

NON-PROJECTIVE COMPLETE LOG CANONICAL SURFACES

OSAMU FUJINO, NAO MORIYAMA, HIROSHI SATO

ABSTRACT. We construct non-projective complete log canonical algebraic surfaces whose canonical divisors are semi-ample over an algebraically closed field of any characteristic other than the algebraic closure of a finite field. We provide a unified framework to construct such surfaces for any given non-negative Kodaira dimension, namely, zero, one, or two. Furthermore, we show that any complete log canonical algebraic surface with Kodaira dimension minus infinity is automatically projective. This projectivity result confirms that our construction covers all possible values for the Kodaira dimension of non-projective complete log canonical surfaces.

1. INTRODUCTION

In [Nag58], Nagata constructed the first example of a non-projective complete normal algebraic surface, which is a landmark result in the development of algebraic geometry. The surface he constructed has very bad singularities from the viewpoint of the minimal model theory. In this paper, we revisit the existence problem of non-projective complete normal algebraic surfaces from the perspective of the minimal model theory.

Let k be an algebraically closed field and let X be a complete normal algebraic surface defined over k . It is easy to see that X is always projective if X is \mathbb{Q} -factorial. Since X is known to be \mathbb{Q} -factorial when it has only rational singularities, X is automatically projective in this case as well. Consequently, if X has only log terminal singularities, it is always projective. Furthermore, it is a classical result of Artin that X is always \mathbb{Q} -factorial when $k = \overline{\mathbb{F}}_p$, which implies that X is always projective in this setting. On the other hand, it is well known that X is not necessarily \mathbb{Q} -factorial if we allow X to have log canonical singularities. Therefore, it is natural to ask whether there exists a non-projective complete log canonical algebraic surface. Although this problem has already been discussed in [Fuj21a, Section 12], here we would like to take a step further and establish the following theorem.

Theorem 1.1. *Let k be an algebraically closed field such that $k \neq \overline{\mathbb{F}}_p$. Then there exists a non-projective complete log canonical algebraic surface X defined over k such that K_X is semi-ample. Furthermore, the Kodaira dimension $\kappa(X, K_X)$ can be chosen to be any value in $\{0, 1, 2\}$.*

The assertion of Theorem 1.1 is sharp when combined with the following theorem.

Theorem 1.2. *Let X be a complete log canonical algebraic surface over an algebraically closed field. If $\kappa(X, K_X) = -\infty$, then X is projective.*

To clarify the position of this work within the framework of the minimal model theory, let us briefly review the context. The first author and Tanaka [Fuj12, Tan14] established

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the minimal model theory for log surfaces, proving that the MMP and the abundance theorem hold for both projective \mathbb{Q} -factorial log surfaces and projective log canonical surfaces. Since a complete \mathbb{Q} -factorial log surface is always projective, attention naturally turns to the existence of non-projective complete log canonical surfaces. While such an example with $\kappa(X, K_X) = 0$ was previously constructed in [Fuj21a, Section 12], it is natural to ask whether such examples exist for other Kodaira dimensions, particularly for general type where $\kappa(X, K_X) = 2$. Our theorems provide a clear picture of this situation. Together with [Mor26], this work provides insight into the scope and limitations of the minimal model theory for log surfaces. Finally, we remark that our construction is based on the examples presented in [Sch99].

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2. PRELIMINARIES

In the remainder of this paper, all surfaces are assumed to be algebraic varieties; we do not need to consider algebraic spaces. In this section, we collect some basic results. For the details on the definitions of singularities of pairs, see [Fuj12, Section 2.4].

The following fact is used to construct an example verifying Theorem 1.1 for $k \neq \overline{\mathbb{F}}_p$.

Fact 2.1 (cf. [Tan14, Fact 2.3]). Let k be an algebraically closed field of arbitrary characteristic and let C be an elliptic curve over k . If $k \neq \overline{\mathbb{F}}_p$, then $\text{Pic}^0(C)$ has a non-torsion element.

In this paper, to construct contraction morphisms, we rely on two tools available for log canonical surfaces: the relative abundance theorem and the basepoint-free theorem.

Theorem 2.2 (Relative abundance for log canonical log surfaces). *Let k be an algebraically closed field of arbitrary characteristic. Let T be a normal surface over k and Δ be a boundary \mathbb{Q} -divisor on T such that $K_T + \Delta$ is \mathbb{Q} -Cartier. Let $\pi : T \rightarrow Y$ be a projective surjective morphism onto a variety Y . Assume that (T, Δ) is log canonical. Further, assume that $K_T + \Delta$ is π -nef. Then $K_T + \Delta$ is π -semi-ample.*

Proof. The case where $k = \mathbb{C}$ is given by [Fuj12, Theorem 7.2], while the positive characteristic case is covered by [Tan14, Theorem 6.9]. For the remaining cases in characteristic zero, one can refer to [Fuj21b, Theorem 4.1]. \square

Theorem 2.3 (Basepoint-freeness for non-klt log surfaces, see [Mor26, Theorem 3.1]). *Let T be a projective normal surface over an algebraically closed field k of arbitrary characteristic and let Δ be an effective \mathbb{Q} -divisor such that $K_T + \Delta$ is \mathbb{Q} -Cartier. Let D be a nef Cartier divisor. If $\text{char}(k) > 0$, we further assume that D is not numerically trivial. Suppose that $aD - (K_T + \Delta)$ is nef and big for some $a \in \mathbb{Z}_{>0}$, and that the restriction $D|_{\text{Nklt}(T, \Delta)}$ is semi-ample, where $\text{Nklt}(T, \Delta)$ denotes the non-klt locus of the pair (T, Δ) . Then D is semi-ample.*

3. PROOF OF THEOREM 1.1

We will work over an algebraically closed field k . Suppose that $k \neq \overline{\mathbb{F}}_p$. The goal of this section is to construct non-projective complete log canonical algebraic surfaces over k with Kodaira dimension 0, 1 and 2. We start with the case $\kappa = 2$, which is the main highlight of this section.

3.1. $\kappa = 2$. The following construction is based on the example given in [Sch99, Section 4.1]. Start with $\mathbb{P}^1 \times C$, where C is an elliptic curve. Let $p_1: \mathbb{P}^1 \times C \rightarrow \mathbb{P}^1$ and $p_2: \mathbb{P}^1 \times C \rightarrow C$ be the first and second projections, respectively. We take two distinct points $c_1, c_2 \in C$. Let $f: T \rightarrow \mathbb{P}^1 \times C$ be the blowup at $(0, c_1)$, (∞, c_2) , and (∞, c_1) . Let E_{0,c_1} , E_{∞,c_2} , and E_{∞,c_1} be the corresponding exceptional divisors. Let $B_1, B_2, C_0, C_1, C_\infty \subset T$ be the strict transforms of $H_1 := \mathbb{P}^1 \times \{c_1\}$, $H_2 := \mathbb{P}^1 \times \{c_2\}$, $F_0 := \{0\} \times C$, $F_1 := \{1\} \times C$, and $F_\infty := \{\infty\} \times C$, respectively.

Step 1. We show that $B_1 + 3C_1$ is a big and semi-ample divisor on T , which induces a birational contraction morphism $g: T \rightarrow S$ onto a projective surface S that contracts only C_0 , C_∞ , and E_{∞,c_2} .

To this end, we first rewrite the divisor as follows:

$$\begin{aligned} B_1 + 3C_1 &\sim B_1 + f^*F_0 + f^*F_\infty + C_1 \\ &\sim B_1 + C_0 + E_{0,c_1} + C_\infty + E_{\infty,c_1} + E_{\infty,c_2} + C_1 \\ &\sim f^*(H_1 + F_1) + C_0 + C_\infty + E_{\infty,c_2}. \end{aligned}$$

The divisor $H_1 + F_1$ on $\mathbb{P}^1 \times C$ is ample. It follows that any irreducible curve on T having non-positive intersection number with $B_1 + 3C_1$ must be either f -exceptional or contained in the support of $C_0 + C_\infty + E_{\infty,c_2}$. To verify the nefness and identify the contracted curves, we compute the intersection numbers for all such candidates. Since

$$(B_1 + 3C_1) \cdot C_0 = (B_1 + 3C_1) \cdot C_\infty = (B_1 + 3C_1) \cdot E_{\infty,c_2} = 0,$$

and

$$(B_1 + 3C_1) \cdot E_{0,c_1} = (B_1 + 3C_1) \cdot E_{\infty,c_1} = 1,$$

we conclude that $B_1 + 3C_1$ is nef, and the irreducible curves whose intersection number with $B_1 + 3C_1$ equals 0 are precisely C_0 , C_∞ , and E_{∞,c_2} . Here, we recall that

$$K_{\mathbb{P}^1 \times C} \sim p_1^*K_{\mathbb{P}^1} + p_2^*K_C \sim -2F_1.$$

Then, the canonical divisor K_T satisfies

$$\begin{aligned} K_T &\sim -2f^*F_1 + E_{0,c_1} + E_{\infty,c_1} + E_{\infty,c_2} \\ &\sim -2f^*F_1 + f^*F_0 + f^*F_\infty - C_0 - C_\infty \\ &\sim -C_0 - C_\infty. \end{aligned}$$

This implies that

$$(B_1 + 3C_1) - (K_T + C_0 + C_\infty) \sim B_1 + 3C_1,$$

which is nef and big. In addition, the restriction

$$(B_1 + 3C_1)|_{\text{Nklt}(T, C_0 + C_\infty)} = (B_1 + 3C_1)|_{\text{Supp}(C_0 + C_\infty)} = 0$$

is trivially semi-ample. Therefore, by Theorem 2.3, $B_1 + 3C_1$ is semi-ample. Consequently, the divisor $B_1 + 3C_1$ on T is big and semi-ample, and the corresponding birational morphism $g: T \rightarrow S$ contracts only C_0 , C_∞ , and E_{∞,c_2} . The resulting surface S is projective.

Step 2. Our goal in this step is to construct a global birational contraction $h: T \rightarrow Z$ via gluing, which naturally factors the morphism $g: T \rightarrow S$.

Let U_0 be the open subset of T defined by

$$U_0 := (p_1 \circ f)^{-1}(\mathbb{P}^1 \setminus \{\infty\}) = T \setminus (C_\infty \cup E_{\infty, c_1} \cup E_{\infty, c_2}),$$

and let $\varphi_0: U_0 \rightarrow \mathbb{P}^1 \setminus \{\infty\}$ be the restriction of $p_1 \circ f$ to U_0 . A direct calculation shows that the restriction

$$(K_T + C_0 + B_1)|_{U_0} \sim (-2C_1 + E_{0, c_1} + C_0 + B_1)|_{U_0} \sim (f^*(-2F_1 + H_1) + C_0)|_{U_0}$$

is φ_0 -nef and φ_0 -big. Thus, Theorem 2.2 applied to the pair $(U_0, (C_0 + B_1)|_{U_0})$ implies that $(K_T + C_0 + B_1)|_{U_0}$ is φ_0 -semi-ample. This induces a relative birational contraction morphism $h_0: U_0 \rightarrow Z_0$ over $\mathbb{P}^1 \setminus \{\infty\}$. Since

$$(K_T + C_0 + B_1)|_{U_0} \cdot E_{0, c_1}|_{U_0} = 1 \quad \text{and} \quad (K_T + C_0 + B_1)|_{U_0} \cdot C_0|_{U_0} = 0,$$

the morphism h_0 contracts precisely C_0 . Similarly, let U_∞ be the open subset of T defined by

$$U_\infty := (p_1 \circ f)^{-1}(\mathbb{P}^1 \setminus \{0\}) = T \setminus (C_0 \cup E_{0, c_1}),$$

and let $\varphi_\infty: U_\infty \rightarrow \mathbb{P}^1 \setminus \{0\}$ be the restriction of $p_1 \circ f$ to U_∞ . The restriction

$$\begin{aligned} (K_T + C_\infty + B_1 + B_2)|_{U_\infty} &\sim (-2C_1 + E_{\infty, c_1} + E_{\infty, c_2} + C_\infty + B_1 + B_2)|_{U_\infty} \\ &\sim (f^*(-2F_1 + H_1 + H_2) + C_\infty)|_{U_\infty} \end{aligned}$$

is φ_∞ -nef and φ_∞ -big. By the relative abundance theorem for the pair $(U_\infty, (C_\infty + B_1 + B_2)|_{U_\infty})$, the divisor $(K_T + C_\infty + B_1 + B_2)|_{U_\infty}$ is φ_∞ -semi-ample, yielding a relative birational contraction morphism $h_\infty: U_\infty \rightarrow Z_\infty$ over $\mathbb{P}^1 \setminus \{0\}$. Since

$$(K_T + C_\infty + B_1 + B_2)|_{U_\infty} \cdot E_{\infty, c_1}|_{U_\infty} = (K_T + C_\infty + B_1 + B_2)|_{U_\infty} \cdot E_{\infty, c_2}|_{U_\infty} = 1$$

and $(K_T + C_\infty + B_1 + B_2)|_{U_\infty} \cdot C_\infty|_{U_\infty} = 0$, the morphism h_∞ contracts precisely C_∞ . By gluing h_0 and h_∞ together, we obtain a contraction morphism $h: T \rightarrow Z$ over \mathbb{P}^1 which contracts only C_0 and C_∞ , and a normal complete surface Z (cf. [Har77, II, Corollary 4.8]). Finally, since every curve contracted by $h: T \rightarrow Z$ is also contracted by $g: T \rightarrow S$, the morphism g factors through h . This induces a birational morphism $\phi: Z \rightarrow S$ such that $g = \phi \circ h$.

Step 3. We show that Z is a complete log canonical surface with $K_Z \sim 0$, and that it is non-projective if we further arrange the choice of $c_1, c_2 \in C$ so that $\mathcal{O}_C(c_1 - c_2) \in \text{Pic}^0(C)$ is a non-torsion element. The existence of such a choice is guaranteed by our assumption on k and Fact 2.1.

The linear equivalence $K_T + C_0 + C_\infty \sim 0$ implies $K_Z \sim 0$, and $K_T = h^*(K_Z) - C_0 - C_\infty$ implies that Z is log canonical. To show that Z is non-projective, we fix a choice of c_1 and c_2 such that $\mathcal{O}_C(c_1 - c_2) \in \text{Pic}^0(C)$ is a non-torsion element, and suppose for contradiction that there exists an ample divisor on Z . Then, there exists a very ample effective divisor D on Z whose support avoids the points $h(C_0)$ and $h(C_\infty)$ on Z . Since h does not contract E_{0, c_1} , E_{∞, c_1} or E_{∞, c_2} , we have $h^*D \cdot E_{0, c_1} > 0$, $h^*D \cdot E_{\infty, c_1} > 0$, and $h^*D \cdot E_{\infty, c_2} > 0$. Let $\tilde{D} := f_*h^*D$. Here, we can write $f^*\tilde{D} = h^*D + n_1E_{0, c_1} + n_2E_{\infty, c_1} + n_3E_{\infty, c_2}$ for some integers n_1, n_2, n_3 . Since the f -exceptional divisors are disjoint from each other, we have the equation $0 = f^*\tilde{D} \cdot E_{0, c_1} = h^*D \cdot E_{0, c_1} - n_1$, which implies $n_1 > 0$. Similarly, we obtain $n_2, n_3 > 0$. By our choice of D , the effective divisor h^*D does not contain C_0 or C_∞ in its support, which implies that \tilde{D} does not contain the fibers F_0 or F_∞ . Hence the intersections satisfy $\tilde{D} \cap F_0 = \{(0, c_1)\}$ and $\tilde{D} \cap F_\infty = \{(\infty, c_1), (\infty, c_2)\}$ set-theoretically.

Since F_0 and F_∞ are isomorphic via f to the curve C_0 and C_∞ respectively, identifying F_0 and F_∞ with the curve C , the scheme-theoretic restrictions yield the relations $\tilde{D}|_{F_0} = n_1c_1$ and $\tilde{D}|_{F_\infty} = n_2c_1 + n_3c_2$ as divisors on C . We note that $\mathcal{O}_{\mathbb{P}^1 \times C}(\tilde{D}) \cong p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$ for some invertible sheaves \mathcal{L}_1 on \mathbb{P}^1 and \mathcal{L}_2 on C . This implies the linear equivalence $\tilde{D}|_{F_0} \sim \tilde{D}|_{F_\infty}$ as divisors on C . On the other hand, by identifying both F_0 and F_∞ with the curve C , the scheme-theoretic restrictions yield the relations $\tilde{D}|_{F_0} = n_1c_1$ and $\tilde{D}|_{F_\infty} = n_2c_1 + n_3c_2$ on C . Therefore, we obtain $(n_1 - n_2)c_1 - n_3c_2 \sim 0$. By comparing the degrees of these two restrictions, we have $n_1 = n_2 + n_3$, which yields $n_3(c_1 - c_2) \sim 0$. Since $\mathcal{O}_C(c_1 - c_2)$ is a non-torsion element, we must have $n_3 = 0$. This contradicts the fact that $n_3 > 0$. Consequently, Z is non-projective.

From now on, we assume that the points c_1 and c_2 are chosen so that Z is non-projective.

Step 4. We construct a finite cyclic cover $\tilde{Z} \rightarrow Z$ branched along a smooth divisor pulled back from S , and show that \tilde{Z} is the desired surface by exploiting the non-projectivity of Z .

Let $J \subset S$ be the finite set consisting of the singular points of S . We choose a very ample Cartier divisor A on the projective surface S . For a positive integer m , there exists a member $A' \in |mA|$ that also avoids J and has a smooth support. We define $A'' := \phi^*A'$. By construction, A'' is an effective Cartier divisor on Z that is naturally identified with A' via ϕ . In particular, A'' is smooth and avoids the singular locus of Z . Next, we consider the cyclic cover of degree m defined by

$$\mu: \tilde{Z} := \text{Spec}_Z \left(\bigoplus_{l=0}^{m-1} \mathcal{O}_Z(-l\phi^*A) \right) \rightarrow Z,$$

which is ramified along A'' (cf. [KM98, Definition 2.50]). Here, we set $m = 2$ if the base field k has characteristic zero; in positive characteristic, we choose m to be coprime to $\text{char}(k)$. By the ramification formula for finite covers, we obtain the equality of \mathbb{Q} -divisors:

$$K_{\tilde{Z}} = \mu^* \left(K_Z + \frac{m-1}{m} A'' \right).$$

Since Z is log canonical and A'' is a smooth divisor avoiding the singular points of Z , the pair $(Z, \frac{m-1}{m}A'')$ is log canonical. Consequently, the finiteness of μ together with [KM98, Proposition 5.20] implies that \tilde{Z} is a log canonical surface. If there were an ample invertible sheaf \mathcal{L} on \tilde{Z} , its image $\text{Norm}_\mu(\mathcal{L})$ via the norm map associated with the cyclic cover $\mu: \tilde{Z} \rightarrow Z$ would yield an ample invertible sheaf on Z (cf. [Sta26, Tag 0BD4]). This contradicts the fact that Z is non-projective. Hence, \tilde{Z} is a non-projective surface. Since $K_Z \sim 0$, the ramification formula yields $\mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}}) \cong \mu^*\phi^*\mathcal{O}_S((m-1)A)$. Since the divisor A is very ample on the projective surface S , the canonical divisor $K_{\tilde{Z}}$ is semi-ample. Furthermore, the Kodaira dimension satisfies

$$\kappa(\tilde{Z}, K_{\tilde{Z}}) = \kappa(S, \mathcal{O}_S((m-1)A)) = 2.$$

Therefore, \tilde{Z} is a normal non-projective complete log canonical surface with a semi-ample canonical divisor $K_{\tilde{Z}}$ and Kodaira dimension $\kappa(\tilde{Z}, K_{\tilde{Z}}) = 2$.

3.2. $\kappa = 0, 1$. Examples with $\kappa = 0$ can be essentially found in [Fuj21a, Section 12], [Kol07, Aside 3.46], and [Sch99, Section 2.5]. Here, for convenience, we construct examples with $\kappa = 0$ and 1 by slightly modifying the case $\kappa = 2$ above. First, Step 3 directly implies that the surface Z itself is the desired example with $\kappa(Z, K_Z) = 0$. Next, we construct an

example with $\kappa = 1$. In the construction of Step 4, we use a general fiber of the morphism $Z \rightarrow \mathbb{P}^1$ instead of the smooth divisor pulled back from the ample divisor A on S . Then, we consider a cyclic cover ramified along m general fibers. For the resulting surface \tilde{Z} , the invertible sheaf $\mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}})$ is isomorphic to the pull-back of $\mathcal{O}_{\mathbb{P}^1}(m-1)$. This implies that $K_{\tilde{Z}}$ is semi-ample and satisfies $\kappa(\tilde{Z}, K_{\tilde{Z}}) = 1$.

4. PROOF OF THEOREM 1.2

The first author proved a theorem similar to Theorem 1.2 in [Fuj21a, Theorem 1.3]. Since we could not find an explicit proof of Theorem 1.2 in the literature, we include a proof here for the reader's convenience. We use a classical argument due to Goodman (see [Goo69]) to prove the projectivity of X . Note that the argument used in the proof of [Fuj21a, Theorem 1.3] does not directly apply to our setting, since a sufficiently developed minimal model theory for two-dimensional algebraic spaces is not currently available.

Proof of Theorem 1.2. Let $f: Y \rightarrow X$ be the minimal resolution of singularities. Then Y is a smooth projective surface with $\kappa(Y, K_Y) = -\infty$. Since X is a normal surface, it is Cohen–Macaulay. Moreover, the assumption $\kappa(X, K_X) = -\infty$ implies that

$$H^0(X, \mathcal{O}_X(K_X)) = 0.$$

Hence Serre duality (see, for example, [Sta26, Tag 0FW0]) yields

$$H^2(X, \mathcal{O}_X) = 0.$$

Therefore, the Leray spectral sequence gives the following short exact sequence:

$$(4.1) \quad 0 \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow H^0(X, R^1 f_* \mathcal{O}_Y) \rightarrow 0.$$

Step 1. We show that X has at most one non-rational singular point.

Assume that X has a non-rational singular point $P \in X$. If Y is rational, namely birational to \mathbb{P}^2 , then

$$H^1(Y, \mathcal{O}_Y) = 0.$$

By (4.1), this implies that

$$R^1 f_* \mathcal{O}_Y = 0,$$

and hence all singularities of X are rational, a contradiction. Therefore, Y is irrational. Running the minimal model program for Y , we may assume that Y is obtained from a \mathbb{P}^1 -bundle $\mathbb{P}_C(\mathcal{E}) \rightarrow C$ over a smooth projective curve C with $g(C) \geq 1$ by a sequence of blow-ups. By the classification of two-dimensional log canonical singularities, the exceptional divisor $f^{-1}(P)$ cannot be mapped to a point of C . Indeed, every non-rational two-dimensional log canonical singularity is either a simple elliptic singularity or a cusp singularity. Hence $f^{-1}(P)$ dominates C . It follows that C is either an elliptic curve or \mathbb{P}^1 . Since P is non-rational, we have

$$R^1 f_* \mathcal{O}_Y \neq 0.$$

Thus (4.1) implies that

$$\dim H^1(Y, \mathcal{O}_Y) \neq 0.$$

Therefore $g(C) = 1$, and consequently

$$\dim H^1(Y, \mathcal{O}_Y) = 1.$$

Again by (4.1), the sheaf $R^1 f_* \mathcal{O}_Y$ is supported only at P . Hence X has at most one non-rational singular point.

Step 2 (Goodman). We prove that X is projective.

By Step 1, there exists an affine open subset $U \subset X$ such that every point of $X \setminus U$ is a rational singularity. In particular, every irreducible component of $X \setminus U$ is a \mathbb{Q} -Cartier Weil divisor. Then, by the same argument as in the proof of [Goo69, Theorem 2], one can construct an effective Cartier divisor D on X such that

$$\text{Supp } D = X \setminus U$$

and D is ample. Therefore X is projective.

This completes the proof. □

Step 1 in the proof of Theorem 1.2 establishes the following corollary.

Corollary 4.1. *Let X be a complete log canonical algebraic surface over an algebraically closed field. If $\kappa(X, K_X) = -\infty$, then X has at most one non-rational singular point.*

We conclude this paper with a remark on Nagata's example.

Remark 4.2. In [Nag58, Example 1], Nagata constructed a non-projective complete surface S^* . By construction, S^* is birational to \mathbb{P}^2 and is not log canonical. Furthermore, $H^0(S^*, \mathcal{O}_{S^*}(K_{S^*})) \neq 0$, which can be verified by an application of the Leray spectral sequence as in the proof of Theorem 1.2.

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email address: fujino@math.kyoto-u.ac.jp

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email address: moriyama.nao.22s@st.kyoto-u.ac.jp

DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCES, FUKUOKA UNIVERSITY, 8-19-1,
NANAKUMA, JONAN-KU, FUKUOKA 814-0180, JAPAN

Email address: `hirosato@fukuoka-u.ac.jp`