1. Y. Takano and H. Uehara pointed out that Example 4.4.2 in [F] is incorrect. The vector $e_1$ is contained in the cone $\langle e_2, e_4, e_5 \rangle$. I overlooked this mistake for a long time. I thank Y. Takano and H. Uehara.

Let $\varphi : X \to Y$ be a 3-dimensional toric flipping contraction such that $X$ has only terminal singularities and that $Y$ is affine. Then we can prove that $X$ is $\mathbb{Q}$-factorial and the unique rational curve that is contracted by $\varphi$ passes through only one singular point of $X$. Therefore, $\varphi : X \to Y$ is the flip described in [M, Example-Claim 14-2-5].

2. Here, we give one example of 3-dimensional non-$\mathbb{Q}$-factorial toric flips. Please replace [F, Example 4.4.2] with the following example. Note that there are no 3-dimensional non-$\mathbb{Q}$-factorial terminal flips!

Example 3 (3-dimensional non-$\mathbb{Q}$-factorial flip). We fix a lattice $N = \mathbb{Z}^3$. Pick lattice points $v_1 = (1, 0, 1), v_2 = (-1, 1, 1), v_3 = (-1, 0, 1), v_4 = (0, -1, 1), v_5 = (1, 2, 0)$. We consider the following fans.

$$
\Delta_X = \{\langle v_1, v_2, v_3, v_4 \rangle, \langle v_1, v_2, v_5 \rangle, \text{and their faces}\},
\Delta_W = \{\langle v_1, v_2, v_3, v_4, v_5 \rangle, \text{and its faces}\}, \text{and}
\Delta_{X^+} = \{\langle v_1, v_4, v_5 \rangle, \langle v_2, v_3, v_5 \rangle, \langle v_3, v_4, v_5 \rangle, \text{and their faces}\}.
$$

We put $X := X(\Delta_X), X^+ := X(\Delta_{X^+}), \text{and } W := X(\Delta_W)$. Then we have a commutative diagram of toric varieties:

$$
\xymatrix{ X \ar[d] \ar[r] & X^+ \ar[d] \\
W & }
$$

such that

(i) $\varphi : X \to W$ and $\varphi^+ : X^+ \to W$ are small projective toric morphisms,

(ii) $\rho(X/W) = 1$ and $\rho(X^+/W) = 2$,

(iii) $X$ has two isolated singular points and $X^+$ has only one terminal quotient singularity.

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(iv) $-K_X$ is $\varphi$-ample and $K_{X^+}$ is $\varphi^+$-ample, and
(v) $X$ is not $\mathbb{Q}$-factorial, but $X^+$ is $\mathbb{Q}$-factorial.

Thus, this diagram is a toric flip. Note that the ampleness of $-K_X$ (resp. $K_{X^+}$) follows from the convexity (resp. concavity) of the roofs of the maximal cones in $\Delta_X$ (resp. $\Delta_{X^+}$). The figure below should help to understand this example.

One can check the following properties:

1. $X$ has one isolated non-quotient canonical Gorenstein singularity and one terminal quotient singularity,
2. the flipping locus is $\mathbb{P}^1$ and it passes through the singular points of $X$,
3. $X^+$ has only one terminal quotient singularity, and
4. the flipped locus is $\mathbb{P}^1 \cup \mathbb{P}^1$ and these two $\mathbb{P}^1$s intersect each other at the singular point of $X^+$.

This example implies that the relative Picard number may increase after a flip when $X$ is not $\mathbb{Q}$-factorial. So, we do not use the Picard number directly to prove the termination of the log MMP.

4. We can construct a 3-dimensional toric flipping diagram

$$
\begin{array}{c}
X \\
\downarrow \\
W
\end{array} \quad \longrightarrow \quad 
\begin{array}{c}
X^+ \\
\downarrow \\
W
\end{array}
$$

with the following properties,

i. $X$ has only canonical Gorenstein singularities,
ii. $\rho(X/W) = 1$ and $\rho(X^+/W) = n$ for any $n \geq 2$, and
(iii) $X$ is smooth.
I will discuss this example elsewhere.

REFERENCES


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