MINIMAL MODEL THEORY FOR LOG SURFACES

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1. Minimal model theory for log surfaces

In this section, we quickly recall the main result of [F16], where the minimal model theory for log surfaces is discussed in full generality.

thm-ls1 **Theorem 1.1** (Minimal model theory for log surfaces)**.** *Let X be a normal surface and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $f : X \to S$ be a projective morphism onto an algebraic variety S. Assume that one of the following conditions holds*:

(A) *X is $\mathbb{Q}$-factorial, or*
(B) *(X, Δ) is log canonical.*

Then we can run the log minimal model program over S with respect to $K_X + \Delta$ and obtain a sequence of extremal contractions

$$(X, \Delta) = (X_0, \Delta_0) \overset{\varphi_0}{\to} (X_1, \Delta_1) \overset{\varphi_1}{\to} \cdots \overset{\varphi_{k-1}}{\to} (X_k, \Delta_k) = (X^*, \Delta^*)$$

over S such that

(1) (Minimal model) *$K_{X^*} + \Delta^*$ is semi-ample over S if $K_X + \Delta$ is f-pseudo-effective, and*
(2) (Mori fiber space) *there is a morphism $g : X^* \to C$ over S such that $-(K_{X^*}+\Delta^*)$ is g-ample, $\dim C < 2$, and the relative Picard number $\rho(X^*/C) = 1$, if $K_X + \Delta$ is not f-pseudo-effective.*

Note that (X, Δ) is not necessarily log canonical in the case (A).

A key point of Theorem 1.1 is as follows. Let X be a normal surface and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the non-lc locus $\mathrm{Nlc}(X, \Delta)$ is empty or consists of points.

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I will add this note to my book.

Therefore, $\mathrm{Nlc}(X, \Delta)$ contains no curves. Anyway, we recommend the reader to see [F16] for the details.

Remark 1.2. In Theorem 1.1, if X is $\mathbb{Q}$-factorial, then X_i is always $\mathbb{Q}$-factorial for every i. If (X, Δ) is log canonical, then (X_i, Δ_i) is log canonical for every i.

Remark 1.3 (Surfaces with only rational singularities)**.** If X is an algebraic surface with only rational singularities, then it is well known that X is $\mathbb{Q}$-factorial. Therefore, we can apply Theorem 1.1 for surfaces with only rational singularities. Moreover, we can prove that X_i has only rational singularities for every i if X has only rational singularities. Thus, the minimal model program for log surfaces is closed for surfaces with only rational singularities.

As a corollary of Theorem 1.1, we obtain the following theorem.

Corollary 1.4 (Finite generation of log canonical rings for log surfaces)**.** *Let X be a normal surface and let Δ be a boundary $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to S$ be a projective morphism onto an algebraic variety S. Assume that (X, Δ) is log canonical or X is $\mathbb{Q}$-factorial. Then the relative log canonical ring*

$$R(X/S, \Delta) = \bigoplus_{m \geq 0} f_*\mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner)$$

is a finitely generated $\mathcal{O}_S$-algebra.

Theorem 1.1 contains the following result, which is a generalization of Fujita's abundance theorem for log surfaces in [Fujita].

thm-ls5 **Theorem 1.5** (Abundance theorem for log surfaces)**.** *Let X be a normal surface and let Δ be a boundary $\mathbb{R}$-divisor on X such that (X, Δ) is $\mathbb{R}$-Cartier. Let $f : X \to S$ be a projective surjective morphism onto an algebraic variety S. Assume that $K_X + \Delta$ is f-nef. Then $K_X + \Delta$ is f-semi-ample.*

The proof of Theorem 1.5 is the main part of [F16]. We can also prove the following theorem.

thm-ls6 **Theorem 1.6.** *Let X be a projective surface with only rational singularities. Then*

$$\mathrm{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner))$$

has only rational singularities.

By Theorem 1.6, we know that the notion of rational singularities is suited for the minimal model theory for surfaces.

References

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