ON GENERAL KODAIRA VANISHING THEOREM

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Abstract. We give a proof of the general Kodaira vanishing theorem in [KM].

1. General Kodaira Vanishing Theorem

The following vanishing theorem is stated in [KM, Theorem 2.70] without proof.

Theorem 1.1 (General Kodaira Vanishing Theorem, see [KM, Theorem 2.70])**.** *Let (X, Δ) be a proper klt pair. Let N be a $\mathbb{Q}$-Cartier Weil divisor on X such that $N \equiv M + \Delta$, where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $H^i(X, \mathcal{O}_X(-N)) = 0$ for $i < \dim X$.*

In this short note, we will prove Theorem 1.1 for the sake of completeness. More precisely, we will see that Theorem 1.1 easily follows from the usual Kawamata–Viehweg vanishing theorem for klt pairs (see, for example, [KMM, Theorem 1-2-5 and Remark 1-2-6] and [F, Corollary 5.7.7]). Here we recall it for the reader's convenience.

Theorem 1.2 (Kawamata–Viehweg vanishing theorem for klt pairs, see [F, Corollary 5.7.7])**.** *Let (X, Δ) be a klt pair and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L - (K_X + \Delta)$ is nef and big over V, where $\pi \colon X \to V$ is a proper morphism. Then $R^q \pi_* \mathcal{O}_X(L) = 0$ for every $q > 0$.*

We do not know the reason why the authors in [KM] adopted the formulation of Theorem 1.1. The usual Kawamata–Viehweg vanishing theorem for klt pairs, that is, Theorem 1.2, seems to be more natural and useful than Theorem 1.1 (see also Remark 2.2 below).

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2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by using Theorem 1.2.

Proof of Theorem 1.1. In Step 1, we will explain a special case of the usual Kawamata–Viehweg vanishing theorem for klt pairs. In Step 2, we will see that $\mathcal{O}_X(-N)$ is a Cohen–Macaulay sheaf on X. Then, in Step 3, we will prove the desired vanishing theorem by using Serre duality for Cohen–Macaulay sheaves under the extra assumption that X is projective. In Step 4, we will treat the general case by using Grothendieck duality.

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Step 1 (Kawamata–Viehweg vanishing theorem)**.** Since M and $M+\Delta$ are $\mathbb{Q}$-Cartier by assumption, we see that Δ is $\mathbb{Q}$-Cartier. Therefore, K_X is a $\mathbb{Q}$-Cartier Weil divisor on X. Thus, K_X+N is a $\mathbb{Q}$-Cartier Weil divisor such that

$$K_X+N-(K_X+\Delta)\equiv M$$

is nef and big. By the usual Kawamata–Viehweg vanishing theorem for klt pairs (see Theorem 1.2), we have

$$H^i(X,\mathcal{O}_X(K_X+N))=0$$

for every $i>0$.

Step 2. Since (X,Δ) is klt, X has only rational singularities (see, for example, [KMM, Theorem 1-3-6] and [F, Theorems 3.13.1 and 3.15.1]). In particular, X is Cohen–Macaulay. Since N is a $\mathbb{Q}$-Cartier Weil divisor and (X,Δ) is klt, it is known that $\mathcal{O}_X(-N)$ is a Cohen–Macaulay sheaf on X (see, for example, [KM, Corollary 5.25] and [K, Corollary 2.88 (2)]).

Step 3 (X is projective)**.** In this step, we will prove the desired vanishing theorem under the extra assumption that X is projective.

In this case, by Serre duality for Cohen–Macaulay sheaves (see, for example, [KM, Theorem 5.71]), the vector space $H^i(X,\mathcal{O}_X(-N))$ is dual to

$$H^{\dim X-i}(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-N),\omega_X))=H^{\dim X-i}(X,\mathcal{O}_X(K_X+N)).$$

Hence $H^i(X,\mathcal{O}_X(-N))=0$ holds for $i<\dim X$ by Step 1. So we finish the proof when X is projective.

Step 4 (X is proper)**.** Let $\pi\colon X\to Y:=\operatorname{Spec}\mathbb{C}$ be the structure morphism. By Grothendieck duality (see, for example, [H]), we have

$$R\mathcal{H}om_Y(R\pi_*\mathcal{O}_X(-N),\omega_Y^\bullet)\simeq R\pi_*R\mathcal{H}om_X(\mathcal{O}_X(-N),\omega_X^\bullet). \tag{2.1}$$

We note that $\omega_X^\bullet\simeq\mathcal{O}_X(K_X)[\dim X]$ since X is Cohen–Macaulay. Let $x\in X$ be any closed point. Then, by [K, Proposition 2.66],

$$\left(\mathcal{E}xt_X^{-i}(\mathcal{O}_X(-N),\omega_X^\bullet)\right)_x=0$$

for $i\neq\dim X$ since $\mathcal{O}_X(-N)$ is Cohen–Macaulay. Hence, by (2.1), $H^i(X,\mathcal{O}_X(-N))$ is dual to $H^{\dim X-i}(X,\mathcal{O}_X(K_X+N))$. Thus, we have the desired vanishing theorem by Step 1.

We finish the proof of Theorem 1.1. □

Note that Theorem 1.1 holds for $\mathbb{R}$-divisors since the usual Kawamata–Viehweg vanishing theorem for klt pairs holds for $\mathbb{R}$-divisors (see Theorem 1.2).

Remark 2.1. In Theorem 1.1, it is sufficient to assume that M is a nef and big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and that (X,Δ) is klt such that $K_X+\Delta$ is $\mathbb{R}$-Cartier.

We close this short note with a small remark on lc pairs.

Remark 2.2. The usual Kawamata–Viehweg vanishing theorem for klt pairs (see Theorem 1.2) can be generalized to the Reid–Fukuda vanishing theorem for lc pairs (see [F, Theorem 5.7.6]). However, by [KM, Corollary 5.72], the formulation of Theorem 1.1 can not be generalized for lc pairs since lc pairs are not necessarily Cohen–Macaulay. This is one of the reasons why we did not treat Theorem 1.1 in [F].

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