Math. Z.
DOI 10.1007/s00209-007-0140-5

Mathematische Zeitschrift

Multiplication maps and vanishing theorems for Toric varieties

Osamu Fujino

Received: 6 July 2006 / Accepted: 20 December 2006

Abstract We use multiplication maps to give a characteristic-free approach to vanishing theorems on toric varieties. Our approach is very elementary but is enough powerful to prove vanishing theorems.

Mathematics Subject Classification (2000) Primary 14F17; Secondary 14F25

1 Introduction

The main purpose of this paper is to understand various vanishing theorems on toric varieties through multiplication maps. We give an elementary and unified approach to vanishing theorems on toric varieties. The following theorem is the main theorem of this paper. Some important special cases were already investigated in various papers. See, for example, [5, 7.5.2. Theorem], [2, Sect. 7], [4, Theorem 5], and [15, Sect. 2].

Theorem 1.1 (Main theorem I) *Let X be a toric variety defined over a field k of arbitrary characteristic and B a reduced torus invariant Weil divisor on X. Let L be a line bundle on X. If $H^i(X, \widetilde{\Omega}^a_X(\log B) \otimes L^l) = 0$ for some positive integer l, then $H^i(X, \widetilde{\Omega}^a_X(\log B) \otimes L) = 0$. In particular, if X is projective and L is ample, then $H^i(X, \widetilde{\Omega}^a_X(\log B) \otimes L) = 0$ for any $i > 0$.*

Before we go further, let us recall the definition of $\widetilde{\Omega}^a_X(\log B)$ (cf. [5, 15.2]).

Definition 1.2 Let W be any Zariski open set of X such that W is non-singular and $\operatorname{codim}_X(X \setminus W) \geq 2$. Furthermore, we assume that B is a simple normal crossing divisor on W. On this assumption, $\Omega^a_W(\log B)$ is a well-defined locally free sheaf on W. Let $\iota : W \hookrightarrow X$ be the natural open immersion. Then we put $\widetilde{\Omega}^a_X(\log B) = \iota_* \Omega^a_W(\log B)$ for any $a \geq 0$. It is easy to see that the reflexive sheaf $\widetilde{\Omega}^a_X(\log B)$ on X does not depend

O. Fujino (✉)
Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan
e-mail: fujino@math.nagoya-u.ac.jp

on the choice of W. Note that B is a simple normal crossing divisor on W if W is a non-singular toric variety. If $B = 0$, then we write $\widetilde{\Omega}^a_X = \widetilde{\Omega}^a_X(\log B)$ for any $a \geq 0$.

The above theorem contains the following important vanishing theorems. If $B = 0$, then we obtain the famous Bott type vanishing theorem for toric varieties. It was first claimed in [5, 7.5.2 Theorem] without proof. See [4, Theorem 5]. The readers can find that this famous vanishing theorem is stated in the standard reference [17, p. 130] without proof.

Corollary 1.3 (Bott, Danilov, $\cdots$) *Let X be a projective toric variety over k and L an ample line bundle on X. Then $H^i(X, \widetilde{\Omega}^a_X \otimes L) = 0$ for any $i > 0$ and $a \geq 0$.*

In the main theorem: Theorem 1.1, if we put $a = \dim X$, then we obtain the toric version of Norimatsu type vanishing theorem. It is nothing but Mustaţă's vanishing theorem in [15, Corollary 2.5 (iii)]. The readers can find the original formulation in Corollary 2.10. One of my motivations is to give an elementary proof to Mustaţă's vanishing theorem.

Corollary 1.4 (Norimatsu, Mustaţă, $\cdots$) *Let X be a projective toric variety over k and B a reduced torus invariant Weil divisor on X. Let L be an ample line bundle on X. Then $H^i(X, \mathcal{O}_X(K_X + B) \otimes L) = 0$ for any $i > 0$.*

Note that K_X is the canonical divisor of X. It is well known that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-\sum_i D_i)$, where the summation $\sum_i D_i$ runs over all the torus invariant prime divisors on X. The final one is the Kodaira type vanishing theorem for toric varieties. It is sufficiently powerful in the toric geometry (see Sect. 4).

Corollary 1.5 (Kodaira, $\cdots$) *Let X be a projective toric variety over k and L an ample line bundle on X. Then $H^i(X, \mathcal{O}_X(K_X) \otimes L) = 0$ for any $i > 0$.*

The next theorem is another main theorem of this paper. It contains the Kawamata-Viehweg type vanishing theorem obtained by Mustaţă (see [15, Corollary 2.5 (i) and (ii)]). Our formulation is very similar to Mustaţă's theorem: [15, Theorem 0.1], but is slightly different. We will quickly see the relationship between Mustaţă's original statement and Theorem 1.6 in Sect. 2.17. His statement is a special case of our theorem (see Corollary 2.18). See also Proposition 2.16, where we will treat a variant of Theorem 1.6.

Theorem 1.6 (Main theorem II) *Let X be a toric variety defined over a field k of arbitrary characteristic and D a torus invariant $\mathbb{Q}$-Weil divisor on X. Assume that lD is an integral Weil divisor for some positive integer l. If $H^i(X, \mathcal{O}_X(lD)) = 0$ (resp. $H^i(X, \mathcal{O}_X(K_X + lD)) = 0$), then we have $H^i(X, \mathcal{O}_X(\llcorner D \lrcorner)) = 0$ (resp. $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$).*

The following corollary easily follows from Theorem 1.6. However, [15, Theorem 0.1] produced it only when D is an ample $\mathbb{Q}$-Cartier divisor (see [15, Corollary 2.5 (i) and (ii)]).

Corollary 1.7 (Kawamata-Viehweg, Mustaţă, $\cdots$) *Let X be a complete toric variety over k and D a nef $\mathbb{Q}$-Cartier torus invariant $\mathbb{Q}$-Weil divisor on X with the Iitaka dimension $\kappa(X, D) = \kappa$. Then we obtain $H^i(X, \mathcal{O}_X(\llcorner D \lrcorner)) = 0$ for $i \neq 0$ and $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for $i \neq n - \kappa$, where $n = \dim X$.*

We note that for a $\mathbb{Q}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ on X, we define the round-up $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp. the round-down $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for any real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \le \ulcorner x \urcorner < x+1$ (resp. $x-1 < \llcorner x \lrcorner \le x$). The fractional part $\{D\}$ of the $\mathbb{Q}$-Weil divisor D denotes $D - \llcorner D \lrcorner$.

We summarize the contents of this paper: In Sect. 2, we will prove Theorems 1.1 and 1.6. The main ingredient of our proof is the *multiplication map*. It is a mystery that no standard references on the toric geometry treat the multiplication map systematically. Let us introduce the l times multiplication map for toric varieties. We consider $\mathbb{P}^n$ and a finite surjective morphism $F : \mathbb{P}^n \to \mathbb{P}^n : [X_0 : \cdots : X_n] \mapsto [X_0^l : \cdots : X_n^l]$. It is the simplest example of l times multiplication maps for projective toric varieties. On the big torus $T \subset \mathbb{P}^n$, the restriction $F_T := F|_T : T \to T$ is nothing but the group homomorphism expressed by $(x_1, \ldots, x_n) \mapsto (x_1^l, \ldots, x_n^l)$. For an arbitrary n-dimensional toric variety X, $F_T : T \to T$ naturally extends to a finite surjective toric morphism $F : X \to X$. We call this $F : X \to X$ the *l times multiplication map* of X. I believe that the multiplication map will play important roles in the toric geometry. Here, I will show its usefulness by proving various vanishing theorems. Our approach is very elementary but is sufficiently powerful to prove vanishing theorems. Related topics are in [1, Sect. 7]. We do not use Frobenius morphisms (cf. [3] and [4]) nor Cox's homogeneous coordinate rings (cf. [15]). We do not need any cumbersome combinatorial arguments nor the Hodge theory (cf. [2]). We recommend the readers to compare our proof with the others (cf. [2], [4], [15], etc.). In Sect. 3, we consider slight generalizations of the main theorems: Theorems 1.1 and 1.6. Let $\mathcal{E}$ be a reflexive sheaf on X. Roughly speaking, we treat $(\mathcal{E} \otimes \widetilde{\Omega}^a(\log B))^{**}$, $(\mathcal{E} \otimes \mathcal{O}_X(\llcorner D \lrcorner))^{**}$, and $(\mathcal{E} \otimes \mathcal{O}_X(K_X + \ulcorner D \urcorner))^{**}$ instead of $\widetilde{\Omega}^a(\log B)$, $\mathcal{O}_X(\llcorner D \lrcorner)$, and $\mathcal{O}_X(K_X + \ulcorner D \urcorner)$ respectively. We note that $\mathcal{E}$ is not assumed to be equivariant. In Sect. 4, we will treat Kollár's injectivity theorem for toric varieties. For toric varieties, it easily follows from the Kodaira type vanishing theorem. In Sect. 5, which is an appendix, we will state relative vanishing theorems explicitly for future uses.

We note that our reference list does not cover all the papers treating the related topics. We apologize in advance to the colleagues whose works are not appropriately mentioned in this paper.

Let k be a fixed field of arbitrary characteristic p (p may be zero). In this paper, everything is defined over k. We do not assume that k is algebraically closed.

2 Multiplication maps and vanishing theorems

We fix our notation and define the multiplication map.

2.1 Let $N \simeq \mathbb{Z}^n$ be a lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the dual lattice. For a fan Δ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, we have the associated toric variety $X = X(\Delta)$. We put $N' = \frac{1}{l} N$ and $M' = \mathrm{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$ for any positive integer l. We note that $M' = lM$. Since $N_{\mathbb{R}} = N'_{\mathbb{R}}$, Δ is also a fan in $N'_{\mathbb{R}}$. We write Δ' to express the fan Δ in $N'_{\mathbb{R}}$. Let $X' = X(\Delta')$ be the associated toric variety. We note that $X \simeq X'$ as toric varieties. We consider the natural inclusion $\varphi : N \to N'$. Then φ induces a finite surjective toric morphism $F : X \to X'$. We call it the *l times multiplication map of X*. The following is the most important example of l times multiplication maps.

Example 2.2 The finite surjective morphism $F : \mathbb{A}^n \to \mathbb{A}^n$ given by $(a_1, \ldots, a_n) \mapsto (a_1^l, \ldots, a_n^l)$ is the l times multiplication map of $\mathbb{A}^n$.

Let us start the proof of the main theorem I: Theorem 1.1.

2.3 Let $\mathcal{A}$ be an object on X. Then we write $\mathcal{A}'$ to indicate the corresponding object on X'. Let T be the big torus of X. We construct a split injection $\Omega^1_{T'} \to F_*\Omega^1_T$. Note that Ω^1_T is nothing but a $k[M]$-module $M \otimes_{\mathbb{Z}} k[M]$.

We recall the toric description of Ω^1_T more precisely. For the details, see [5, Sect. 4] and [13].

2.4 By choosing a base suitably, we have $k[M] \simeq k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. We can write $x^m = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ for $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n = M$. Then we have the isomorphism of $k[M]$-modules $M \otimes_{\mathbb{Z}} k[M] \to H^0(T, \Omega^1_T)$ induced by $m \otimes x^{\widetilde{m}} \mapsto \frac{dx^m}{x^m} \cdot x^{\widetilde{m}} = x^{\widetilde{m}-m} dx^m$, where $m, \widetilde{m} \in \mathbb{Z}^n = M$. Note that $\wedge^a M \otimes_{\mathbb{Z}} k[M] \simeq H^0(T, \Omega^a_T)$ as $k[M]$-modules for any $a \geq 0$.

We go back to the proof of the main theorem.

2.5 Therefore, $F_*\Omega^1_T$ corresponds to a $k[M']$-module $M \otimes_{\mathbb{Z}} k[M]$. We consider the $k[M']$-module homomorphism $M' \otimes_{\mathbb{Z}} k[M'] \to M \otimes_{\mathbb{Z}} k[M]$ induced by $m_\alpha \otimes x^{m_\beta} \mapsto m_\alpha \otimes x^{lm_\beta}$. This gives an injection $\Omega^1_{T'} \to F_*\Omega^1_T$. We also consider the $k[M']$-module homomorphism $M \otimes_{\mathbb{Z}} k[M] \to M' \otimes_{\mathbb{Z}} k[M']$ obtained from $m_\alpha \otimes x^{m_\gamma} \mapsto m_\alpha \otimes x^{m_\beta}$ if $m_\gamma = lm_\beta$ and $m_\alpha \otimes x^{m_\gamma} \mapsto 0$ otherwise. By this homomorphism, the above injection $\Omega^1_{T'} \to F_*\Omega^1_T$ splits. We can generalize the above construction to $\wedge^a M' \otimes {}_{\mathbb{Z}} k[M']$ and $\wedge^a M \otimes {}_{\mathbb{Z}} k[M]$. More precisely, we consider the $k[M']$-module homomorphisms $\wedge^a M' \otimes_{\mathbb{Z}} k[M'] \to \wedge^a M \otimes_{\mathbb{Z}} k[M]$ given by $m_{\alpha_1} \wedge \cdots \wedge m_{\alpha_a} \otimes x^{m_\beta} \mapsto m_{\alpha_1} \wedge \cdots \wedge m_{\alpha_a} \otimes x^{lm_\beta}$, and $\wedge^a M \otimes_{\mathbb{Z}} k[M] \to \wedge^a M' \otimes_{\mathbb{Z}} k[M']$ induced by $m_{\alpha_1} \wedge \cdots \wedge m_{\alpha_a} \otimes x^{m_\gamma} \mapsto m_{\alpha_1} \wedge \cdots \wedge m_{\alpha_a} \otimes x^{m_\beta}$ if $m_\gamma = lm_\beta$ and $m_{\alpha_1} \wedge \cdots \wedge m_{\alpha_a} \otimes x^{m_\gamma} \mapsto 0$ otherwise. So, we obtain split injections $\Omega^a_{T'} \to F_*\Omega^a_T$ for any $a \geq 0$.

2.6 Let Δ_V be the fan in $N_{\mathbb{R}}$ that is obtained from Δ by removing the cones with dimensions ≥ 2. Then $V = X(\Delta_V)$ is a non-singular toric variety such that $\mathrm{codim}_X(X \setminus V) \geq 2$. Let B be a reduced torus invariant Weil divisor on X. Then we can construct split injections $\psi : \Omega^a_{V'}(\log B') \to F_*\Omega^a_V(\log B)$ for all $a \geq 0$, which are induced by $\Omega^a_{T'} \to F_*\Omega^a_T$. Note that we can see $\Omega^a_W(\log B) \subset \Omega^a_T$ for each affine toric open set W of V. To check that $\psi : \Omega^a_{V'}(\log B') \to F_*\Omega^a_V(\log B)$ is a split injection, it is sufficient to check it on $U = k \times (k^\times)^{n-1} \subset V$ since V is covered by finitely many $k \times (k^\times)^{n-1}$. On the open set U, it is easy to see that ψ is a split injection by direct local computations.

2.7 Let $\iota : V \hookrightarrow X$ be the natural open immersion. Since the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\iota} & X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ V' & \xrightarrow{\iota'} & X' \end{array}$$

is commutative, we obtain split injections $\widetilde{\psi} = \iota'_*\psi : \widetilde{\Omega}^a_{X'}(\log B') \to F_*\widetilde{\Omega}^a_X(\log B)$ for all a. Note that $\widetilde{\Omega}^a_X(\log B) = \iota_*\Omega^a_V(\log B)$ by Definition 1.2.

2.8 Let L be a line bundle on X. Since $L \simeq \mathcal{O}_X(G)$ for some torus invariant Cartier divisor G, we can see that $F^*L' \simeq L^l$. By combining these results,

$$\begin{aligned} H^i(X, \widetilde{\Omega}^a_X(\log B) \otimes L) &\simeq H^i(X', \widetilde{\Omega}^a_{X'}(\log B') \otimes L') \\ &\subset H^i(X', F_*\widetilde{\Omega}^a_X(\log B) \otimes L') \\ &\simeq H^i(X, \widetilde{\Omega}^a_X(\log B) \otimes L^l). \end{aligned}$$

This inclusion and Serre's vanishing theorem imply Theorem 1.1.

2.9 The corollaries in Sect. 1 directly follow from the main theorem I: Theorem 1.1. We note that Corollary 1.4 is equivalent to the following statement. This formulation seems to be more useful for various applications.

Corollary 2.10 (cf. [15, Corollary 2.5 (iii)]) *Let X be a projective toric variety over k and L an ample line bundle on X. If $D_{j_1}, \ldots, D_{j_r}$ are distinct torus invariant prime divisors, then $H^i(X, L \otimes \mathcal{O}_X(-D_{j_1} - \cdots - D_{j_r})) = 0$ for every $i > 0$.*

Let us go to the proofs of the main theorem II: Theorem 1.6, and Corollary 1.7.

2.11 (Proof of Theorem 1.6) Let $F : X \to X'$ be the l times multiplication map constructed in 2.1. Then there exist natural split injections $\mathcal{O}_{V'}(\llcorner D' \lrcorner) \to F_*\mathcal{O}_V(lD)$ and $\mathcal{O}_{V'}(K_{V'} + \ulcorner D' \urcorner) \to F_*\mathcal{O}_V(K_V + lD)$, which are induced by the split injections $\mathcal{O}_{T'} \to F_*\mathcal{O}_T$ and $\Omega^n_{T'} \to F_*\Omega^n_T$ (see 2.5). By pushing them forward to X', we obtain split injections $\mathcal{O}_{X'}(\llcorner D' \lrcorner) \to F_*\mathcal{O}_X(lD)$ and $\mathcal{O}_{X'}(K_{X'} + \ulcorner D' \urcorner) \to F_*\mathcal{O}_X(K_X + lD)$. So, we obtain

$$\begin{aligned} H^i(X, \mathcal{O}_X(\llcorner D \lrcorner)) &\simeq H^i(X', \mathcal{O}_{X'}(\llcorner D' \lrcorner)) \\ &\subset H^i(X', F_*\mathcal{O}_X(lD)) \simeq H^i(X, \mathcal{O}_X(lD)) \end{aligned}$$

and

$$\begin{aligned} H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) &\simeq H^i(X', \mathcal{O}_{X'}(K_{X'} + \ulcorner D' \urcorner)) \\ &\subset H^i(X', F_*\mathcal{O}_X(K_X + lD)) \\ &\simeq H^i(X, \mathcal{O}_X(K_X + lD)). \end{aligned}$$

Thus, Theorem 1.6 is obvious.

2.12 (Proof of Corollary 1.7) We take a positive integer l such that lD is integral and Cartier. Then $\mathcal{O}_X(K_X + lD) \simeq \mathcal{O}_X(K_X) \otimes \mathcal{O}_X(lD)$ since $\mathcal{O}_X(lD)$ is locally free. Thus, $H^i(X, \mathcal{O}_X(K_X) \otimes \mathcal{O}_X(lD)) = 0$ for $i \neq n - \kappa$ (see Theorem 4.1) and $H^i(X, \mathcal{O}_X(lD)) = 0$ for $i \neq 0$ since lD is a nef Cartier divisor (see, for example, [12, p. 74 Corollary]). This implies the desired vanishing theorems in Corollary 1.7.

Remark 2.13 Note that there are complete toric varieties that have no non-trivial nef line bundles (see [8] and [10]).

The next remark is due to Nakayama.

Remark 2.14 In Theorem 1.6 and Corollary 1.7, the assumption that D is a torus invariant $\mathbb{Q}$-Weil divisor on X can be slightly weakened. It is sufficient to assume that the fractional part $\{D\}$ is a torus invariant $\mathbb{Q}$-Weil divisor on X. We note that the integral part $\llcorner D \lrcorner$ is always linearly equivalent to a torus invariant Weil divisor on X. Similar modifications work for Propositions 2.16, 3.5, Corollary 2.18, Theorems 5.3 and 5.4. We leave the details for the readers' exercises.

2.15 The following proposition is a variant of Theorem 1.6.

Proposition 2.16 *We use the same notation as in* Theorem 1.6. *Let B be a reduced torus invariant Weil divisor on X such that B and $\{D\}$ have no common irreducible components. If $H^i(X, \mathcal{O}_X(K_X+B+lD)) = 0$, then $H^i(X, \mathcal{O}_X(K_X+B+\ulcorner D \urcorner)) = 0$. We further assume that X is projective and D is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor. Then $H^i(X, \mathcal{O}_X(K_X+B+\ulcorner D \urcorner)) = 0$ for $i > 0$.*

The proof is essentially the same as that of Theorem 1.6 if we use Corollary 1.4. We leave it for the readers' exercise.

2.17 Let us compare Mustaţă's original vanishing theorem: [15, Theorem 0.1] with Theorem 1.6. The following corollary is nothing but a reformulation of Theorem 1.6, which is a slight but important generalization of Mustaţă's vanishing theorem. We note that Doctor Sam Payne independently obtained the first part of Corollary 2.18 by another method.

Corollary 2.18 *Let X be a toric variety defined over k and D a torus invariant Weil divisor on X. Suppose that we have $E = \sum_{j=1}^{d} a_j D_j$ with $0 \le a_j \le 1$, where $D_1, \ldots, D_d$ are distinct torus invariant prime divisors on X, such that mE is an integral Weil divisor for some integer $m \ge 1$. If $H^i(X, \mathcal{O}_X(D + m(D+E))) = 0$ for some $i \ge 0$, then $H^i(X, \mathcal{O}_X(D)) = 0$. Moreover, if $H^i(X, \mathcal{O}_X(K_X + D + m(D+E))) = 0$ for some $i \ge 0$, then $H^i(X, \mathcal{O}_X(K_X + D + \ulcorner E \urcorner)) = 0$.*

Proof We put $l := m+1$ and consider a $\mathbb{Q}$-Weil divisor $D^\dagger := D + \frac{m}{m+1}E$. Then, apply Theorem 1.6. We note that $lD^\dagger = D + m(D+E)$, $\llcorner D^\dagger \lrcorner = D$, and $\ulcorner D^\dagger \urcorner = D + \ulcorner E \urcorner$. □

Remark 2.19 In Corollary 2.18, we do not assume that $m(D+E)$ is Cartier. So, the first statement is slightly better than Mustaţă's original one: [15, Theorem 0.1]. This difference may look very small. However, it causes big differences in various applications (see Corollary 1.7 and Remark 2.20). The latter statement is new. As we saw in Remark 2.14, we do not have to assume that D is torus invariant.

Remark 2.20 Let D be a torus invariant $\mathbb{Q}$-Weil divisor on X such that lD is integral. If we put $D^\spadesuit := \llcorner D \lrcorner$ and $E^\spadesuit := \frac{l}{l-1}\{D\}$, and apply Corollary 2.18 to $D^\spadesuit$ and $E^\spadesuit$ with $m := l-1$, then we can recover Theorem 1.6 from Corollary 2.18. To recover Theorem 1.6 from Mustaţă's theorem: [15, Theorem 0.1], we have to assume that $m(D^\spadesuit + E^\spadesuit) = lD - \llcorner D \lrcorner$ is Cartier. It seems to be a very artificial assumption. Thus, I believe that our theorem is much better.

2.21 In [20], Viehweg obtained his vanishing theorems as applications of the Bogomolov type vanishing theorem (cf. [20, Theorem III]). For toric varieties, we can easily check the following Bogomolov type vanishing theorem.

Theorem 2.22 (Bogomolov, $\cdots$) *Let X be a complete toric variety defined over a field k and B a reduced torus invariant Weil divisor on X. Let L be a line bundle on X with the Iitaka dimension $\kappa(X, L) \ge 0$. Then $H^0(X, \widetilde{\Omega}^a_X(\log B) \otimes L^{-1}) = 0$ for any $a \ge 0$ unless $L \simeq \mathcal{O}_X$.*

Proof Assume that $H^0(X, \widetilde{\Omega}^a_X(\log B) \otimes L^{-1}) \neq 0$. Since $\widetilde{\Omega}^a_X(\log B) \subset \wedge^a M \otimes \mathcal{O}_X$, we obtain $H^0(X, L^{-1}) \neq 0$. Therefore, $L \simeq \mathcal{O}_X$ by the assumption $\kappa(X, L) \ge 0$. □

We think that the Kawamata–Viehweg type vanishing theorem for toric varieties (cf. Corollary 1.7) does not directly follow from Theorem 2.22.

2.23 We close this section with the following three remarks.

Remark 2.24 In [2, Theorem 7.1], Corollary 1.3 was proved under the assumption that the toric variety is $\mathbb{Q}$-factorial, equivalently, has only quotient singularities. Batyrev and Cox proved it as a special case of [2, Theorem 7.2]. We note that we can easily prove [2, Theorem 7.2] by [2, Theorem 5.4], which is [5, 15.7], and Corollary 1.3 using induction on k (not on $p-k$). For k and $p-k$, see the proof of [2, Theorem 7.2]. Therefore, we can obtain [2, Lemma 7.4] as a corollary of the vanishing theorem: Corollary 1.3. Here, we do not pursue this subject anymore since we need the Hodge theory.

Remark 2.25 (Frobenius morphisms) If $l = p > 0$ and k is a perfect field, then $F : V \to V'$ is the relative Frobenius morphism and ψ induces the inverse Cartier isomorphisms $\wedge^a C^{-1} : \Omega^a_{V'/k}(\log B') \simeq \mathcal{H}^a(F_* \Omega^\bullet_{V/k}(\log B))$ for any $a \geq 0$. All the computations we need were described in [6, 9.14. Theorem]. We note that this technique produces the E_1-degeneration of the spectral sequence $E_1^{ij} = H^j(X, \widetilde{\Omega}^i_X(\log B)) \Rightarrow \mathbb{H}^{i+j}(X, \widetilde{\Omega}^\bullet_X(\log B))$ (see [4, Remark 1]). We do not pursue this topic since it was already treated in [3] and [4].

Remark 2.26 (Applications of vanishing theorems) In [15, Sect. 4], Mustaţă obtained various results on linear systems on toric varieties as applications of his vanishing theorem (cf. [15, Corollary 2.5 (iii)] or Corollaries 1.4 and 2.10). In those applications, the considered toric varieties are always non-singular. In [7], Mustaţă's results in [15, Sect. 4] were reproved and some of them were generalized for singular toric varieties. See [7, Sect. 4 and Remark 3.3]. However, the proofs in [7] are quite different from Mustaţă's. They depend on the toric Mori theory. Note that the foundation of the toric Mori theory was constructed without using vanishing theorems (see [18], [11], [9], and [19]). See also [19, Sect. 4, Applications] for some generalizations of Mustaţă's results for the relative setting.

3 Variants of the main vanishing theorems

In this section, we treat slight generalizations of the main vanishing theorems. We need no new arguments.

3.1 The following theorem is a small generalization of Theorem 1.1. It may be useful in the future. So, we state it here.

Proposition 3.2 *Let X and B be the same as in* Theorem 1.1. *Let D be a* (*not necessarily torus invariant*) *Weil divisor on X and $\mathcal{E}$ a reflexive sheaf on X. We consider the l times multiplication map $F : X \to X' \simeq X$, which was defined in 2.1, for some positive integer l. If $H^i(X, (\widetilde{\Omega}^a_X(\log B) \otimes F^*\mathcal{E} \otimes \mathcal{O}_X(lD))^{**}) = 0$, then $H^i(X, (\widetilde{\Omega}^a_X(\log B) \otimes \mathcal{E} \otimes \mathcal{O}_X(D))^{**}) = 0$. In particular, $H^i(X, (\widetilde{\Omega}^a_X(\log B) \otimes \mathcal{O}_X(lD))^{**}) = 0$ implies $H^i(X, (\widetilde{\Omega}^a_X(\log B) \otimes \mathcal{O}_X(D))^{**}) = 0$.*

We will prove this proposition after the proof of Proposition 3.5.

Remark 3.3 In Proposition 3.2, if $\mathcal{E}$ is locally free and D (resp. lD) is Cartier, or X is non-singular, then $(\widetilde{\Omega}^a_X(\log B)\otimes\mathcal{E}\otimes\mathcal{O}_X(D))^{**}\simeq\widetilde{\Omega}^a_X(\log B)\otimes\mathcal{E}\otimes\mathcal{O}_X(D)$ (resp. $(\widetilde{\Omega}^a_X(\log B)\otimes F^*\mathcal{E}\otimes\mathcal{O}_X(lD))^{**}\simeq\widetilde{\Omega}^a_X(\log B)\otimes F^*\mathcal{E}\otimes\mathcal{O}_X(lD)$) in Proposition 3.2. See Remark 3.7 (ii).

3.4 We treat a similar variant of Theorem 1.6 here. Doctor Sam Payne independently obtained a special case of the following theorem under the extra assumption that $\mathcal{E}$ is equivariant. I was inspired by his private notes.

Proposition 3.5 *We use the same notation as in* Theorem 1.6*. Let $\mathcal{E}$ be a reflexive sheaf on X. Let $F: X\to X'\simeq X$ be the l times multiplication map as in* 2.1*. If $H^i(X,(F^*\mathcal{E}\otimes\mathcal{O}_X(lD))^{**})=0$ (resp. $H^i(X,(F^*\mathcal{E}\otimes\mathcal{O}_X(K_X+lD))^{**})=0$), then we have $H^i(X,(\mathcal{E}\otimes\mathcal{O}_X(\llcorner D\lrcorner))^{**})=0$ (resp. $H^i(X,(\mathcal{E}\otimes\mathcal{O}_X(K_X+\ulcorner D\urcorner))^{**})=0$).*

Remark 3.6 If $\mathcal{E}$ is a locally free sheaf or X is non-singular, then we do not need to take double duals in Proposition 3.5. See Remark 3.7 (ii) below. If $\mathcal{E}\simeq\mathcal{O}_X$, then Proposition 3.5 is nothing but Theorem 1.6.

Before we go to the proofs, we make some remarks on reflexive sheaves.

Remark 3.7 (i) Let $\mathcal{F}$ be a coherent sheaf on a normal variety X. Then $\mathcal{F}^{**}$ denotes the double dual of $\mathcal{F}$. (ii) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be reflexive sheaves on a normal variety X. Then $(\mathcal{F}_1\otimes\mathcal{F}_2)^{**}\simeq\mathcal{F}_1\otimes\mathcal{F}_2$ if one of the $\mathcal{F}_i$ is locally free.

Proof of Proposition 3.5 Let V' be the Zariski open set of X' as in 2.6. We take a Zariski open set W' of V' such that $\mathcal{E}'$ is locally free on W' and $\operatorname{codim}_{X'}(X'\setminus W')\geq 2$. Note that W' is not torus invariant when $W'\neq V'$. We put $W=F^{-1}(W')\subset V$. Then we obtain the following commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ W' & \longrightarrow & X' \end{array}$$

as in 2.7, where the horizontal arrows are natural open immersions. We have split injections

$$\mathcal{E}'\otimes\mathcal{O}_{W'}(\llcorner D'\lrcorner)\to\mathcal{E}'\otimes F_*\mathcal{O}_W(lD)\simeq F_*(F^*\mathcal{E}'\otimes\mathcal{O}_W(lD))$$

and

$$\begin{aligned}\mathcal{E}'\otimes\mathcal{O}_{W'}(K_{W'}+\ulcorner D'\urcorner)&\to\mathcal{E}'\otimes F_*\mathcal{O}_W(K_W+lD)\\ &\simeq F_*(F^*\mathcal{E}'\otimes\mathcal{O}_W(K_W+lD))\end{aligned}$$

on W' by 2.11 and the projection formula. By pushing them forward to X', we obtain the desired vanishing theorems by the same arguments as in 2.11. □

Proof of Proposition 3.2 We note that we can replace D by a linearly equivalent torus invariant Weil divisor. So, we assume that D is torus invariant. By the arguments in 2.6 and 2.8, we can check that there exist split injections

$$\Omega^a_{W'}(\log B')\otimes\mathcal{E}\otimes\mathcal{O}_{W'}(D')\to F_*(\Omega^a_W(\log B)\otimes F^*\mathcal{E}\otimes\mathcal{O}_W(lD))$$

for any $a\geq 0$, where W' (resp. W) is the Zariski open set of V' (resp. V) defined in the proof of Proposition 3.5. Note that $F^*\mathcal{O}_{V'}(D')\simeq\mathcal{O}_V(lD)$ since D is Cartier on

V. It is because V is non-singular. Therefore, by pushing the above split injections to X', we have split injections $(\widetilde{\Omega}^a_{X'}(\log B') \otimes \mathcal{E} \otimes \mathcal{O}_{X'}(D'))^{**} \to F_*((\widetilde{\Omega}^a_X(\log B) \otimes F^*\mathcal{E} \otimes \mathcal{O}_X(lD))^{**})$ for all $a \geq 0$ (see 2.7). This obviously implies Proposition 3.2. □

By applying Proposition 3.2 in place of Theorem 1.1, some vanishing theorems in this paper can be generalized slightly. We leave the details for the readers' exercise. We also leave to the interested readers the pleasure of combining the latter part of Proposition 3.5 with Proposition 2.16.

4 Kollár's injectivity theorem

In this section, we treat Kollár's injectivity theorem (cf. [14, Theorem 2.2]) for toric varieties. It is an application of Corollary 1.5.

Theorem 4.1 *Let X be a complete toric variety defined over k and L a nef line bundle on X. Let s be a non-zero holomorphic section of L^l, where $l \geq 0$. Then*

$$\times s : H^i(X, \mathcal{O}_X(K_X) \otimes L^m) \to H^i(X, \mathcal{O}_X(K_X) \otimes L^{m+l})$$

is injective for any $m \geq 1$ and $i \geq 0$, where $\times s$ is the morphism induced by the tensor product with s. More precisely, $H^i(X, \mathcal{O}_X(K_X) \otimes L^m) = 0$ for any $m \geq 1$ when $i \neq n - \kappa$. Here, $n = \dim X$ and $\kappa = \kappa(X, L)$.

The following lemma is well known. The readers can find it in any text book on the toric geometry (see, for example, [12, p. 76 Proposition and p. 89 Proposition]).

Lemma 4.2 *Let $f : X \to Y$ be a proper birational toric morphism. Then $R^if_*\mathcal{O}_X = 0$ and $R^if_*\mathcal{O}_X(K_X) = 0$ for all $i > 0$, $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$, and $f_*\mathcal{O}_X(K_X) \simeq \mathcal{O}_Y(K_Y)$.*

The next lemma is a slight generalization of Lemma 4.2.

Lemma 4.3 *Let $f : X \to Y$ be a proper surjective toric morphism with connected fibers. Then $R^if_*\mathcal{O}_X = 0$ for $i > 0$ and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Moreover, $R^{n-m}f_*\mathcal{O}_X(K_X) \simeq \mathcal{O}_Y(K_Y)$ and $R^if_*\mathcal{O}_X(K_X) = 0$ for $i \neq n - m$, where $n = \dim X$ and $m = \dim Y$.*

Sketch of the proof The former statement is an exercise if we use Lemma 4.2. For the proof, see, for example, [13, Theorem 3.2]. The latter part follows from the Grothendieck duality and the former statement. □

Lemma 4.3 implies that Kollár's torsion-freeness (cf. [14, Theorem 2.1 (i)]) is obvious for toric varieties and Kollár's vanishing theorem (cf. [14, Theorem 2.1 (iii)]) is a special case of Corollary 1.5 in the toric geometry.

Proof of Theorem 4.1 Since L is nef, there exists a proper surjective toric morphism with connected fibers $f : X \to Y$ such that $L \simeq f^*H$, where H is an ample line bundle on Y. By the definition of κ, we have $\dim Y = \kappa$. We consider the spectral sequence $H^i(Y, R^jf_*\mathcal{O}_X(K_X) \otimes H^b) \Rightarrow H^{i+j}(X, \mathcal{O}_X(K_X) \otimes L^b)$ for any integer b. By Lemma 4.3, we obtain $H^i(Y, \mathcal{O}_Y(K_Y) \otimes H^b) \simeq H^{i+n-\kappa}(X, \mathcal{O}_X(K_X) \otimes L^b)$. Therefore, we have $H^{n-\kappa}(X, \mathcal{O}_X(K_X) \otimes L^b) \simeq H^0(Y, \mathcal{O}_Y(K_Y) \otimes H^b)$ and $H^i(X, \mathcal{O}_X(K_X) \otimes L^m) = 0$ for $i \neq n - \kappa$ and $m \geq 1$ by Corollary 1.5. Note that $H^0(X, L^l) \simeq H^0(Y, H^l)$. So, there exists a non-zero $t \in H^0(Y, H^l)$ such that $s = f^*t$. Thus, $\times s : H^{n-\kappa}(X, \mathcal{O}_X(K_X) \otimes L^m) \to H^{n-\kappa}(X, \mathcal{O}_X(K_X) \otimes L^{m+l})$ is nothing but $\times t : H^0(Y, \mathcal{O}_Y(K_Y) \otimes H^m) \to H^0(Y, \mathcal{O}_Y(K_Y) \otimes H^{m+l})$. Therefore, $\times s$ is injective since $\times t$ is injective. □

4.4 As we saw in Theorem 4.1, the Kodaira type vanishing theorem (cf. Corollary 1.5) holds for nef and big line bundles. However, the Norimatsu type vanishing theorem (cf. Corollary 1.4) does not always hold for nef and big line bundles by the next example.

Example 4.5 In this example, we assume $k = \mathbb{C}$, the complex number field, for simplicity. Let $P \in \mathbb{P}^2$ be a torus invariant closed point and let $f : X \to \mathbb{P}^2$ be the blow-up at P. Let B be the f-exceptional curve on X. Then we obtain $0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + B) \to \mathcal{O}_B(K_B) \to 0$ by adjunction. By applying R^if_*, we obtain $f_*\mathcal{O}_X(K_X + B) \simeq \mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2})$ and $R^1f_*\mathcal{O}_X(K_X + B) \simeq \mathbb{C}(P)$ since $R^if_*\mathcal{O}_X(K_X) = 0$ for $i > 0$. We put $H = \mathcal{O}_{\mathbb{P}^2}(1)$ and $L = f^*H$. Note that L is nef and big. Then, by the Leray spectral sequence, we have the following exact sequence: $0 \to H^1(\mathbb{P}^2, f_*\mathcal{O}_X(K_X + B) \otimes H) \to H^1(X, \mathcal{O}_X(K_X + B) \otimes L) \to H^0(\mathbb{P}^2, R^1f_*\mathcal{O}_X(K_X + B) \otimes H) \to H^2(\mathbb{P}^2, f_*\mathcal{O}_X(K_X + B) \otimes H) \to \cdots$. Since the first and the last terms are zero, $H^1(X, \mathcal{O}_X(K_X + B) \otimes L) \simeq H^0(\mathbb{P}^2, \mathbb{C}(P) \otimes H) \simeq \mathbb{C}$.

5 Appendix: Relative vanishing theorems

In this appendix, we state relative vanishing theorems explicitly for future uses. All the vanishing theorems easily follow from the main theorems and their proofs. We only give a proof of Theorem 5.3 for the readers' convenience. The others are similar and easier to prove.

5.1 Let $f : X \to Y$ be a proper surjective toric morphism and B a reduced torus invariant Weil divisor on X. We put $n = \dim X$ and $m = \dim Y$.

Theorem 5.2 *Let L be an f-ample line bundle on X. Then we have $R^if_*(\widetilde{\Omega}^a_X(\log B) \otimes L) = 0$ for $i > 0$. In particular, $R^if_*(\widetilde{\Omega}^a_X \otimes L) = 0$, $R^if_*(\mathcal{O}_X(K_X + B) \otimes L) = 0$, and $R^if_*(\mathcal{O}_X(K_X) \otimes L) = 0$ for $i > 0$.*

Theorem 5.3 *Let D be a torus invariant $\mathbb{Q}$-Weil divisor on X. Assume that D is $\mathbb{Q}$-Cartier and f-nef with the relative Iitaka dimension $\kappa(X/Y, D) = \kappa$. Then $R^if_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$ for $i \neq n - m - \kappa$ and $R^if_*\mathcal{O}_X(\llcorner D \lrcorner) = 0$ for $i \neq 0$.*

Proof Let l be a positive integer such that lD is integral and Cartier. We note that $f^*f_*\mathcal{O}_X(lD) \to \mathcal{O}_X(lD)$ is surjective since lD is an f-nef Cartier divisor on X (see, for example, [16, Chapter IV, 1.13 Lemma]). We can assume that Y is affine since the problem is local. By 2.11, it is sufficient to prove that $R^if_*(\mathcal{O}_X(K_X) \otimes \mathcal{O}_X(lD)) = 0$ for $i \neq n - m - \kappa$ and $R^if_*\mathcal{O}_X(lD) = 0$ for $i \neq 0$. First, we prove $R^if_*\mathcal{O}_X(lD) = 0$ for $i > 0$. In this case, $R^if_*\mathcal{O}_X(lD) \simeq H^i(X, \mathcal{O}_X(lD)) = 0$ by [12, p. 74 Corollary] since $\mathcal{O}_X(lD)$ is generated by its global sections and the support of the fan associated to X is convex. Next, we prove $R^if_*(\mathcal{O}_X(K_X) \otimes \mathcal{O}_X(lD)) = 0$ for $i \neq n - m - \kappa$. Let $g : X \to Z$ be a proper surjective toric morphism over Y with connected fibers such that $\mathcal{O}_X(lD) \simeq g^*\mathcal{O}_Z(H)$, where H is a Cartier divisor on Z which is ample over Y. We note that $\dim Z = m + \kappa$. By Lemma 4.3 and Leray's spectral sequence, we obtain $R^if_*(\mathcal{O}_X(K_X) \otimes \mathcal{O}_X(lD)) \simeq R^{i-(n-m-\kappa)}h_*(\mathcal{O}_Z(K_Z) \otimes \mathcal{O}_Z(H))$, where $h : Z \to Y$. In particular, $R^if_*(\mathcal{O}_X(K_X) \otimes \mathcal{O}_X(lD)) = 0$ for $i < n - m - \kappa$. By the same arguments as in 2.8, we have $R^{i-(n-m-\kappa)}h_*(\mathcal{O}_Z(K_Z) \otimes \mathcal{O}_Z(H)) \subset R^{i-(n-m-\kappa)}h_*(\mathcal{O}_Z(K_Z) \otimes \mathcal{O}_Z(l'H))$ for any positive integer l'. If $l' \gg 0$, then $R^{i-(n-m-\kappa)}h_*(\mathcal{O}_Z(K_Z) \otimes \mathcal{O}_Z(l'H)) = 0$ for

$i > n-m-\kappa$ by Serre's vanishing theorem. Thus, $R^{i-(n-m-\kappa)}h_*(\mathcal{O}_Z(K_Z) \otimes \mathcal{O}_Z(H)) = 0$ for $i > n-m-\kappa$. Therefore, we obtain the desired vanishing theorems. □

Theorem 5.4 *Let D be an f-ample $\mathbb{Q}$-Cartier torus invariant $\mathbb{Q}$-Weil divisor on X such that B and $\{D\}$ have no common irreducible components. Then $R^if_*\mathcal{O}_X(K_X + B + \ulcorner D \urcorner) = 0$ for $i > 0$.*

We note that Theorem 4.1 can be generalized for the relative setting if we use Theorem 5.2 instead of Corollary 1.5.

Acknowledgment I would like to express my gratitude to Professors Kazuhiro Fujiwara and Takeshi Abe for giving me much advice and encouraging me during the preparation of this paper. I thank Dr. Hiroshi Sato for valuable discussions and Professor Noboru Nakayam for pointing out a little mistake. I was partially supported by The Sumitomo Foundation and by the Grant-in-Aid for Young Scientists (A) ♯17684001 from JSPS. I would like to thank Professor Donu Arapura and Doctor Sam Payne, who gave me comments by e-mails after I circulated the first version of this paper. Professor Donu Arapura informed me of his recent paper [1]. Doctor Sam Payne sent me his private notes on vanishing theorems for toric varieties. The discussions with him helped me revise this paper.

References

1. Arapura, D.: Partial regularity and amplitude. J. Math. Am. **128**(4), 1025–1056 (2006)
2. Batyrev, V., Cox, D.: On the Hodge structure of projective hypersurfaces in toric varieties. Duke Math. J. **75**(2), 293–338 (1994)
3. Blickle, M.: Cartier isomorphism for toric varieties. J. Algebra **237**(1), 342–357 (2001)
4. Buch, A., Thomsen, J.F., Lauritzen, N., Mehta, V.: The Frobenius morphism on a toric variety. Tohoku Math. J. (2) **49**(3), 355–366 (1997)
5. Danilov, V.I.: The geometry of toric varieties. Russ. Math. Surv. **33**(2), 97–154 (1978)
6. Esnault, H., Viehweg, E.: Lectures on vanishing theorems, DMV seminar, vol. 20, vi+164 pp. Birkhäuser Verlag, Basel (1992)
7. Fujino, O.: Notes on toric varieties from Mori theoretic viewpoint. Tohoku Math. J.(2) **55**(4), 551–564 (2003)
8. Fujino, O.: On the Kleiman-Mori cone. Proc. Jpn. Acad. Ser. A Math. Sci. **81**(5), 80–84 (2005)
9. Fujino, O.: Equivariant completions of toric contraction morphisms. Tohoku Math. J. (2) **58**(3), 303–321 (2006)
10. Fujino, O., Payne, S.: Smooth complete toric threefolds with no nontrivial nef line bundles. Proc. Jpn. Acad. Ser. A Math. Sci. **81**(10), 174–179 (2005)
11. Fujino, O., Sato, H.: Introduction to the toric Mori theory. Mich. Math. J. **52**(3), 649–665 (2004)
12. Fulton, W.: Introduction to toric varieties. Ann. Math. Stud. In: The William H. Roever Lectures in Geometry, vol. 131, xii+157 pp. Princeton University Press, Princeton (1993)
13. Ishida, M.: Torus embeddings and de Rham complexes, commutative algebra and combinatorics (Kyoto, 1985). Adv. Stud. Pure Math. vol. 11, pp. 111–145. North-Holland, Amsterdam (1987)
14. Kollár, J.: Higher direct images of dualizing sheaves. I. Ann. Math. (2) **123**(1), 11–42 (1986)
15. Mustaţă, M.: Vanishing theorems on toric varieties. Tohoku Math J. (2) **54**(3), 451–470 (2002)
16. Nakayama, N.: Zariski-decomposition and abundance, MSJ memoirs. Math. Soc. Jpn. **14**, xiv+277 (2004)
17. Oda, T.: Convex bodies and algebraic geometry, an introduction to the theory of toric varieties. Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, viii+212 pp. Springer, Berlin (1988)
18. Reid, M.: Decomposition of toric morphisms, Arithmetic and geometry, vol. II. Progr. Math. vol. 36, pp. 395–418. Birkhäuser Boston (1983)
19. Sato, H.: Combinatorial descriptions of toric extremal contractions. J. Nagoya Math. **180**, 111–120 (2005)
20. Viehweg, E.: Vanishing theorems. J. Reine Angew. Math. **335**, 1–8 (1982)