A LETTER OF A LOG MINIMAL MODELER –WHAT IS LOG TERMINAL ?–

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Abstract. The main purpose of this letter is to attract log minimal modelers' attention to the subtleties of various kinds of *log terminal*.

1. What is log terminal?

In seminars, I sometimes have a question: What is log terminal? So I decided to write this letter to log minimal modelers.

The main purpose of this letter is to attract log minimal modelers' attention to the subtleties of various kinds of log terminal singularities. Needless to say, my opinion is not necessarily the best. I hope that this letter will help you to understand *log terminal*. We note that this letter is not self-contained and contains no proofs.

In this world, there are too many variants of *log terminal*. We have troubles whenever we treat log terminal singularities. We, log minimal modelers, have three bibles: [KMM], [FA], and [KM] [1]. It is unpleasant for us that each bible adopted different definitions of log terminal and *log resolutions*.

Historically, Shokurov introduced various kinds of log terminal singularities in his famous paper [Sh, §1]. However, we don't mention [Sh] anymore for simplicity. We only treat the above three bibles.

2. Resolution Lemma

I think that one of the most useful log terminal singularities is *divisorial log terminal* (dlt, for short), which was introduced by Shokurov (see [FA, (2.13.3)]). We will define dlt in Definition 3.1. By Szabó's work [Sz], the notion of dlt coincides with that of *weakly Kawamata log terminal* (wklt, for short). For the definition of wklt, see [FA, (2.13.4)].

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[1] We are preparing the 4th bible now. I hope that it will overthrow this chaotic situation about log terminal.

This fact is non-trivial and based on the deep results about desingularization theorem. For the details, see the original fundamental paper [Sz]. The key result is Szabó's resolution lemma [Sz, Resolution Lemma]. The following is a weak version of Resolution Lemma, but it contains the essential part of Szabó's result. For the precise statement, see [Sz, Resolution Lemma]. By combining Theorem 2.1 with usual desingularization arguments, we can recover the original Resolution Lemma without any difficulty. The details are left to you as an easy exercise. See also Example 4.1.

Theorem 2.1. *Let X be a smooth variety over a field of characteristic zero and D a reduced divisor. Then there exists a proper birational morphism $f : Y \to X$ with the following properties:*

(1) *f is a composition of blowing ups of smooth subvarieties,*
(2) *Y is smooth,*
(3) *$f_*^{-1}D \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor,*
(4) *f is an isomorphism over U, where U is the largest open set of X such that the restriction $D|_U$ is a simple normal crossing divisor on U.*

In my previous paper [F3], I quoted [BM, Theorem 12.4]. However, I think that it is not appropriate. For desingularization theorems, we recommend you to see [Ma], which explains the proof of the desingularization theorem by Encinas and Villamayor. The following is the special case of [Ma, Remark 6-2 (iii)].

Theorem 2.2. *Let F be a reduced and irreducible divisor on a smooth variety X, which is defined over a field of characteristic zero, and E a simple normal crossing divisor on X. Assume that E does not contain F. Then there exists a proper birational morphism $f : Y \to X$ with the following properties:*

(1) *f is a composition of blowing ups of smooth subvarieties,*
(2) *Y is smooth,*
(3) *$f_*^{-1}F \cup f_*^{-1}E \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor,*
(4) *f is an isomorphism over U, where U is the largest open set of X such that the restriction $(F \cup E)|_U$ is a simple normal crossing divisor on U.*

By applying Theorem 2.2 to each irreducible component of D, we can prove Theorem 2.1 without any difficulty. The details are left to you as an easy exercise. If you can not prove it, see the proof of [Sz, Resolution Lemma], which uses Hironaka's paper.

3. What is a log resolution?

We often use the words *good resolution* or *log resolution* without defining them precisely. However, it caused some serious problems. We will define our log resolution later (see Definition 3.2). Let us recall the definition of dlt.

Definition 3.1 (Divisorial log terminal)**.** Let X be a normal variety and D a boundary on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. There exists a *log resolution* $f : Y \to X$ such that $a(E, X, D) > -1$ for every f-exceptional divisor E.

There are three questions about the above definition.

(1) f is projective?
(2) the exceptional locus $\mathrm{Exc}(f)$ is of pure codimension one?
(3) $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a *simple normal crossing* divisor or only a *normal crossing* divisor?

In [KM, Notaion 0.4 (10)], they assume that $\mathrm{Exc}(f)$ is of pure codimension one and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a simple normal crossing divisor. We note that, in [FA, 2.9 Definition], $\mathrm{Exc}(f)$ is not necessarily of pure codimenison one. So, the definition of lt in [FA, (2.13.1)] is the same as Definition 3.1, but lt in the sense of [FA] is different from dlt. See Example 4.4 below. The difference exists in the definition of log resolutions!

Our definition of a log resolution is the following, which is [KM, Notaion 0.4 (10)]. By Hironaka, log resolutions exist for varieties over a field of characteristic zero.

Definition 3.2 (Log resolution)**.** Let X be a variety and D a $\mathbb{Q}$-divisor on X. A *log resolution* of (X, D) is a *proper* birational morphism $f : Y \to X$ such that Y is smooth, $\mathrm{Exc}(f)$ is a divisor and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}D)$ is a *simple* normal crossing divisor.

We note that, by Szabó's resolution lemma, we don't need the projectivity of f in the definition of dlt. In [KMM], they always use normal crossing divisors instead of simple normal crossing divisors. So there is a difference between wklt and weak log-terminal. We note that wklt is *weak log-terminal* in the sense of [KMM, Definition 0-2-10 (2)], but weak log-terminal is not necessarily wklt. See Section 4, especially, Example 4.2. In my experience, dlt, which is equivalent to wklt, is easy to treat and useful for inductive arguments, but weak log-terminal is very difficult to use. I think that [KM, Corollary 5.50] makes dlt useful. For the usefulness of dlt, see [F1], [F2], [F3], and [F4]. See also Example 4.2 and Remark 4.3. We summarize;

Conclusion 3.3. *The notion of dlt coincides with that of wklt by* [Sz]. *In particular, dlt is weak log-terminal in the sense of* [KMM]. *Therefore, we can freely apply the results that were proved for weak log-terminal pairs in* [KMM] *to dlt pairs.*

For other characterizations of dlt, see [Sz, Divisorical Log Terminal Theorem]. The definition [KM, Definition 2.37] looks different from our definition. However, it is the same by [Sz, Divisorical Log Terminal Theorem]. See also [KM, Theorem 2.44].

By the way, I have a question about lt in the sense of [FA].

Question 3.4. Does [FA, (2.16.2)] hold without the projectivity assumption? That is, the notion of $\mathbb{Q}$-factorial lt in the sense of [FA] is equivalent to that of $\mathbb{Q}$-factorial dlt?

One solution of the above question is that we don't use lt in the sense of [FA]. In my experience, this is one of the best solutions.

4. Examples

In this section, we collect some examples. The following example says that Szabó's resolution lemma is not true if we replace simple normal crossing with normal crossing.

Example 4.1. Let $X := \mathbb{C}^3$ and D a Whitney umbrella, that is, $D = (x^2 - zy^2 = 0)$. We can not make D normal crossing by blowing up only over the locus where D is not normal crossing divisor.

It is obvious that D is normal crossing outside the origin. We recommend you to blow up the origin of Example 4.1. The following says that weak log-terminal is not necessarily wklt.

Example 4.2 (SNC vs NC)**.** Let X be a smooth surface and D a nodal curve on X. Then the pair (X, D) is not wklt but weak log-terminal.

The following fact is crucial for inductive arguments.

Remark 4.3. Let (X, D) be a dlt (resp. weak log-terminal) pair and S an irreducible component of $\lfloor D \rfloor$. Then $(S, \mathrm{Diff}(D-S))$ is dlt (resp. not necessarily weak log-terminal).

In Example 4.2, $S := \lfloor D \rfloor$ is not normal. This makes weak log-terminal difficult to use for inductive arguments. The next example explains that we have to assume that $\mathrm{Exc}(f)$ is a divisor in Definition 3.1.

Example 4.4 (Small resolution). Let $X := (xy - uv = 0) \subset \mathbb{C}^4$. It is well-known that this X is a toric variety. We take the torus invariant divisor D; the complement of the big torus. Then (X, D) is not dlt but lt in the sense of [FA]. Especially, it is log-canonical. We note that there exists a small resolution.

The following is a variant of the above example.

Example 4.5 ([FA, 17.5.2 Example]). Let $X := (xy - uv = 0) \subset \mathbb{C}^4$ and

$$D = (x = u = 0) + (y = v = 0) + \frac{1}{2}\sum_{i=1}^{4}(x + 2^i u = y + 2^{-i} v = 0).$$

Then (X, D) is lt in the sense of [FA]. In particular (X, D) is log-canonical. The divisor $\lfloor D \rfloor$ is two planes intersecting at a single point. Thus it is not S_2.

Example 4.5 says that [FA, (16.9.1)] is not true. The problem is that S does not necessarily satisfy Serre's condition S_2.

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