ON GENERAL KODAIRA VANISHING THEOREM

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Abstract. We give a proof of the general Kodaira vanishing theorem stated in [KM].

Contents

1. General Kodaira Vanishing Theorem

The following vanishing theorem is stated in [KM, Theorem 2.70] without proof.

Theorem 1.1 (General Kodaira Vanishing Theorem, see [KM, Theorem 2.70])**.** *Let (X, Δ) be a proper klt pair. Let N be a $\mathbb{Q}$-Cartier Weil divisor on X such that $N \equiv M + \Delta$, where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $H^i(X, \mathcal{O}_X(-N)) = 0$ for $i < \dim X$.*

In this short note, we prove Theorem 1.1 for the sake of completeness. More precisely, we show that Theorem 1.1 follows easily from the usual Kawamata–Viehweg vanishing theorem for klt pairs (see, for example, [KMM, Theorem 1-2-5 and Remark 1-2-6] and [F, Corollary 5.7.7]). For the reader's convenience, we recall it here.

Theorem 1.2 (Kawamata–Viehweg vanishing theorem for klt pairs, see [F, Corollary 5.7.7])**.** *Let (X, Δ) be a klt pair and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L - (K_X + \Delta)$ is nef and big over V, where $\pi\colon X \to V$ is a proper morphism. Then $R^q\pi_*\mathcal{O}_X(L) = 0$ for every $q > 0$.*

We do not know why the authors of [KM] adopted the formulation of Theorem 1.1. The usual Kawamata–Viehweg vanishing theorem for klt pairs, that is, Theorem 1.2, seems to be more natural and useful than Theorem 1.1 (see also Remark 2.2 below).

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2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using Theorem 1.2.

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Proof of Theorem 1.1. In Step 1, we explain a special case of the usual Kawamata–Viehweg vanishing theorem for klt pairs. In Step 2, we show that $\mathcal{O}_X(-N)$ is a Cohen–Macaulay sheaf on X. In Step 3, we prove the desired vanishing theorem using Serre duality for Cohen–Macaulay sheaves under the additional assumption that X is projective. Finally, in Step 4, we treat the general case using Grothendieck duality.

Step 1 (Kawamata–Viehweg vanishing theorem)**.** Since M and $M+\Delta$ are $\mathbb{Q}$-Cartier by assumption, it follows that Δ is $\mathbb{Q}$-Cartier. Therefore, K_X is a $\mathbb{Q}$-Cartier Weil divisor on X. Thus, K_X+N is a $\mathbb{Q}$-Cartier Weil divisor such that

$$K_X+N-(K_X+\Delta)\equiv M$$

is nef and big. By the Kawamata–Viehweg vanishing theorem for klt pairs (see Theorem 1.2), we obtain

$$H^i(X,\mathcal{O}_X(K_X+N))=0$$

for every $i>0$.

Step 2. Since (X,Δ) is klt, X has only rational singularities (see, for example, [KMM, Theorem 1-3-6] and [F, Theorems 3.13.1 and 3.15.1]). In particular, X is Cohen–Macaulay. Since N is a $\mathbb{Q}$-Cartier Weil divisor and (X,Δ) is klt, it is known that $\mathcal{O}_X(-N)$ is a Cohen–Macaulay sheaf on X (see, for example, [KM, Corollary 5.25] and [K, Corollary 2.88 (2)]).

Step 3 (X is projective)**.** In this step, we prove the desired vanishing theorem under the additional assumption that X is projective.

In this case, by Serre duality for Cohen–Macaulay sheaves (see, for example, [KM, Theorem 5.71]), the vector space $H^i(X,\mathcal{O}_X(-N))$ is dual to

$$H^{\dim X-i}(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-N),\omega_X))\simeq H^{\dim X-i}(X,\mathcal{O}_X(K_X+N)).$$

Hence $H^i(X,\mathcal{O}_X(-N))=0$ for $i<\dim X$ by Step 1. This completes the proof when X is projective.

Step 4 (X is proper)**.** Let $\pi\colon X\to Y:=\operatorname{Spec}\mathbb{C}$ be the structure morphism. By Grothendieck duality (see, for example, [H]), we have

$$R\mathcal{H}om_Y(R\pi_*\mathcal{O}_X(-N),\omega_Y^\bullet)\simeq R\pi_*R\mathcal{H}om_X(\mathcal{O}_X(-N),\omega_X^\bullet). \tag{2.1}$$

Since X is Cohen–Macaulay, we have

$$\omega_X^\bullet\simeq\mathcal{O}_X(K_X)[\dim X].$$

Let $x\in X$ be any closed point. Then, by [K, Proposition 2.66],

$$\left(\mathcal{E}xt_X^{-i}(\mathcal{O}_X(-N),\omega_X^\bullet)\right)_x=0$$

for $i\neq\dim X$ since $\mathcal{O}_X(-N)$ is Cohen–Macaulay. Therefore, $R\mathcal{H}om_X(\mathcal{O}_X(-N),\omega_X^\bullet)$ is quasi-isomorphic to $\mathcal{O}_X(K_X+N)[\dim X]$. Hence, by (2.1), $H^i(X,\mathcal{O}_X(-N))$ is dual to $H^{\dim X-i}(X,\mathcal{O}_X(K_X+N))$. Thus, the desired vanishing follows from Step 1.

This completes the proof of Theorem 1.1. □

Note that Theorem 1.1 holds for $\mathbb{R}$-divisors, since the usual Kawamata–Viehweg vanishing theorem for klt pairs holds for $\mathbb{R}$-divisors (see Theorem 1.2).

Remark 2.1. In Theorem 1.1, it is sufficient to assume that M is a nef and big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and that (X,Δ) is klt with $K_X+\Delta$ $\mathbb{R}$-Cartier.

We close this short note with a brief remark on lc pairs.

Remark 2.2. The usual Kawamata–Viehweg vanishing theorem for klt pairs (see Theorem 1.2) can be generalized to the Reid–Fukuda vanishing theorem for lc pairs (see [F, Theorem 5.7.6]). However, by [KM, Corollary 5.72], the formulation of Theorem 1.1 cannot be generalized to lc pairs, since lc pairs are not necessarily Cohen–Macaulay. This is one reason why we do not treat Theorem 1.1 in [F].

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