

ON IMAGES OF WEAK FANO MANIFOLDS II

OSAMU FUJINO AND YOSHINORI GONGYO

Dedicated to Professor Klaus Hulek on the occasion of his sixtieth birthday

ABSTRACT. We consider a smooth projective surjective morphism between smooth complex projective varieties. We give a Hodge theoretic proof of the following well-known fact: If the anti-canonical divisor of the source space is nef, then so is the anti-canonical divisor of the target space. We do not use mod p reduction arguments. In addition, we make some supplementary comments on our paper: On images of weak Fano manifolds.

CONTENTS

1. Introduction	1
2. Proof of the main theorem	2
References	7

1. INTRODUCTION

We will work over \mathbb{C} , the complex number field. The following theorem is the main result of this paper. It is a generalization of [D, Corollary 3.15 (a)].

Theorem 1.1 (Main theorem). *Let $f : X \rightarrow Y$ be a smooth projective surjective morphism between smooth projective varieties. Let D be an effective \mathbb{Q} -divisor on X such that (X, D) is log canonical, $\text{Supp} D$ is a simple normal crossing divisor, and $\text{Supp} D$ is relatively normal crossing over Y . Let Δ be a (not necessarily effective) \mathbb{Q} -divisor on Y . Assume that $-(K_X + D) - f^* \Delta$ is nef. Then $-K_Y - \Delta$ is nef.*

By setting $D = 0$ and $\Delta = 0$ in Theorem 1.1, we obtain the following corollary.

Date: 2012/11/5, version 1.22.

2010 Mathematics Subject Classification. Primary 14J45; Secondary 14N30, 14E30.

Key words and phrases. anti-canonical divisors, weak positivity.

Corollary 1.2. *Let $f : X \rightarrow Y$ be a smooth projective surjective morphism between smooth projective varieties. Assume that $-K_X$ is nef. Then $-K_Y$ is nef.*

By setting $D = 0$ and assuming that Δ is a small ample \mathbb{Q} -divisor, we can recover [KMM, Corollary 2.9] by Theorem 1.1. Note that Theorem 1.1 is also a generalization of [FG, Theorem 4.8].

Corollary 1.3 (cf. [KMM, Corollary 2.9]). *Let $f : X \rightarrow Y$ be a smooth projective surjective morphism between smooth projective varieties. Assume that $-K_X$ is ample. Then $-K_Y$ is ample.*

In this paper, we give a proof of Theorem 1.1 without mod p reduction arguments. Our proof is Hodge theoretic. We use a generalization of Viehweg’s weak positivity theorem following [CZ]. In our previous paper [FG], we just use Kawamata’s positivity theorem. We note that Theorem 1.1 is better than [FG, Theorem 4.1] (see Theorem 2.4 below). We also note that Kawamata’s positivity theorem (cf. [FG, Theorem 2.2]) and Viehweg’s weak positivity theorem (and its generalization in [C, Theorem 4.13]) are obtained by the Fujita–Kawamata semi-positivity theorem and its generalization, which follow from the theory of the variation of (mixed) Hodge structure. We recommend the readers to compare the proof of Theorem 1.1 with the arguments in [FG, Section 4]. By the Lefschetz principle, all the results in this paper hold over any algebraically closed field k of characteristic zero. In this paper, we do not discuss the case when the characteristic of the base field is positive.

Acknowledgments. The first author was partially supported by the Grant-in-Aid for Young Scientists (A) #20684001 from JSPS. The second author was partially supported by the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. The authors would like to thank Professor Sebastien Boucksom for informing them of Berndtsson’s results [B]. They also would like to thank the Erwin Schrödinger International Institute for Mathematical Physics in Vienna for its hospitality.

2. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1.1. We closely follow the arguments in [CZ].

Lemma 2.1. *Let $f : Z \rightarrow C$ be a projective surjective morphism from a $(d + 1)$ -dimensional smooth projective variety Z to a smooth projective curve C . Let B be an ample Cartier divisor on Z such that*

$R^i f_* \mathcal{O}_Z(kB) = 0$ for every $i > 0$ and $k \geq 1$. Let H be a very ample Cartier divisor on C such that $B^{d+1} < f^*(H - K_C) \cdot B^d$ and $B^{d+1} \leq f^*H \cdot B^d$. Then

$$(f_* \mathcal{O}_Z(kB))^* \otimes \mathcal{O}_C(lH)$$

is generated by global sections for $l > k \geq 1$.

Proof. By the Grothendieck duality

$$R\mathcal{H}om(Rf_* \mathcal{O}_Z(kB), \omega_C^\bullet) \simeq Rf_* R\mathcal{H}om(\mathcal{O}_Z(kB), \omega_Z^\bullet),$$

we obtain

$$(f_* \mathcal{O}_Z(kB))^* \simeq R^d f_* \mathcal{O}_Z(K_{Z/C} - kB)$$

for $k \geq 1$ and

$$R^i f_* \mathcal{O}_Z(K_{Z/C} - kB) = 0$$

for $k \geq 1$ and $i \neq d$. We note that $f_* \mathcal{O}_Z(kB)$ is locally free and $(f_* \mathcal{O}_Z(kB))^*$ is its dual locally free sheaf. Therefore, we have

$$\begin{aligned} & H^1(C, (f_* \mathcal{O}_Z(kB))^* \otimes \mathcal{O}_C((l-1)H)) \\ & \simeq H^1(C, R^d f_* \mathcal{O}_Z(K_{Z/C} - kB) \otimes \mathcal{O}_C((l-1)H)) \\ & \simeq H^{d+1}(Z, \mathcal{O}_Z(K_Z - f^*K_C - kB + f^*(l-1)H)) \end{aligned}$$

for $k \geq 1$. By the Serre duality,

$$H^{d+1}(Z, \mathcal{O}_Z(K_Z - f^*K_C - kB + f^*(l-1)H))$$

is dual to

$$H^0(Z, \mathcal{O}_Z(kB + f^*K_C - f^*(l-1)H)).$$

On the other hand, by the assumptions

$$(kB + f^*K_C - f^*(l-1)H) \cdot B^d < 0$$

if $l-1 \geq k$. Thus, we obtain

$$H^0(Z, \mathcal{O}_Z(kB + f^*K_C - f^*(l-1)H)) = 0$$

for $l > k$. This means that

$$H^1(C, (f_* \mathcal{O}_Z(kB))^* \otimes \mathcal{O}_C((l-1)H)) = 0$$

for $k \geq 1$ and $l > k$. Therefore, $(f_* \mathcal{O}_Z(kB))^* \otimes \mathcal{O}_C(lH)$ is generated by global sections for $k \geq 1$ and $l > k$. \square

The following lemma directly follows from [C, Theorem 4.3]. It is a key lemma for the proof of Theorem 1.1.

Lemma 2.2. *Let $f : V \rightarrow W$ be a proper surjective morphism between smooth projective varieties with connected fibers. Let Δ be an effective \mathbb{Q} -divisor on V such that (V, Δ) is log canonical. Assume that $m\Delta$ is Cartier for some positive integer m . Then $f_*\mathcal{O}_V(m(K_{V/W} + \Delta))$ is weakly positive over some non-empty Zariski open set U of W .*

For the basic properties of weakly positive sheaves, see, for example, [V, Section 2.3]. Although the original proof of [C, Theorem 4.3] depends on Kawamata's difficult result (see [K, Theorem 32]), [F, Theorem 3.9] and [FF, Theorem 1.1] are sufficient for the proof of our lemma: Lemma 2.2.

Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1. We note that, by the Stein factorization, we may assume that f has connected fibers (see [FG, Lemma 2.4]). We prove the following claim.

Claim. *Let $\pi : C \rightarrow Y$ be a projective morphism from a smooth projective curve C and let L be an ample Cartier divisor on C . Then $(-\pi^*K_Y - \pi^*\Delta + 2\varepsilon L) \cdot C \geq 0$ for any positive rational number ε .*

Let us start the proof of Claim. We fix an arbitrary positive rational number ε . We may assume that $\pi(C)$ is a curve, that is, π is finite. We consider the following base change diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ g \downarrow & & \downarrow f \\ C & \xrightarrow{\pi} & Y \end{array}$$

where $Z = X \times_Y C$. Then $g : Z \rightarrow C$ is smooth, Z is smooth, $\text{Supp}(p^*D)$ is relatively normal crossing over C , and $\text{Supp}(p^*D)$ is a simple normal crossing divisor on Z . Let A be a very ample Cartier divisor on X and let δ be a small positive rational number such that $0 < \delta \ll \varepsilon$. Since $-(K_X + D) - f^*\Delta + \delta A$ is ample, we can take a general effective \mathbb{Q} -divisor F on X such that $-(K_X + D) - f^*\Delta + \delta A \sim_{\mathbb{Q}} F$. Then we have

$$K_{X/Y} + D + F \sim_{\mathbb{Q}} \delta A - f^*K_Y - f^*\Delta.$$

By taking the base change, we obtain

$$K_{Z/C} + p^*D + p^*F \sim_{\mathbb{Q}} \delta p^*A - g^*\pi^*K_Y - g^*\pi^*\Delta.$$

Without loss of generality, we may assume that $\text{Supp}(p^*D + p^*F)$ is a simple normal crossing divisor, p^*D and p^*F have no common irreducible components, and $(Z, p^*D + p^*F)$ is log canonical. Let m be a

sufficiently divisible positive integer such that $m\delta$ and $m\varepsilon$ are integers, mp^*D , mp^*F , and $m\Delta$ are Cartier divisors, and

$$m(K_{Z/C} + p^*D + p^*F) \sim m(\delta p^*A - g^*\pi^*K_Y - g^*\pi^*\Delta).$$

We apply the weak positivity theorem (see Lemma 2.2) and obtain that

$$g_*\mathcal{O}_Z(m(K_{Z/C} + p^*D + p^*F)) \simeq g_*\mathcal{O}_Z(m(\delta p^*A - g^*\pi^*K_Y - g^*\pi^*\Delta))$$

is weakly positive over some non-empty Zariski open set U of C . Therefore,

$$\begin{aligned} \mathcal{E}_1 &:= S^n(g_*\mathcal{O}_Z(m(\delta p^*A - g^*\pi^*K_Y - g^*\pi^*\Delta))) \otimes \mathcal{O}_C(nm\varepsilon L) \\ &\simeq S^n(g_*\mathcal{O}_Z(m\delta p^*A)) \otimes \mathcal{O}_C(-nm\pi^*K_Y - nm\pi^*\Delta + nm\varepsilon L) \end{aligned}$$

is generated by global sections over U for every $n \gg 0$. On the other hand, by Lemma 2.1, if $m\delta \gg 0$, then we have that

$$\mathcal{E}_2 := \mathcal{O}_C(nm\varepsilon L) \otimes S^n((g_*\mathcal{O}_Z(m\delta p^*A))^*)$$

is generated by global sections because $0 < \delta \ll \varepsilon$ and p^*A is ample on Z . We note that

$$\mathcal{E}_2 \simeq S^n(\mathcal{O}_C(m\varepsilon L) \otimes (g_*\mathcal{O}_Z(m\delta p^*A))^*).$$

Thus there is a homomorphism

$$\alpha : \bigoplus_{\text{finite}} \mathcal{O}_C \rightarrow \mathcal{E} := \mathcal{E}_1 \otimes \mathcal{E}_2$$

which is surjective over U . By using the non-trivial trace map

$$S^n(g_*\mathcal{O}_Z(m\delta p^*A)) \otimes S^n((g_*\mathcal{O}_Z(m\delta p^*A))^*) \rightarrow \mathcal{O}_C,$$

we have a non-trivial homomorphism

$$\bigoplus_{\text{finite}} \mathcal{O}_C \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{O}_C(-nm\pi^*K_Y - nm\pi^*\Delta + 2nm\varepsilon L),$$

where β is induced by the above trace map. We note that $g_*\mathcal{O}_Z(m\delta p^*A)$ is locally free and

$$S^n((g_*\mathcal{O}_Z(m\delta p^*A))^*) \simeq (S^n(g_*\mathcal{O}_Z(m\delta p^*A)))^*.$$

Thus we obtain

$$(-nm\pi^*K_Y - nm\pi^*\Delta + 2nm\varepsilon L) \cdot C = nm(-\pi^*K_Y - \pi^*\Delta + 2\varepsilon L) \cdot C \geq 0.$$

We finish the proof of Claim.

Since ε is an arbitrary small positive rational number, we obtain $\pi^*(-K_Y - \Delta) \cdot C \geq 0$. This means that $-K_Y - \Delta$ is nef on Y . \square

Remark 2.3. In Theorem 1.1, if $-(K_X + D)$ is semi-ample, then we can simply prove that $-K_Y$ is nef as follows. First, by the Stein factorization, we may assume that f has connected fibers (see [FG, Lemma 2.4]). Next, in the proof of Theorem 1.1, we can take $\delta = 0$ and $\Delta = 0$ when $-(K_X + D)$ is semi-ample. Then

$$g_*\mathcal{O}_Z(m(K_{Z/C} + p^*D + p^*F)) \simeq \mathcal{O}_C(-m\pi^*K_Y)$$

is weakly positive over some non-empty Zariski open set U of C . This means that $-m\pi^*K_Y$ is pseudo-effective. Since C is a smooth projective curve, $-\pi^*K_Y$ is nef. Therefore, $-K_Y$ is nef. In this case, we do not need Lemma 2.1. The proof given here is simpler than the proof of [FG, Theorem 4.1].

We apologize the readers of [FG] for misleading them on [FG, Theorem 4.1]. A Hodge theoretic proof of [FG, Theorem 4.1] is implicitly contained in Viehweg's theory of weak positivity (see, for example, [V]). Here we give a proof of [FG, Theorem 4.1] following Viehweg's arguments.

Theorem 2.4 ([FG, Theorem 4.1]). *Let $f : X \rightarrow Y$ be a smooth projective surjective morphism between smooth projective varieties. If $-K_X$ is semi-ample, then $-K_Y$ is nef.*

Proof. By the Stein factorization, we may assume that f has connected fibers (see [FG, Lemma 2.4]). Note that a locally free sheaf \mathcal{E} on Y is nef, equivalently, semi-positive in the sense of Fujita–Kawamata, if and only if \mathcal{E} is weakly positive over Y (see, for example, [V, Proposition 2.9 e)). Since f is smooth and $-K_X$ is semi-ample, $f_*\mathcal{O}_X(K_{X/Y} - K_X)$ is locally free and weakly positive over Y (cf. [V, Proposition 2.43]). Therefore, we obtain that $\mathcal{O}_Y(-K_Y) \simeq f_*\mathcal{O}_X(K_{X/Y} - K_X)$ is nef. \square

Note that our Hodge theoretic proof of [FG, Theorem 4.1], which depends on Kawamata's positivity theorem, is different from the proof given above and plays important roles in [FG, Remark 4.2] and the proof of Theorem 2.5 below.

Theorem 2.5 (see [BC, Theorem 1.3]). *Let $f : X \rightarrow Y$ be a smooth projective surjective morphism between smooth projective varieties. If $-K_X$ is semi-ample, then $-K_Y$ is also semi-ample.*

Theorem 2.5 is a complete solution of our conjecture: [FG, Conjecture 1.3]. The proof of Theorem 2.5 in [BC] uses the minimal model theory. For the details, see [BC].

2.6 (Analytic method). Sebastien Boucksom pointed out that the following theorem, which is a special case of [B, Theorem 1.2], implies [FG, Theorem 4.1] and [KMM, Corollary 2.9].

Theorem 2.7 (cf. [B, Theorem 1.2]). *Let $f : X \rightarrow Y$ be a proper smooth morphism from a compact Kähler manifold X to a compact complex manifold Y . If $-K_X$ is semi-positive (resp. positive), then $-K_Y$ is semi-positive (resp. positive).*

The proof of [B, Theorem 1.2] is analytic and does not use mod p reduction arguments. For the details, see [B].

2.8 (Varieties of Fano type). Let X be a normal projective variety. If there is an effective \mathbb{Q} -divisor on X such that (X, Δ) is klt and that $-(K_X + \Delta)$ is ample, then X is said to be of *Fano type*.

In [PS, Theorem 2.9] and [FG, Corollary 3.3], the following statement was proved.

Let $f : X \rightarrow Y$ be a proper surjective morphism between normal projective varieties with connected fibers. If X is of Fano type, then so is Y .

It is indispensable for the proof of the main theorem in [FG] (see [FG, Theorem 1.1]). The proofs in [PS] and [FG] need the theory of the variation of Hodge structure. It is because we use Ambro's canonical bundle formula or Kawamata's positivity theorem. In [GOST], Okawa, Sannai, Takagi, and the second author give a new proof of the above result without using the theory of the variation of Hodge structure. It deeply depends on the minimal model theory and the theory of F -singularities.

We close this paper with a remark on [D]. By modifying the proof of Theorem 1.1 suitably, we can generalize [D, Corollary 3.14] without any difficulties. We leave the details as an exercise for the readers.

Corollary 2.9 (cf. [D, Corollary 3.14]). *Let $f : X \rightarrow Y$ be a projective surjective morphism from a smooth projective variety X such that Y is smooth in codimension one. Let D be an effective \mathbb{Q} -divisor on X such that $\text{Supp} D^{\text{hor}}$, where D^{hor} is the horizontal part of D , is a simple normal crossing divisor on X and that (X, D) is log canonical over the generic point of Y . Let Δ be a not necessarily effective \mathbb{Q} -Cartier \mathbb{Q} -divisor on Y .*

- (a) *If $-(K_X + D) - f^*\Delta$ is nef, then $-K_Y - \Delta$ is generically nef.*
- (b) *If $-(K_X + D) - f^*\Delta$ is ample, then $-K_Y - \Delta$ is generically ample.*

REFERENCES

- [B] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, *Ann. of Math. (2)* **169** (2009), no. 2, 531–560.

- [BC] C. Birkar, Y. Chen, On the moduli part of the Kawamata–Kodaira canonical bundle formula, preprint (2012).
- [C] F. Campana, Orbifolds, special varieties and classification theory, *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 3, 499–630.
- [CZ] M. Chen, Q. Zhang, On a question of Demailly–Peternell–Schneider, to appear in *J. Eur. Math. Soc.*
- [D] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext. Springer-Verlag, New York, 2001.
- [F] O. Fujino, Higher direct images of log canonical divisors, *J. Differential Geom.* **66** (2004), no. 3, 453–479.
- [FF] O. Fujino, T. Fujisawa, Variations of mixed Hodge structure and semi-positivity theorems, preprint (2012).
- [FG] O. Fujino, Y. Gongyo, On images of weak Fano manifolds, *Math. Z.* **270** (2012), no. 1-2, 531–544.
- [GOST] Y. Gongyo, S. Okawa, A. Sannai, S. Takagi, Characterization of varieties of Fano type via singularities of Cox rings, (2012), preprint
- [K] Y. Kawamata, Characterization of abelian varieties, *Compositio Math.* **43** (1981), no. 2, 253–276.
- [KMM] J. Kollár, Y. Miyaoka, S. Mori, Rational connectedness and boundedness of Fano manifolds, *J. Differential Geom.* **36** (1992), no. 3, 765–779.
- [PS] Yu. G. Prokhorov, V. V. Shokurov, Towards the second main theorem on complements, *J. Algebraic Geom.* **18** (2009), no. 1, 151–199.
- [V] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* [Results in Mathematics and Related Areas (3)], **30**. Springer-Verlag, Berlin, 1995.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY,
KYOTO 606-8502, JAPAN

E-mail address: fujino@math.kyoto-u.ac.jp

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO,
3-8-1 KOMABA, MEGURO, TOKYO, 153-8914 JAPAN

E-mail address: gongyo@ms.u-tokyo.ac.jp

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, 180 QUEEN'S
GATE, LONDON SW7 2AZ, UK

E-mail address: y.gongyo@imperial.ac.uk