = 0$. We have

$$\lambda_{-1} := 1 \geq \lambda_0 \geq \lambda_1 \geq \cdots$$

such that this sequence is

- finite with $\lambda_{N-1} > \lambda_N = 0$, or
- infinite with $\lim_{i\to\infty} \lambda_i = 0$.

Of course, it is conjectured that the above minimal model program always terminates after finitely many steps. Unfortunately, however, it is still widely open even when $\pi\colon X \to Y$ is algebraic.

Anyway, for the details of the minimal model theory for projective morphisms of complex analytic spaces, see [6].

3. Zariski's subspace theorem In this short section, we quickly review *Zariski's subspace theorem* following [1].

3.1 (see [1, (1.1)]). Let R_1 and R_2 be noetherian local rings. Then we say that R_2 *dominates* R_1 if R_1 is a subring of R_2 and $m_{R_1} \subset m_{R_2}$ holds, where m_{R_1} (resp. m_{R_2}) is the maximal ideal of R_1 (resp. R_2).

3.2 (see [1, (1,1)]). Let R_1 and R_2 be noetherian local rings such that R_1 is a subring of R_2. We say that R_1 is a *subspace* of R_2 if R_1 with its Krull topology is a subspace of R_2 with its Krull topology. This means that R_2 dominates R_1 and there exists a sequence of non-negative integers $a(n)$ such that $a(n)$ tends to infinity with n and $R_1 \cap m^n_{R_2} \subset m^{a(n)}_{R_1}$ holds for every $n \geq 0$.

3.3 (see [1, (1.1)]). Let R_1 and R_2 be noetherian local domains such that R_1 is a subring of R_2. Then $\operatorname{trdeg}_{R_1} R_2$ denotes the *transcendence degree* of the quotient field of R_2 over the quotient field of R_1. Let $h\colon R_2 \to R_2/m_{R_2}$ be the canonical surjection, where m_{R_2} is the maximal ideal of R_2. Let k be the quotient field of $h(R_1)$ in $h(R_2)$. Then $\operatorname{trdeg}_k h(R_2)$ is called the *residual transcendence degree* of R_2 over R_1 and is denoted by $\operatorname{restrdeg}_{R_1} R_2$.

We need the following form of Zariski's subspace theorem.

Theorem 3.4 (see, for example, [1, (10.13)]). *Let R_1 and R_2 be noetherian local domains such that R_1 is analytically irreducible, R_2 dominates R_1, $\operatorname{trdeg}_{R_1} R_2 < \infty$, and $\dim R_1 + \operatorname{trdeg}_{R_1} R_2 = \dim R_2 + \operatorname{restrdeg}_{R_1} R_2$. Then R_1 is a subspace of R_2.*

Here we do not prove Theorem 3.4. For the details, see [1, §10].

4. Proof of Theorem 1.1 Let us prove Theorem 1.1 following the argument in [7], where the log canonical inversion of adjunction was established for log canonical centers of arbitrary dimension. Our proof given below uses Zariski's subspace theorem as in [8].

Proof of Theorem 1.1. In this proof, we will closely follow the argument in [7] with some suitable modifications. If $(X, S+B)$ is log canonical in a neighborhood of S, then it is easy to see that (S^ν, B_{S^ν}) is log canonical by adjunction. Therefore, it is sufficient to prove that $(X, S+B)$ is log canonical near S under the assumption that (S^ν, B_{S^ν}) is log canonical. Without loss of generality, we may assume that S is irreducible. We take an arbitrary point $P \in S$. We can replace X with a relatively compact Stein open neighborhood of P since the statement is local. From now on, we will freely shrink X around P suitably throughout the proof without mentioning it explicitly.

Step 1. In this step, we will see that we can reduce the problem to the case where $K_X + S + B$ is $\mathbb{Q}$-Cartier.

The argument here is more or less well known to the experts and is standard in the theory of minimal models. Hence we will only give a sketch of the proof. As usual, we can write

$$K_X + S + B = \sum_{p=1}^{q} r_p (K_X + S + B_p)$$

such that $K_X + S + B_p$ is $\mathbb{Q}$-Cartier, $0 < r_p < 1$ for every p with $\sum_{p=1}^{q} r_p = 1$, and (S^ν, B^ν_p) is log canonical for every p, where $K_{S^\nu} + B^\nu_p = \nu^*(K_X + S + B_p)$. Note that if $(X, S+B_p)$ is log canonical near S for every p then $(X, S+B)$ is log canonical in a suitable neighborhood of S. Therefore, we can replace $(X, S+B)$ with $(X, S+B_p)$ and assume that $K_X + S + B$ is $\mathbb{Q}$-Cartier. This is what we wanted.

Step 2. In this step, we will make a good partial resolution of singularities of the pair $(X, S+B)$ by using the minimal model program established in [6] (see also Section 2).

Let W be a Stein compact subset of X such that $\Gamma(W, \mathcal{O}_X)$ is noetherian and that W contains some open neighborhood of P. By [6, Theorem 1.21], we can take a projective bimeromorphic morphism $\mu\colon Y \to X$ with $K_Y + \Delta_Y = \mu^*(K_X + S + B)$ such that

(i) Y is $\mathbb{Q}$-factorial over W,

(ii) Δ_Y is effective and $\Delta_Y = \sum_j d_j \Delta_j$ is the irreducible decomposition,

(iii) the pair

$$\left(Y, \Delta'_Y := \sum_{d_j \leq 1} d_j \Delta_j + \sum_{d_j > 1} \Delta_j\right)$$

is divisorial log terminal, and

(iv) every μ-exceptional divisor appears in $(\Delta'_Y)^{=1} := \sum_{d_j \geq 1} \Delta_j$.

Note that $\mu\colon Y \to X$ is sometimes called a dlt blow-up of $(X, S+B)$ in the literature (see [6, Theorem 1.21]). We write $\Delta'_Y = T + \Gamma$, where T is the strict transform of S and $\Gamma := \Delta'_Y - T$, and put

$$\Sigma := \Delta_Y - T - \Gamma = \Delta_Y - \Delta'_Y.$$

We take an effective Cartier divisor E on Y such that $-E$ is μ-ample and $K_Y + T + \Gamma - E$ is μ-nef over W. We note that we can choose E such that E and T have no common components. Then we run the $(K_Y + T + \Gamma)$-minimal model program with scaling of $-E$ over X around W. We obtain a sequence of flips and divisorial contractions:

$$\begin{aligned}(Y, T+\Gamma) =: (Y_0, T_0 + \Gamma_0) &\overset{\phi_0}{\dashrightarrow} (Y_1, T_1 + \Gamma_1) \\ &\overset{\phi_1}{\dashrightarrow} (Y_2, T_2 + \Gamma_2) \overset{\phi_2}{\dashrightarrow} \cdots \\ &\overset{\phi_{i-1}}{\dashrightarrow} (Y_i, T_i + \Gamma_i) \overset{\phi_i}{\dashrightarrow} \cdots .\end{aligned}$$

Note that each step exists only after shrinking X around W suitably. Let $\mu_i \colon Y_i \to X$ be the induced morphism. For any divisor G on Y, we let G_i denote the pushforward of G on Y_i. We put $\lambda_{-1} := 1$. By construction, there exists a non-increasing sequence of rational numbers $\lambda_i \geq \lambda_{i+1}$ with $i \geq 0$ that is either

- finite with $\lambda_{N-1} > \lambda_N = 0$, or
- infinite with $\lim_{i\to\infty} \lambda_i = 0$

such that $K_{Y_i} + T_i + \Gamma_i - \lambda E_i$ is nef over W for all $\lambda_{i-1} \geq \lambda \geq \lambda_i$. Without loss of generality, we may assume that each ϕ_i is a flip for every $i \geq i_0$ or that $i_0 = N$, that is, the minimal model program stops at $i_0 = N$. For any positive rational number t, there is an effective $\mathbb{Q}$-divisor Θ_t on Y such that $\Theta_t \sim_{\mathbb{Q}} \Gamma - tE$ and $(Y, T + \Theta_t)$ is purely log terminal with $\lfloor T + \Theta_t \rfloor = T$. In this case, we see that if $t < \lambda_{i-1}$ then $(Y_i, T_i + \Theta_{t,i})$ is purely log terminal. In particular, $(Y_i, \Theta_{t,i})$ is kawamata log terminal.

Step 3. In this step, we will check that $T_i \cap \Sigma_i = \emptyset$ holds for every i.

We note that T_i is normal since $(Y_i, T_i + \Gamma_i)$ is a divisorial log terminal pair. Therefore, $\mu_i \colon T_i \to S$ factors through $\nu \colon S^\nu \to S$. By construction, we have $K_{Y_i} + T_i + \Gamma_i + \Sigma_i = \mu_i^*(K_X + S + B)$. Hence

(4.1)
$$\begin{aligned}K_{T_i} + \mathrm{Diff}_{T_i}(\Gamma_i + \Sigma_i) &:= (K_{Y_i} + T_i + \Gamma_i + \Sigma_i)\,|_{T_i} \\ &= (\mu_i')^*(K_{S^\nu} + B_{S^\nu})\end{aligned}$$

holds, where $\mu_i' \colon T_i \to S^\nu$. Assume that $T_i \cap \Sigma_i$ is not empty. Then we see that $(T_i, \mathrm{Diff}_{T_i}(\Gamma_i + \Sigma_i))$ is not log canonical. By (4.1), this is a contradiction since (S^ν, B_{S^ν}) is log canonical by assumption. This implies that $T_i \cap \Sigma_i = \emptyset$ holds for every i. In particular, we have

$$\begin{aligned}K_{T_i} + \mathrm{Diff}_{T_i}(\Gamma_i + \Sigma_i) &= (K_{Y_i} + T_i + \Gamma_i + \Sigma_i)\,|_{T_i} \\ &= (K_{Y_i} + T_i + \Gamma_i)\,|_{T_i} \\ &=: K_{T_i} + \mathrm{Diff}_{T_i}(\Gamma_i).\end{aligned}$$

Step 4. In this step, we will show that $\phi_i|_{T_i} \colon T_i \dashrightarrow T_{i+1}$ is an isomorphism for every i. Moreover, we will prove that if ϕ_i is a flip then ϕ_i is an isomorphism on some open neighborhood of T_i.

First, we assume that ϕ_i is a flip. We consider the following flipping diagram

$$\begin{array}{ccc}(Y_i, T_i + \Gamma_i) & \overset{\phi_i}{\dashrightarrow} & (Y_{i+1}, T_{i+1} + \Gamma_{i+1}) \\ & {\scriptstyle\varphi_i}\searrow \quad \swarrow{\scriptstyle\varphi_i^+} & \\ & Z_i & \end{array}$$

and we let W_i denote the normalization of $\varphi_i(T_i)$. Let C be any flipping curve. If C is contained in T_i, then we obtain

(4.2) $(K_{Y_i} + T_i + \Gamma_i) \cdot C = (K_{Y_i} + T_i + \Gamma_i + \Sigma_i) \cdot C = 0$

since $T_i \cap \Sigma_i = \emptyset$ by Step 3. This is absurd. Hence this implies that the natural map $T_i \to W_i$ is an isomorphism. By the same argument, we see that the natural map $T_{i+1} \to W_i$ is also an isomorphism. This means that $\phi_i|_{T_i} \colon T_i \dashrightarrow T_{i+1}$ is an isomorphism when ϕ_i is a flip. By the above argument, we see that T_{i+1} (resp. T_i) does not contain any flipped (resp. flipping) curves. Note that if $T_i \cdot C > 0$ holds for some flipping curve C then $-T_{i+1}$ is φ_i^+-ample. Hence T_i is disjoint from the flipping locus. This implies that ϕ_i is an isomorphism near T_i when ϕ_i is a flip.

Next, we assume that ϕ_i is a divisorial contraction. In this case, $\phi_i|_{T_i} \colon T_i \dashrightarrow T_{i+1}$ is obviously a projective bimeromorphic morphism between normal complex varieties. Let C be any curve contracted by ϕ_i. Assume that C is contained in T_i. Then, by the same computation as in (4.2), we get a contradiction. This means that $\phi_i|_{T_i} \colon T_i \to T_{i+1}$ does not contract any curves. Thus, $\phi_i|_{T_i} \colon T_i \dashrightarrow T_{i+1}$ is an isomorphism.

We get the desired statement.

Step 5. In this step, we will prove that the natural restriction map

$$(\mu_{i_0})_* \mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0}) \to (\mu_{i_0})_* \mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

is surjective over some open neighborhood of P for every positive integer $m \geq a/\lambda_{i_0 - 1}$ such that $m\Sigma$ is an integral divisor, where a is the smallest positive integer such that aE_{i_0} is Cartier.

By definition, aE_{i_0} is Cartier. By Step 4, $Y_{i_0} \dashrightarrow Y_i$ is an isomorphism on some open neighborhood of T_{i_0} for every $i \geq i_0$. Therefore, aE_i is Cartier on some open neighborhood of T_i for every $i \geq i_0$. Since $(Y_i, T_i + \Gamma_i)$ is divisorial log terminal and T_i is a $\mathbb{Q}$-Cartier integral divisor, we have the following short exact sequence:

(4.3)
$$\begin{aligned}0 \to \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i - T_i) &\to \mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) \\ &\to \mathcal{O}_{T_i}(-aE_i) \to 0\end{aligned}$$

for every $i \geq i_0$ and every m such that $m\Sigma_i$ is integral (cf. [9, Proposition 5.26]). Here, we used the fact that $T_i \cap \Sigma_i = \emptyset$ (see Step 3). Let U be an open neighborhood of P contained in W. For every positive integer $m \geq a$ such that $m\Sigma$ is an integral divisor, there exists i such that $\lambda_{i-1} \geq a/m \geq \lambda_i$. If further $m \geq a/\lambda_{i_0-1}$, then $i \geq i_0$. Since

$$\begin{aligned}&-m\Sigma_i - aE_i - T_i - \left(K_{Y_i} + \Theta_{\frac{a}{m},i}\right)\\ &\sim_{\mathbb{Q},\mu_i} (m-1)\left(K_{Y_i} + T_i + \Gamma_i - \frac{a}{m}E_i\right),\end{aligned}$$

$\left(Y_i, \Theta_{\frac{a}{m},i}\right)$ is kawamata log terminal, $K_{Y_i}+T_i+\Gamma_i-\frac{a}{m}E_i$ is nef over U, we obtain that

$$R^1(\mu_i)_*\mathcal{O}_{Y_i}(-m\Sigma_i - aE_i - T_i) = 0 \tag{4.4}$$

on U by the Kawamata–Viehweg vanishing theorem for projective bimeromorphic morphisms of complex analytic spaces. Hence the natural restriction map

$$\begin{aligned}(\mu_i)_*\mathcal{O}_{Y_i}(-m\Sigma_i - aE_i) \to &(\mu_i)_*\mathcal{O}_{T_i}(-m\Sigma_i - aE_i)\\ &= (\mu_i)_*\mathcal{O}_{T_i}(-aE_i)\end{aligned}$$

is surjective on U by (4.3) and (4.4). Note that

$$\begin{aligned}&(\mu_i)_*\mathcal{O}_{Y_i}(-m\Sigma_i - aE_i)\\ &= (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0})\end{aligned}$$

and

$$(\mu_i)_*\mathcal{O}_{T_i}(-aE_i) = (\mu_{i_0})_*\mathcal{O}_{T_{i_0}}(-aE_{i_0})$$

hold because $Y_{i_0} \dashrightarrow Y_i$ is an isomorphism in codimension one and $Y_{i_0} \dashrightarrow Y_i$ is an isomorphism on some open neighborhood of T_{i_0} by Step 4, respectively. Thus, the natural restriction map

$$(\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(-m\Sigma_{i_0} - aE_{i_0}) \to (\mu_{i_0})_*\mathcal{O}_{T_{i_0}}(-aE_{i_0}) \tag{4.5}$$

is surjective on U for every positive integer $m \geq a/\lambda_{i_0-1}$ such that $m\Sigma$ is an integral divisor. This is what we wanted.

Step 6. In this final step, we will get a contradiction by assuming that $(X, S+B)$ is not log canonical at P. Here, we will use Zariski's subspace theorem as in [8].

The assumption implies that $P \in \mu(\Sigma)$. Note that the non-log canonical locus of $(X, S+B)$ is $\mu(\Sigma)$ set theoretically. By construction, $(Y_i, T_i + \Gamma_i)$ is divisorial log terminal. Therefore, the non-log canonical locus of $(Y_i, T_i + \Gamma_i + \Sigma_i)$ is nothing but the support of Σ_i. Therefore, $\mu(\Sigma) = \mu_i(\Sigma_i)$ holds set theoretically for every i. Hence we have $P \in \mu_{i_0}(\Sigma_{i_0})$.

Claim. *Let $\mathcal{O}_{X,P}$ be the localization of $\mathcal{O}_X$ at P and let m_P denote the maximal ideal of $\mathcal{O}_{X,P}$. For every positive integer n, there exists a divisible positive integer $\nu(n)$ such that*

$$(\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(-\nu(n)\Sigma_{i_0} - aE_{i_0})_P \subset m_P^n \subset \mathcal{O}_{X,P}$$

holds, where $(\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(-\nu(n)\Sigma_{i_0} - aE_{i_0})_P$ denotes the localization of $(\mu_{i_0})_*\mathcal{O}_{Y_{i_0}}(-\nu(n)\Sigma_{i_0} - aE_{i_0})$ at P.

Proof of Claim. We take $Q \in \Sigma_{i_0}$ such that $\mu_{i_0}(Q) = P$. We consider $\mathcal{O}_{X,P} \hookrightarrow \mathcal{O}_{Y_{i_0},Q}$, where $\mathcal{O}_{Y_{i_0},Q}$ is the localization of $\mathcal{O}_{Y_{i_0}}$ at Q. It is well known that $\mathcal{O}_{X,P}$ is excellent. Therefore, $\mathcal{O}_{X,P}$ is analytically irreducible since X is normal. Since $\mu_{i_0}\colon Y_{i_0} \to X$ is a projective bimeromorphic morphism, the quotient field of $\mathcal{O}_{Y_{i_0},Q}$ coincides with the one of $\mathcal{O}_{X,P}$. We note that the natural map $\mathcal{O}_{X,P} \to \mathcal{O}_{Y_{i_0},Q}/m_Q$ is surjective, where m_Q is the maximal ideal of $\mathcal{O}_{Y_{i_0},Q}$. Hence we can use Zariski's subspace theorem (see Theorem 3.4). Thus we get a large and divisible positive integer $\nu(n)$ with the desired property. □

We consider the localization of the following restriction map $\mathcal{O}_X \simeq (\mu_{i_0})_*\mathcal{O}_{Y_{i_0}} \to (\mu_{i_0})_*\mathcal{O}_{T_{i_0}}$ at P. We put $A = \mathcal{O}_{X,P}$, $M = \left((\mu_{i_0})_*\mathcal{O}_{T_{i_0}}\right)_P$, and $N = \left((\mu_{i_0})_*\mathcal{O}_{T_{i_0}}(-aE_{i_0})\right)_P$. Then, by the surjection (4.5) in Step 5 and Claim, we obtain that $N = (0)$ by Lemma 4.1 below. This is a contradiction.

Hence, we obtain that $(X, S+B)$ is log canonical at P. Since P is an arbitrary point of S, $(X, S+B)$ is log canonical in a neighborhood of S. We finish the proof of Theorem 1.1. □

We used the following easy commutative algebra lemma in the above proof of Theorem 1.1.

Lemma 4.1. *Let $(A, \mathfrak{m})$ be a noetherian local ring, let M be a finitely generated A-module, and let $\varphi\colon A \to M$ be a homomorphism of A-modules. Let $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$ be a chain of ideals of A such that there exists $\nu(n)$ satisfying $I_{\nu(n)} \subset \mathfrak{m}^n$ for every positive integer n. Let N be an A-submodule of M. Assume that $\varphi(I_k) = N$ holds for every positive integer k. Then we have $N = (0)$.*

Proof. Let b be any element of N. Then we can take $a \in I_{\nu(n)} \subset \mathfrak{m}^n$ such that $\varphi(a) = b$. This implies that $b = \varphi(a) \in \mathfrak{m}^n M$. Hence $b \in \mathfrak{m}^n M$ holds for every positive integer n. Thus we obtain $b \in \bigcap_n \mathfrak{m}^n M = (0)$. Therefore, $b = 0$ holds, that is, $N = (0)$. □

We close this short note with a remark.

Remark 4.2. If $(X, S+B)$ is algebraic in Theorem 1.1, then we do not need [6]. It is sufficient to use the minimal model program at the level of [3], the well-known relative Kawamata–Viehweg vanishing theorem, and Zariski's subspace theorem (see, for example, [1, (10.6)]). Our proof given here is longer than Kawakita's one (see [8]). However, it looks more accessible for the experts of the minimal model program since the argument is more or less standard.

Acknowledgments. The author would like to thank Masayuki Kawakita very much for answering his questions. He also would like to thank Shunsuke Takagi very much for answering his questions and giving him many fruitful comments. Finally, he thanks the referee for useful suggestions and comments. He was partially supported by JSPS KAKENHI Grant Numbers JP19H01787, JP20H00111, JP21H00974, JP21H04994.

References

[1] S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[2] C. Bănică, O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, Translated from the Romanian. Editura Academiei, Bucharest; John Wiley & Sons, London–New York–Sydney, 1976.

[3] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[4] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[5] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[6] O. Fujino, Minimal model program for projective morphisms between complex analytic spaces, preprint (2022). arXiv:2201.11315 [math.AG]

[7] C. D. Hacon, On the log canonical inversion of adjunction, Proc. Edinb. Math. Soc. (2) **57** (2014), no. 1, 139–143.

[8] M. Kawakita, Inversion of adjunction on log canonicity, Invent. Math. **167** (2007), no. 1, 129–133.

[9] J. Kollár, S. Mori, *Birational geometry of algebraic varieties.* With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998.

[10] N. Nakayama, The lower semicontinuity of the plurigenera of complex varieties, *Algebraic geometry, Sendai, 1985*, 551–590, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.