Math. Z.
DOI 10.1007/s00209-010-0810-6

Mathematische Zeitschrift

On images of weak Fano manifolds

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Received: 7 March 2010 / Accepted: 27 July 2010

Abstract We consider a smooth projective morphism between smooth complex projective varieties. If the source space is a weak Fano (or Fano) manifold, then so is the target space. Our proof is Hodge theoretic. We do not need mod p reduction arguments. We also discuss related topics and questions.

Keywords Fano manifolds · Weak Fano manifolds · Log Fano varieties · Canonical bundle formula · Mod p reduction

Mathematics Subject Classification (2000) Primary 14J45; Secondary 14N30 · 14E30

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Published online: 20 November 2010

1 Introduction

Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties defined over $\mathbb{C}$. The following theorem is one of the main results of this paper.

Theorem 1.1 (cf. Theorem 4.5) *If X is a weak Fano manifold, that is, $-K_X$ is nef and big, then so is Y.*

Our proof of Theorem 1.1 is Hodge theoretic. We do not need mod p reduction arguments. More precisely, we obtain Theorem 1.1 as an application of Kawamata's positivity theorem (cf. [11]). By the same method, we can recover the well-known result on Fano manifolds.

Theorem 1.2 (cf. Theorem 4.7) *If X is a Fano manifold, that is, $-K_X$ is ample, then so is Y.*

Our proof of Theorem 1.2 is completely different from the original one by Kollár et al. [12]. It is the first proof which does not use mod p reduction arguments. We raise a conjecture on the semi-ampleness of anti-canonical divisors.

Conjecture 1.3 *If $-K_X$ is semi-ample, then so is $-K_Y$.*

We reduce Conjecture 1.3 to another conjecture on canonical bundle formulas and give affirmative answers to Conjecture 1.3 in some special cases (cf. Remark 4.2 and Theorem 4.4). In this paper, we obtain the following theorem, which is a key result for the proof of Theorem 1.1 and Theorem 1.2.

Theorem 1.4 (cf. Theorem 4.1) *If $-K_X$ is semi-ample, then $-K_Y$ is nef.*

We note that the proof of Theorem 1.4 is also an application of Kawamata's positivity theorem. We note that it is the first time that Theorem 1.4 is proved without mod p reduction arguments. The reader will recognize that Kawamata's positivity theorem is very powerful. We can find related topics in [21] and [3, Section 3.6]. Note that both of them depend on mod p reduction arguments.

We summarize the contents of this paper. Section 2 is a preliminary section. We recall Kawamata's positivity theorem (cf. Theorem 2.2) here. In Sect. 3, we treat log Fano varieties with only kawamata log terminal singularities. The result obtained in this section will be used in Sect. 4. In Sect. 4, we prove Theorem 1.1, Theorem 1.2, and some related theorems. In Sect. 5, we give some comments and questions on related topics. In the final section: Sect. 6, which is an appendix, we give a mod p reduction approach to Theorem 1.1.

We will work over $\mathbb{C}$, the complex number field, from Sect. 2 to Sect. 4.

2 Preliminaries

We will make use of the standard notation as in the book [13].

Notation For a $\mathbb{Q}$-divisor $D = \sum_{j=1}^{r} d_j D_j$ on a normal variety X such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define

$$D^+ = \sum_{d_j > 0} d_j D_j \quad \text{and} \quad D^- = -\sum_{d_j < 0} d_j D_j .$$

We denote the *round-up* of D by $\ulcorner D \urcorner$. Furthermore, let $f : X \to Y$ be a surjective morphism of varieties. We define

$$D^h = \sum_{f(D_j)=Y} d_j D_j \quad \text{and} \quad D^v = D - D^h.$$

Let X be a normal variety and Δ an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\varphi : Y \to X$ be a projective resolution such that the union of the exceptional locus of φ and the strict transform of Δ has a simple normal crossing support on Y. We put

$$K_Y = \varphi^*(K_X + \Delta) + \sum_i a_i E_i$$

where E_i is a prime divisor for every i and $E_i \neq E_j$ for $i \neq j$. The pair (X, Δ) is called *kawamata log terminal* (*klt*, for short) (resp. *log canonical* (*lc*, for short)) pair if $a_i > -1$ (resp. $a_i \geq -1$) for every i.

Definition 2.1 (*Relative normal crossing divisors*) Let $f : X \to Y$ be a smooth surjective morphism between smooth varieties with connected fibers and $D = \sum_i D_i$ a reduced divisor on X such that $D^h = D$, where D_i is a prime divisor for every i. We say that D is *relatively normal crossing* if D satisfies the condition that for each closed point $x \in X$, there exits an analytic open neighborhood U and $u_1, \ldots, u_k \in \mathcal{O}_{X,x}$ inducing a regular system of parameter on $f^{-1}f(x)$ at x, where $k = \dim f^{-1}f(x)$, such that $D \cap U = \{u_1 \cdots u_l = 0\}$ for some l with $0 \leq l \leq k$.

Let us recall Kawamata's positivity theorem in [11]. It is the main ingredient of this paper.

Theorem 2.2 (Kawamata's positivity theorem) *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum_j P_j$ and $Q = \sum_l Q_l$ be simple normal crossing divisors on X and Y, respectively, such that $f^{-1}(Q) \subseteq P$ and f is smooth over $Y \setminus Q$. Let $D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor (d_j's may be negative or zero), which satisfies the following conditions*:

(1) $f : \mathrm{Supp} D^h \to Y$ *is relatively normal crossing over* $Y \setminus Q$ *and* $f(\mathrm{Supp} D^v) \subseteq Q$,
(2) $d_j < 1$ *unless* $\mathrm{codim}_Y f(P_j) \geq 2$,
(3) $\dim_{\mathbb{C}(\eta)} f_* \mathcal{O}(\ulcorner -D \urcorner) \otimes_{\mathcal{O}_Y} \mathbb{C}(\eta) = 1$, *where η is the generic point of Y, and*
(4) $K_X + D \sim_{\mathbb{Q}} f^*(K_Y + L)$ *for some $\mathbb{Q}$-divisor L on Y.*

Let

$$\begin{aligned} f^*(Q_l) &= \sum_j w_{lj} P_j, \quad \textit{where } w_{lj} > 0, \\ \bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}} \quad \textit{if } f(P_j) = Q_l, \\ \delta_l &= \max\{\bar{d}_j \mid f(P_j) = Q_l\}, \\ \Delta_0 &= \sum \delta_l Q_l, \quad \textit{and} \\ M &= L - \Delta_0. \end{aligned}$$

Then M is nef. We sometimes call M (resp. Δ_0) the moduli part (*resp.* discriminant part).

Remark 2.3 In Theorem 2.2, we note that δ_l can be characterized as follows. If we put

$$c_l = \sup\{t \in \mathbb{Q} \mid K_X + D + t f^* Q_l \text{ is lc over the generic point of } Q_l\}, \text{ then } \delta_l = 1 - c_l.$$

We give a remark on the Stein factorization. We will use Lemma 2.4 in Sect. 4. See also Remark 5.3 below.

Lemma 2.4 (Stein factorization) *Let $f : X \to Y$ be a smooth projective morphism between smooth varieties. Let*

$$f : X \xrightarrow{h} Z \xrightarrow{g} Y$$

be the Stein factorization. Then $g : Z \to Y$ is étale. Therefore, $h : X \to Z$ is a smooth projective morphism between smooth varieties with connected fibers.

Proof By assumption, $R^i f_*\mathcal{O}_X$ is locally free and

$$R^i f_*\mathcal{O}_X \otimes \mathbb{C}(y) \simeq H^i(X_y, \mathcal{O}_{X_y})$$

for every i and any $y \in Y$. By definition, $Z = \mathrm{Spec}_Y f_*\mathcal{O}_X$. Since $g_*\mathcal{O}_Z \simeq f_*\mathcal{O}_X$ is locally free, g is flat. By construction,

$$Z_y = \mathrm{Spec} H^0(X_y, \mathcal{O}_{X_y})$$

consists of n copies of $\mathrm{Spec}\mathbb{C}$ for any $y \in Y$, where n is the rank of $f_*\mathcal{O}_X$. Therefore, g is unramified. This implies that g is étale. Thus, Z is a smooth variety and $h : X \to Z$ is a smooth morphism with connected fibers. □

3 Log Fano varieties

The proof of the following theorem is essentially the same as [4, Theorem 1.2]. We will use similar arguments in Sect. 4.

Theorem 3.1 *Let $f : X \to Y$ be a proper surjective morphism between normal projective varieties with connected fibers. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Assume that $-(K_X + \Delta + \varepsilon f^*H)$ is semi-ample, where ε is a positive rational number and H is an ample Cartier divisor on Y. Then we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample. In particular, if K_Y is $\mathbb{Q}$-Cartier, then $-K_Y$ is big.*

Proof By replacing H with mH and ε with $\frac{\varepsilon}{m}$ for some sufficiently large positive integer m, we can assume that H is very ample and $\varepsilon < 1$. By replacing H with a general member of $|H|$, we can further assume that $(X, \Delta + \varepsilon f^*H)$ is klt. Let A be a general member of a free linear system $|-m(K_X + \Delta + \varepsilon f^*H)|$ such that $(X, \Delta + \varepsilon f^*H + \frac{1}{m}A)$ is klt and

$$K_X + \Delta + \varepsilon f^*H + \frac{1}{m}A \sim_{\mathbb{Q}} 0.$$

We put $\Gamma = \Delta + \varepsilon f^*H + \frac{1}{m}A$. Then we consider the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\mu]{} & Y, \end{array}$$

where

(1) X' and Y' are smooth projective varieties,
(2) ν and μ are projective birational morphisms,
(3) we put $L = -K_{Y'}$ and define a $\mathbb{Q}$-divisor D on X' as follows:

$$K_{X'} + D = \nu^*(K_X + \Gamma),$$

and
(4) there are simple normal crossing divisors P on X' and Q on Y' which satisfy the conditions (1) of Theorem 2.2 and there exists a set of sufficiently small non-negative rational numbers $\{s_l\}$ such that $\mu^*H - \sum_l s_l Q_l$ is ample.

We see that $f' : X' \to Y'$, D, and L satisfy the conditions (1), (2), and (4) in Theorem 2.2. Now we check the condition (3) in Theorem 2.2. We put $h = f \circ \nu$.

Claim 1 $\mathcal{O}_Y = h_*\mathcal{O}_{X'}(\ulcorner -D \urcorner)$

Proof of Claim 1 Since (X, Γ) is klt, we see that $\ulcorner -D \urcorner$ is effective and ν-exceptional. Thus it holds that $\nu_*\mathcal{O}_{X'}(\ulcorner -D \urcorner) \simeq \mathcal{O}_X$. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, we have $\mathcal{O}_Y = h_*\mathcal{O}_{X'}(\ulcorner -D \urcorner)$. □

By Claim 1, we see that $f' : X' \to Y'$ and D satisfy the condition (3) in Theorem 2.2 since μ is birational. If we take $\mathbb{Q}$-divisors Δ_0 and M on Y' as in Theorem 2.2, then

$$K_{X'} + D \sim_{\mathbb{Q}} f'^*(K_{Y'} + M + \Delta_0)$$

and M is nef. We have the following claim about Δ_0.

Claim 2 $\Delta_0^+ \geq \varepsilon \mu^* H$.

Proof of Claim 2 Since H is general, h^*H is reduced. We set $h^*H = \sum_j P_{k_j}$. Note that the coefficient of P_{k_j} in D is ε for every j by the generality of H and A. By the definition of $\bar{d}_{k_j}$, it holds that

$$\bar{d}_{k_j} = d_{k_j} = \varepsilon.$$

Thus, we have $\Delta_0^+ \geq \varepsilon \mu^* H$. □

We decompose $\varepsilon = \varepsilon' + \varepsilon''$ such that ε' and ε'' are positive rational numbers. Since M is nef, $M + \varepsilon'(\mu^*H - \sum_l s_l Q_l)$ is ample. Hence, there exists an effective $\mathbb{Q}$-divisor B such that $M + \varepsilon'(\mu^*H - \sum_l s_l Q_l) \sim_{\mathbb{Q}} B$, $(Y', B + \varepsilon' \sum_l s_l Q_l + \Delta_0^+ + \varepsilon'' \mu^* H)$ is klt, and $\mathrm{Supp}(B + \varepsilon' \sum_l s_l Q_l + \Delta_0^+ + \varepsilon'' \mu^* H - \Delta_0^-)$ is simple normal crossing. If ε' is a sufficiently small positive rational number, then we see that

$$\mathrm{Supp}\left(B + \varepsilon' \sum_l s_l Q_l + \Delta_0^+ + \varepsilon'' \mu^* H - \Delta_0^-\right)^- = \mathrm{Supp}\, \Delta_0^-.$$

We set

$$\Delta_0' = \Delta_0^+ - \varepsilon \mu^* H \quad \text{and} \quad \Omega' = B + \varepsilon' \sum_l s_l Q_l + \Delta_0' + \varepsilon'' \mu^* H - \Delta_0^-.$$

It holds that

$$K_{Y'} + \Omega' \sim_{\mathbb{Q}} K_{Y'} + L \sim_{\mathbb{Q}} 0.$$

By the following claim, $\mu_*\Omega'$ is effective.

Claim 3 *(cf. Claim (B) in [4])* $\mu_*\Delta_0^- = 0$.

Proof of Claim 3 Let $\Delta_0^- = -\sum_k \delta_{l_k} Q_{l_k}$, where $\delta_{l_k} < 0$. If there exists k and j such that $\ulcorner -d_j \urcorner < w_{l_k j}$, it holds that $-d_j + 1 \le w_{l_k j}$ since $w_{l_k j}$ is an integer. Then we obtain $\delta_{l_k} \ge 0$. Thus, it holds that $\ulcorner -d_j \urcorner \ge w_{l_k j}$ for all k and j. Therefore, we have $\ulcorner -D \urcorner \ge f'^* Q_{l_k}$. Since $\mathcal{O}_{Y'} = f'_*\mathcal{O}_{X'}$, we see that $f'_*\mathcal{O}_{X'}(\ulcorner -D \urcorner) \supseteq \mathcal{O}_{Y'}(Q_{l_k})$. By Claim 1, $\mu_* Q_{l_k} = 0$. We finish the proof of Claim 3. □

We put $\Omega = \mu_*\Omega'$. Then we see that Ω is effective by Claim 3,

$$K_{Y'} + \Omega' = \mu^*(K_Y + \Omega), \quad K_Y + \Omega \sim_{\mathbb{Q}} 0, \quad \text{and} \quad \Omega \ge \varepsilon'' H.$$

Thus (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y) \sim_{\mathbb{Q}} \varepsilon'' H$ is ample if we put $\Delta_Y = \Omega - \varepsilon'' H \ge 0$. We finish the proof of Theorem 3.1. □

Remark 3.2 Let (X, B) be a projective klt pair. Then $-(K_X + B)$ is semi-ample if and only if $-(K_X + B)$ is nef and abundant by [6, Theorem 1.1].

The following corollary is obvious by Theorem 3.1.

Corollary 3.3 (cf. [16, Theorem 2.9]) *Let $f : X \to Y$ be a proper surjective morphism between normal projective varieties with connected fibers. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. Then there is an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

For related topics, see [17, Remark 6.5] and [7, Section 5]. We close this section with an easy corollary of Theorem 3.1.

Corollary 3.4 *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is semi-ample. Let n be a positive integer such that $n(K_X + \Delta)$ is Cartier. Then there is an effective $\mathbb{Q}$-divisor Δ_Y on*

$$Y = \operatorname{Proj} \bigoplus_{m \ge 0} H^0(X, \mathcal{O}_X(-mn(K_X + \Delta)))$$

such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.

Proof By definition, Y is a normal projective variety and there is a projective surjective morphism $f : X \to Y$ with connected fibers such that $-(K_X + \Delta) \sim_{\mathbb{Q}} f^* H$, where H is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Then we can apply Theorem 3.1. □

4 Fano and weak Fano manifolds

In this section, we apply Kawamata's positivity theorem to smooth projective morphisms between smooth projective varieties.

We note that the statement of the following theorem is weaker than [3, Corollary 3.15 (a)]. However, the proof of Theorem 4.1 has potential for further generalizations. We describe it in details.

Theorem 4.1 (cf. [3, Corollary 3.15 (a)]) *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties with connected fibers. If $-K_X$ is semi-ample, then $-K_Y$ is nef.*

Proof Let C be an integral curve on Y. Let A be a general member of the free linear system $|-mK_X|$. Then there is a non-empty Zariski open set U of Y such that $C \cap U \neq \emptyset$ and that A is smooth over U. By construction, $K_X + \frac{1}{m}A \sim_{\mathbb{Q}} 0$. Let $\mu : Y' \to Y$ be a resolution such that μ is an isomorphism over U and $\mu^{-1}(Y \setminus U)$ is a simple normal crossing divisor on Y'. We consider the following commutative diagram.

$$\begin{array}{ccc} \widetilde{X} = X \times_Y Y' & \xrightarrow{\varphi} & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

We note that $\widetilde{f} : \widetilde{X} \to Y'$ is smooth. We write $K_{Y'} = \mu^* K_Y + E$. Then $\mathrm{Supp} E = \mathrm{Exc}(\mu)$, where $\mathrm{Exc}(\mu)$ is the exceptional locus of μ, and E is effective. We put

$$K_{\widetilde{X}} + \widetilde{D} = \varphi^* \left(K_X + \frac{1}{m} A \right) \sim_{\mathbb{Q}} 0.$$

Then

$$\widetilde{D} = -\widetilde{f}^* E + \varphi^* \frac{1}{m} A.$$

Note that $K_{\widetilde{X}} = \varphi^* K_X + \widetilde{f}^* E$. We put $U' = \mu^{-1}(U)$. Then $\mu : U' \to U$ is an isomorphism. Let $\psi : X' \to \widetilde{X}$ be a resolution such that ψ is an isomorphism over $\widetilde{f}^{-1}(U')$ and that $\mathrm{Supp} A' \cup \mathrm{Supp} f'^{-1}(Y' \setminus U')$ is a simple normal crossing divisor, where A' is the strict transform of A on X' and $f' = \widetilde{f} \circ \psi : X' \to Y'$. We define

$$K_{X'} + D = \psi^* (K_{\widetilde{X}} + \widetilde{D}) \sim_{\mathbb{Q}} 0.$$

We can write

$$K_{X'} + D = f'^* (K_{Y'} + \Delta_0 + M)$$

as in Kawamata's positivity theorem (see Theorem 2.2). We put $E = \sum_i e_i E_i$, where E_i is a prime divisor for every i and $E_i \neq E_j$ for $i \neq j$. The coefficient of E_i in Δ_0 is $1 - c_i$, where

$$c_i = \sup\{ t \in \mathbb{Q} \mid K_{X'} + D + t f'^* E_i \text{ is lc over the generic point of } E_i \}.$$

By construction,

$$c_i = \sup\{ t \in \mathbb{Q} \mid K_{\widetilde{X}} + \widetilde{D} + t \widetilde{f}^* E_i \text{ is lc over the generic point of } E_i \}.$$

Since

$$\widetilde{D} = -\widetilde{f}^* E + \varphi^* \frac{1}{m} A,$$

and $\varphi^* \frac{1}{m} A$ is effective, we can write $c_i = e_i + a_i$ for some $a_i \in \mathbb{Q}$ with $a_i \leq 1$. Thus, we have $1 - c_i = 1 - e_i - a_i$. Therefore, the coefficient of E_i in $E + \Delta_0$ is

$$e_i + 1 - e_i - a_i = 1 - a_i \geq 0.$$

So, we can see that $E + \Delta_0$ is effective. Since $K_{Y'} + \Delta_0 + M \sim_{\mathbb{Q}} 0$ and $K_{Y'} = \mu^* K_Y + E$, we have

$$-\mu^* K_Y = -K_{Y'} + E \sim_{\mathbb{Q}} E + \Delta_0 + M.$$

Let C' be the strict transform of C on Y'. Then

$$\begin{aligned} C \cdot (-K_Y) &= C' \cdot (-\mu^* K_Y) \\ &= C' \cdot (E + \Delta_0 + M) \geq 0. \end{aligned}$$

It is because M is nef and $\mathrm{Supp}(E + \Delta_0) \subset Y' \setminus U'$. Therefore, $-K_Y$ is nef. □

We give a very important remark on Theorem 4.1.

Remark 4.2 (Semi-ampleness of $-K_Y$) We use the same notation as in Theorem 4.1 and its proof. It is conjectured that the *moduli part* M is semi-ample (see, for example [1, 0. Introduction]). Some very special cases of this conjecture were treated in [5] before [1]. Unfortunately, the results in [5] are useless for our purposes here. If this semi-ampleness conjecture is solved, then we will obtain that $-K_Y$ is semi-ample.

Let $y \in Y$ be an arbitrary point. We can choose A such that $y \in U$. Since

$$-\mu^* K_Y \sim_{\mathbb{Q}} M + E + \Delta_0,$$

$E + \Delta_0$ is effective, and $\mathrm{Supp}(E + \Delta_0) \subset Y' \setminus U'$, we can find a positive integer m and an effective Cartier divisor D on Y such that $-mK_Y \sim D$ and that $y \notin \mathrm{Supp} D$. It implies that $-K_Y$ is semi-ample.

By [10], M is semi-ample if $\dim Y = \dim X - 1$. Therefore, $-K_Y$ is semi-ample when $\dim Y = \dim X - 1$.

In [2, Theorem 3.3], Ambro proved that M is nef and abundant. So, if Y is a surface, then we can check that $-K_Y$ is semi-ample as follows. If $\nu(Y', M) = \kappa(Y', M) = 0$ or 1, then M is semi-ample. Therefore, we can apply the same argument as above. If $\nu(Y', M) = \kappa(Y', M) = 2$, then M is big. Since

$$-\mu^* K_Y \sim_{\mathbb{Q}} M + E + \Delta_0$$

and $E + \Delta_0$ is effective, $-\mu^* K_Y$ is big. Therefore, $-K_Y$ is nef and big. In this case, $-K_Y$ is semi-ample by the Kawamata–Shokurov base point free theorem. Anyway, for an arbitrary point $y \in Y$, we can always find a positive integer m and an effective Cartier divisor D on Y such that $-mK_Y \sim D$ and that $y \notin \mathrm{Supp} D$. It means that $-K_Y$ is semi-ample.

In the end, in Theorem 4.1, $-K_Y$ is semi-ample if $\dim Y \leq 2$. By combining the above results, we know that $-K_Y$ is semi-ample when $\dim X \leq 4$. We conjecture that $-K_Y$ is semi-ample if $-K_X$ is semi-ample without any assumptions on dimensions.

Remark 4.3 In Remark 4.2, we used Ambro's results in [1] and [2]. When we investigate the moduli part M on Y by the theory of variations of Hodge structures, we note the following construction. Let $\pi : V \to X$ be a cyclic cover associated to $m(K_X + \frac{1}{m}A) \sim 0$. In this case, π is a finite cyclic cover which is ramified only along $\mathrm{Supp} A$. Since $\mathrm{Supp} A$ is relatively normal crossing over U, we can construct a simultaneous resolution $f \circ \pi : V \to Y$ and make the union of the exceptional locus and the inverse image of $\mathrm{Supp} A$ a simple normal crossing divisor and relatively normal crossing over U by the canonical desingularization theorem. Therefore, the moduli part M on X behaves well under pull-backs. It is a very important remark.

The semi-ampleness of $-K_Y$ is not so obvious even when $-K_X \sim_{\mathbb{Q}} 0$. The proof of the following theorem depends on some deep results on the theory of variations of Hodge structures (cf. [2] and [6]).

Theorem 4.4 *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties. Assume that $-K_X \sim_{\mathbb{Q}} 0$. Then $-K_Y$ is semi-ample.*

Proof By the Stein factorization (cf. Lemma 2.4), we can assume that f has connected fibers. In this case, we can write

$$K_X \sim_{\mathbb{Q}} f^*(K_Y + M),$$

where M is the moduli part. By [2, Theorem 3.3], we know that M is nef and abundant. Therefore, $-K_Y$ is nef and abundant. This implies that $-K_Y$ is semi-ample by [6, Theorem 1.1]. □

The following theorem is one of the main results of this paper. We note that it was proved by Yasutake in a special case where $f : X \to Y$ is a $\mathbb{P}^n$-bundle (cf. [20]).

Theorem 4.5 (Weak Fano manifolds) *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties. If X is a weak Fano manifold, then so is Y.*

Proof By taking the Stein factorization, we can assume that f has connected fibers (cf. Lemma 2.4). By Theorem 4.1, $-K_Y$ is nef since $-K_X$ is semi-ample by the Kawamata–Shokurov base point free theorem. By Kodaira's lemma, we can find an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt and that $-(K_X + \Delta)$ is ample. By Theorem 3.1, we can find an effective $\mathbb{Q}$-divisor Δ_Y such that $-(K_Y + \Delta_Y)$ is ample. Therefore, $-K_Y$ is big. So, $-K_Y$ is nef and big. This means that Y is a weak Fano manifold. □

The following example is due to Hiroshi Sato.

Example 4.6 (Sato) Let Σ be the fan in $\mathbb{R}^3$ whose rays are generated by

$$\begin{aligned} &x_1 = (1, 0, 1), \quad x_2 = (0, 1, 0), \quad x_3 = (-1, 3, 0), \quad x_4 = (0, -1, 0), \\ &y_1 = (0, 0, 1), \quad y_2 = (0, 0, -1), \end{aligned}$$

and their maximal cones are

$$\begin{aligned} &\langle x_1, x_2, y_1\rangle, \langle x_1, x_2, y_2\rangle, \langle x_2, x_3, y_1\rangle, \langle x_2, x_3, y_2\rangle, \\ &\langle x_3, x_4, y_1\rangle, \langle x_3, x_4, y_2\rangle, \langle x_4, x_1, y_1\rangle, \langle x_4, x_1, y_2\rangle. \end{aligned}$$

Let Δ be the fan obtained from Σ by successive star subdivisions along the rays spanned by

$$z_1 = x_2 + y_1 = (0, 1, 1)$$

and

$$z_2 = x_2 + z_1 = 2x_1 + y_1 = (0, 2, 1).$$

We see that $V = X(\Sigma)$, the toric threefold corresponding to the fan Σ with respect to the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$, is a $\mathbb{P}^1$-bundle over $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$. We note that the $\mathbb{P}^1$-bundle structure $V \to Y$ is induced by the projection $\mathbb{Z}^3 \to \mathbb{Z}^2 : (x, y, z) \mapsto (x, y)$. The toric variety $X = X(\Delta)$ corresponding to the fan Δ was obtained by successive blow-ups from V. We can check that X is a three-dimensional toric weak Fano manifold and that the induced morphism $f : X \to Y$ is a flat morphism onto Y since every fiber of f is one-dimensional. It is easy to see that $-K_Y$ is big but not nef.

Therefore, if f is only flat, then $-K_Y$ is not always nef even when X is a weak Fano manifold.

Let us give a new proof of the well-known theorem by Kollár, Miyaoka, and Mori (cf. [12]). We note that Y is not always Fano if f is only flat. There exists an example in [19].

Theorem 4.7 (cf. [12, Corollary 2.9]) *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties. If X is a Fano manifold, then so is Y.*

Proof By taking the Stein factorization, we can assume that f has connected fibers (cf. Lemma 2.4). By Theorem 4.5, $-K_Y$ is nef and big. Therefore, $-K_Y$ is semi-ample by the Kawamata–Shokurov base point free theorem. Thus, it is sufficient to see that $C \cdot (-K_Y) > 0$ for every integral curve C on Y. Let C be an integral curve C on Y. We take a general very ample divisor H on Y. Let ε be a small positive rational number. Then $K_X + \varepsilon f^*H$ is anti-ample. Let A be a general member of the free linear system $|-m(K_X + \varepsilon f^*H)|$. We can assume that there is a non-empty Zariski open set U of Y such that H is smooth on U, $\mathrm{Supp}(A + f^*H)$ is simple normal crossing on $f^{-1}(U)$, $\mathrm{Supp}A$ is smooth over U, and $C \cap H \cap U \neq \emptyset$. Apply the same arguments as in the proof of Theorem 4.1 to

$$K_X + \varepsilon f^*H + \frac{1}{m}A \sim_{\mathbb{Q}} 0.$$

Then we obtain a projective birational morphism $\mu : Y' \to Y$ from a smooth projective variety Y' such that μ is an isomorphism over U and $\mathbb{Q}$-divisors Δ_0 and M on Y' as before. By construction, Δ_0 contains $\varepsilon H'$, where H' is the strict transform of H on Y' (cf. the proof of Theorem 3.1). Therefore, we have

$$C \cdot (-K_Y) = C' \cdot (E + \Delta_0 + M) > 0$$

as in the proof of Theorem 4.1. Thus, $-K_Y$ is ample. □

We can prove the following theorem by the same arguments. It is a generalization of Theorem 4.7.

Theorem 4.8 *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties. Let H be an ample Cartier divisor on Y. Assume that $-(K_X + \varepsilon f^*H)$ is semi-ample for some positive rational number ε. Then $-K_Y$ is ample, that is, Y is a Fano manifold.*

Proof By Lemma 2.4, we can assume that f has connected fibers. By Theorem 3.1, we see that $-K_Y$ is big. By the proof of Theorem 4.7, we can see that $C \cdot (-K_Y) > 0$ for every integral curve C on Y. By the Kawamata–Shokurov base point free theorem, $-K_Y$ is semi-ample. Thus, $-K_Y$ is ample. □

5 Comments and questions

In this section, we will work over an algebraically closed field k of arbitrary characteristic. We denote the characteristic of k by $\mathrm{char}k$.

5.1 Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties defined over k.

(A) If $-K_X$ is ample, that is, X is Fano, then so is $-K_Y$.

It was obtained by Kollár et al. [12]. Their proof is an application of the deformation theory of morphisms from curves invented by Mori. It needs mod p reduction arguments even when $\mathrm{char}k = 0$. In the case $\mathrm{char}k = 0$, we gave a Hodge theoretic proof without using mod p reduction arguments in Theorem 4.7.

(N) If $-K_X$ is nef, then so is $-K_Y$.

This result can be proved by the same method as in [12] (cf. [15,21], and [3, Corollary 3.15 (a)]). In the case char$k = 0$, we do not know whether we can prove it without mod p reduction arguments or not.

(NB) If $-K_X$ is nef and big, that is, X is weak Fano, then so is $-K_Y$ when char$k = 0$.

It was proved in Theorem 4.5. We do not know whether this statement holds true or not in the case char$k > 0$. See also Sect. 6: Appendix.

(SA) If $-K_X$ is semi-ample, is $-K_Y$ semi-ample?

We have only some partial answers to this question. For details, see Remark 4.2 and Theorem 4.4. In the case char$k = 0$, we note that $-K$ is semi-ample if and only if $-K$ is nef and abundant (see Remark 3.2).

(B) If $-K_X$ is big, is $-K_Y$ big?

The following example gives a negative answer to this question.

Example 5.2 Let $E \subset \mathbb{P}^2$ be a smooth cubic curve. We consider $f : X = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(1)) \to E = Y$. Then, we see that $-K_X$ is big. However, $-K_Y$ is not big since E is a smooth elliptic curve.

Anyway, it seems to be difficult to construct nontrivial examples. It is because the smoothness of f is a very strong condition.

We close this section with a remark on Lemma 2.4. It may be indispensable when $k \neq \mathbb{C}$.

Remark 5.3 Lemma 2.4 holds true even when $k \neq \mathbb{C}$. We can check it as follows. By the proof of Lemma 2.4, it is sufficient to see that $f_*\mathcal{O}_X$ is locally free and $f_*\mathcal{O}_X \otimes k(y) \simeq H^0(X_y, \mathcal{O}_{X_y})$ for every closed point $y \in Y$. Without loss of generality, we can assume that Y is affine. Let us check that the natural map

$$f_*\mathcal{O}_X \otimes k(y) \to H^0(X_y, \mathcal{O}_{X_y})$$

is surjective for every $y \in Y$. We take an arbitrary closed point $y \in Y$. We can replace Y with Spec$\mathcal{O}_{Y,y}$. Let m_y be the maximal ideal corresponding to $y \in Y$. We note that $f_*\mathcal{O}_X \otimes k(y) \simeq (f_*\mathcal{O}_X)^\wedge_y \otimes k(y)$, where $(f_*\mathcal{O}_X)^\wedge_y$ is the formal completion of $f_*\mathcal{O}_X$ at y. By the theorem on formal functions (cf. [9, Theorem 11.1]), we have

$$(f_*\mathcal{O}_X)^\wedge_y \simeq \varprojlim H^0(X_n, \mathcal{O}_{X_n}),$$

where $X_n = X \times_Y \mathrm{Spec}\mathcal{O}_{Y,y}/m_y^n$. Therefore, we can see that

$$(f_*\mathcal{O}_X)^\wedge_y \otimes k(y) \to H^0(X_y, \mathcal{O}_{X_y})$$

is surjective. It is because $H^0(X_{yi}, \mathcal{O}_{X_{yi}}) = k$ for every i, where $X_y = \coprod_i X_{yi}$ is the irreducible decomposition of a smooth variety X_y. By the base change theorem (cf. [9, Theorem 12.11]), we obtain the desired results.

6 Appendix

In this appendix, we give another proof of Theorem 1.1 depending on mod p reduction arguments. This proof is not related to Kawamata's positivity theorem.

First let us recall various results without proofs for the reader's convenience.

6.1 (Preliminary results) The following theorem was obtained by the same way as in [12].

Theorem 6.2 ([3, Corollary 3.15 (a)]) *Let $f : X \to Y$ be a smooth morphism of smooth projective varieties over an arbitrary algebraic closed field. If $-K_X$ is nef, then so is $-K_Y$.*

In [17], Schwede and Smith established the following results on log Fano varieties and global F-regular varieties. For various definitions and details, see [17] and [18]. See also [8] for related topics.

Theorem 6.3 (cf. [17, Theorem 1.1]) *Let X be a normal projective variety over an F-finite field of prime characteristic. Suppose that X is globally F-regular. Then there exists an effective $\mathbb{Q}$-divisor Δ on X such that $-(K_X + \Delta)$ is ample and that (X, Δ) is klt.*

For the definition of *klt in any characteristic*, see [17, Remark 4.2].

Theorem 6.4 (cf. [17, Theorem 5.1]) *Let X be a normal projective variety defined over a filed of characteristic zero. Suppose that there exists an effective $\mathbb{Q}$-divisor Δ on X such that $-(K_X + \Delta)$ is ample and that (X, Δ) is klt. Then X has globally F-regular type.*

Theorem 6.5 (cf. [17, Corollary 6.4]) *Let $f : X \to Y$ be a projective morphism of normal projective varieties over an F-finite field of prime characteristic. Suppose that $f_*\mathcal{O}_X = \mathcal{O}_Y$. If X is a globally F-regular variety, then so is Y.*

We can find the following lemma in [14, Proposition 3.7 (a)].

Lemma 6.6 *Let C be a smooth projective curve over a field k, let K be an extension field of k, and let D be a Cartier divisor on C. Suppose that $\pi : C_K := C \times_k K \to C$ is the natural projection. Then $\deg_k D = \deg_K \pi^* D$.*

By the above lemma, we see the following lemma.

Lemma 6.7 *Let X be a projective variety over a field k, let K be an extension field of k, and let D be a Cartier divisor on X. Suppose that $\pi^* D$ is nef, where $\pi : X_K := X \times_k K \to X$ is the projection. Then D is nef.*

Proof We take a morphism $f : C \to X$ from a smooth projective curve. We consider the following commutative diagram:

$$
\begin{array}{ccc}
C_K & \xrightarrow{\pi_C} & C \\
{\scriptstyle f_K}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f} \\
X_K & \xrightarrow{\pi} & X \\
\downarrow & \circlearrowleft & \downarrow \\
\mathrm{Spec}\, K & \longrightarrow & \mathrm{Spec}\, k
\end{array}
$$

where $C_K := C \times_k K$. By the assumption, $\deg_K {\pi_C}^*(f^*D) \geq 0$. Hence $\deg_k f^*D \geq 0$ by Lemma 6.6. Thus, D is nef. □

Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1 First, we note that $-K_X$ is semi-ample by the Kawamata–Shokurov base point free theorem and that $-K_Y$ is nef by Theorem 6.2. It is sufficient to show that $(-K_Y)^{\dim Y} > 0$. By the Stein factorization, we can assume that f has connected fibers. We can take a finitely generated $\mathbb{Z}$-algebra A, a non-empty affine open set $U \subseteq \operatorname{Spec} A$, and smooth morphisms $\varphi : \mathcal{X} \to U$ and $\psi : \mathcal{Y} \to U$ such that

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

and $F \simeq f$ over the generic point of U and that $-K_{\mathcal{X}}$ is semi-ample. We take a general closed point $\mathfrak{p} \in U$. Note that the residue field $k := \kappa(\mathfrak{p})$ of $\mathfrak{p}$ has positive characteristic p. Let $f_p : X_p \to Y_p$ be the fiber of F at $\mathfrak{p}$, and let K be an algebraic closure of k. By Theorem 6.4, we may assume that X_p is globally F-regular. Let $\overline{f_p} : \overline{X_p} \to \overline{Y_p}$ be the base change of f_p by $\operatorname{Spec} K$, where $\overline{X_p} := X_p \times_k K$ and $\overline{Y_p} := Y_p \times_k K$. Since $-K_{\mathcal{X}}$ is semi-ample, we see that $-K_{\overline{X_p}}$ is semi-ample. In particular, $-K_{\overline{X_p}}$ is nef. Hence, we obtain that $-K_{\overline{Y_p}}$ is nef by Theorem 6.2. By Lemma 6.7, $-K_{Y_p}$ is nef. By Theorem 6.5, Y_p is globally F-regular. Hence $-K_{Y_p}$ is nef and big. Thus $(-K_{Y_p})^{\dim Y} > 0$. Since ψ is flat, $(-K_Y)^{\dim Y} > 0$. Therefore, $-K_Y$ is nef and big. □

Acknowledgments The first author would like to thank Takeshi Abe and Kazuhiko Yamaki for fruitful discussions. He also thanks Shunsuke Takagi, Kazunori Yasutake, and Karl Schwede for useful comments. He was partially supported by The Inamori Foundation and by the Grant-in-Aid for Young Scientists (A) ♯20684001 from JSPS. The second author would like to express his deep gratitude to his supervisor Professor Hiromichi Takagi for teaching him various techniques of the log minimal model program. He also would like to thank Doctor Kazunori Yasutake, who introduced the question on weak Fano manifolds in the seminar held at the Nihon University in December 2009. The second author thanks Doctor Kiwamu Watanabe for reading a preliminary version of this paper, Professor Shunsuke Takagi and Doctor Takuzo Okada for fruitful discussions. He is indebted to Doctor Daizo Ishikawa. Finally, the authors would like to thank Hiroshi Sato for constructing an interesting example and Professor Nobuo Hara for useful comments.

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