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The next lemma is well known as the negativity lemma.

lem211 **Lemma 0.1** (Negativity lemma). *Let $h : Z \to Y$ be a proper birational morphism between normal varieties. Let $-B$ be an h-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Z. Then we have the following statements.*

(1) *B is effective if and only if h_*B is.*
(2) *Assume that B is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset \mathrm{Supp} B$ or $h^{-1}(y) \cap \mathrm{Supp} B = \emptyset$.*

Sketch of the proof. By Chow's lemma, we can assume that h is projective. We can also assume that Y is affine. By taking general hypersurfaces, we can reduce the problem to the case when $\dim Y = 2$. Then we use the Hodge index theorem on Z. For the details, see km [?, Lemma 3.39]. □

1. Semi-ample $\mathbb{R}$-divisors

defn4949 **Definition 1.1** (Semi-ample $\mathbb{R}$-divisors). An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X is *π-semi-ample* if $D \sim_{\mathbb{R}} \sum_i a_i D_i$, where D_i is a π-semi-ample Cartier divisor on X and a_i is a positive real number for every i.

Remark 1.2. In Definition defn4949 1.1, we can replace $D \sim_{\mathbb{R}} \sum_i a_i D_i$ with $D = \sum_i a_i D_i$ since every principal Cartier divisor on X is π-semi-ample.

The following two lemmas seem to be missing in the literature.

49-1 **Lemma 1.3.** *Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then the following conditions are equivalent.*

(1) *D is π-semi-ample.*
(2) *There exists a morphism $f : X \to Y$ over S such that $D \sim_{\mathbb{R}} f^*A$, where A is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y which is ample over S.*

Proof. It is obvious that (1) follows from (2). If D is π-semi-ample, then we can write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition defn4949 1.1. By replacing D_i with its multiple, we can assume that $\pi^*\pi_*\mathcal{O}_X(D_i) \to \mathcal{O}_X(D_i)$ is

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surjective for every i. Let $f : X \to Y$ be a morphism over S obtained by the surjection $\pi^*\pi_*\mathcal{O}_X(\sum_i D_i) \to \mathcal{O}_X(\sum_i D_i)$. Then it is easy to see that $f : Y \to X$ has the desired property. □

49-2 **Lemma 1.4.** *Let D be a Cartier divisor on X. If D is π-semi-ample in the sense of* Definition defn4949 1.1*, then D is π-semi-ample in the usual sense, that is, $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective for some positive integer m. In particular,* Definition defn4949 1.1 *is well-defined.*

Proof. We write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition defn4949 1.1. Let $f : X \to Y$ be a morphism in Lemma 49-1 1.3 (2). By taking the Stein factorization, we can assume that f has connected fibers. By the construction, $D_i \sim_{\mathbb{Q},f} 0$ for every i. By replacing D_i with its multiple, we can assume that $D_i \sim f^*D'_i$ for some Cartier divisor D'_i on Y for every i. Let U be any Zariski open set of Y on which $D'_i \sim 0$ for every i. On $f^{-1}(U)$, we have $D \sim_{\mathbb{R}} 0$. This implies $D \sim_{\mathbb{Q}} 0$ on $f^{-1}(U)$ since D is Cartier. Therefore, there exists a positive integer m such that $f^*f_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective. By this surjection, we have $mD \sim f^*A$ for a Cartier divisor A on Y which is ample over S. This means that D is π-semi-ample in the usual sense. □

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We recommend the reader to see Cutkosky's interesting example in cutkosky [Cu, Theorem 6], which is a cone over a generic Enriques surface. Our example seems to be slightly simpler.

Remark 2.1. In Example exe88 ??, we have $H^i(S, \mathcal{O}_S) = 0$ for every $i > 0$ since S is rational. By Lemma weakdel 2.2 and Lemma 4371em ?? (2), the cone singularity of X in Example exe88 ?? is a rational singularity.

weakdel **Lemma 2.2.** *Let $E \subset \mathbb{P}^2$ be a smooth cubic curve and $f : S \to \mathbb{P}^2$ the blow-up of nine general points on E. Then*

$$H^i(S, \mathcal{O}_S(A)) = 0$$

for every $i > 0$, where A is an ample Cartier divisor on S.

Proof. It is easy to see that $-K_S \sim E_S$, where E_S is the strict transform of E on S. Since $(E_S)^2 = 0$, we see that $-K_S$ is nef. Therefore, $-K_S + A$ is ample. Thus, $H^i(S, \mathcal{O}_S(A)) = H^i(S, \mathcal{O}_S(K_S - K_S + A)) = 0$ for every $i > 0$ by the Kodaira vanishing theorem. □

References

cutkosky [Cu] S. Cutkosky, Weil divisors and symbolic algebras, Duke Math. J. **57** (1988), no. 1, 175–183.

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