ON MIXED-ω-SHEAVES

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ABSTRACT. We introduce the notion of mixed-ω-sheaves and use it for the study of a relative version of Fujita's freeness conjecture. It is related to the Iitaka conjecture. We note that the notion of mixed-ω-sheaves is a generalization of that of Nakayama's ω-sheaves in some sense. One of the main motivations of this paper is to make Nakayama's theory of ω-sheaves more accessible and make it applicable to the study of log canonical pairs.

CONTENTS

1. INTRODUCTION

Let us recall Fujita's famous freeness conjecture on adjoint bundles. Note that everything is defined over $\mathbb{C}$, the complex number field, in this paper.

Conjecture 1.1 (Takao Fujita). *Let X be a smooth projective variety with $\dim X = n$ and let $\mathcal{L}$ be any ample invertible sheaf on X. Then $\omega_X \otimes \mathcal{L}^{\otimes l}$ is generated by global sections for every $l \geq n + 1$.*

Although there have already been many related results, Conjecture 1.1 is still open. As a generalization of Conjecture 1.1, Popa and Schnell proposed the following conjecture, which is a relative version of Fujita's freeness conjecture.

Conjecture 1.2 (Popa and Schnell, see [PS, Conjecture 1.3]). *Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties with $\dim Y = n$. Let $\mathcal{L}$ be any ample invertible sheaf on Y. Then, for every positive integer k, the sheaf*

$$f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}$$

is generated by global sections for $l \geq k(n+1)$.

We can find some interesting results on Conjecture 1.2 in [De], [Du], [DuM], and [I]. In this paper, we do not directly treat Conjecture 1.2. When the base space Y is a curve in Conjecture 1.2, we have the following stronger result. We note that we are mainly interested in the case where $k \geq 2$ for some geometric applications.

Theorem 1.3. *Let $f\colon X \to C$ be a surjective morphism from a smooth projective variety X onto a smooth projective curve C. Let $\mathcal{H}$ be an ample invertible sheaf on C with $\deg \mathcal{H} \geq 2$ and let k be any positive integer. Then the sheaf*

$$f_*\omega_{X/C}^{\otimes k} \otimes \omega_C \otimes \mathcal{H}$$

is generated by global sections. Let $\mathcal{L}$ be an ample invertible sheaf on C. Then

$$f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l}$$

is generated by global sections for $l \geq 2k$. In particular, Conjecture 1.2 holds true when the base space is a smooth projective curve.

Here, we give a detailed proof of Theorem 1.3 in order to explain our idea.

Proof of Theorem 1.3. If $f_*\omega_{X/C}^{\otimes k} = 0$, then there is nothing to prove. So we assume that $f_*\omega_{X/C}^{\otimes k} \neq 0$. We take any closed point P.

Claim. $H^1(C, f_*\omega_{X/C}^{\otimes k} \otimes \omega_C \otimes \mathcal{H} \otimes \mathcal{O}_C(-P)) = 0$.

Proof of Claim. By [Ka, Theorem 1], $f_*\omega_{X/C}^{\otimes k}$ is a nef locally free sheaf. This fact also follows from Viehweg's weak positivity theorem since C is a smooth projective curve. Therefore, $\mathcal{E} := f_*\omega_{X/C}^{\otimes k} \otimes \mathcal{H} \otimes \mathcal{O}_C(-P)$ is ample. If $H^1(C, \mathcal{E} \otimes \omega_C) \neq 0$, then we get $H^0(C, \mathcal{E}^*) \neq 0$ by Serre duality. This implies that there is a nontrivial inclusion $0 \to \mathcal{O}_C \to \mathcal{E}^*$. By taking the dual of this inclusion, we have the following surjection $\mathcal{E} \to \mathcal{O}_C \to 0$. This is a contradiction since $\mathcal{E}$ is ample. Hence we have $H^1(C, \mathcal{E} \otimes \omega_C) = 0$. $\qquad\square$

By the Claim, the natural restriction map

$$H^0(C, f_*\omega_{X/C}^{\otimes k} \otimes \omega_C \otimes \mathcal{H}) \to f_*\omega_{X/C}^{\otimes k} \otimes \omega_C \otimes \mathcal{H} \otimes \mathbb{C}(P)$$

is surjective. This means that $f_*\omega_{X/C}^{\otimes k} \otimes \omega_C \otimes \mathcal{H}$ is generated by global sections. Since C is a smooth projective curve, $\omega_C \otimes \mathcal{L}^{\otimes 2}$ is generated by global sections. This implies that

$$f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l} \simeq f_*\omega_{X/C}^{\otimes k} \otimes \omega_C \otimes \mathcal{L}^{\otimes(l-2(k-1))} \otimes \left(\omega_C \otimes \mathcal{L}^{\otimes 2}\right)^{\otimes(k-1)}$$

is generated by global sections because $\deg \mathcal{L}^{\otimes(l-2(k-1))} \geq 2$ by assumption. $\qquad\square$

A key point of Theorem 1.3 is the fact that $f_*\omega_{X/C}^{\otimes k}$ is a nef locally free sheaf on C for every positive integer k. When $f\colon X \to Y$ is a weakly semistable morphism in the sense of Abramovich–Karu (see [AK]), it is conjectured that $f_*\omega_{X/Y}^{\otimes k}$ is a nef locally free sheaf on Y for every positive integer k (see [Fn4] and [Fn8, Conjecture 3.14]). Hence it is natural to consider the case where $f\colon X \to Y$ is weakly semistable.

Theorem 1.4 (see Theorem 8.2). *Let $f\colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y with connected fibers. Assume that f is weakly semistable in the sense of Abramovich–Karu and that the geometric generic fiber $X_{\bar{\eta}}$ of $f\colon X \to Y$ has a good minimal model. Let H be an ample Cartier divisor on Y, let k be a positive integer with $k \geq 2$, and let A be an ample Cartier divisor on Y such that $|A|$ is free. Then*

$$\left(\bigotimes^{s} f_* \omega_{X/Y}^{\otimes k} \right) \otimes \omega_Y \otimes \mathcal{O}_Y(H + nA)$$

is generated by global sections for all integers $s \geq 1$, where $n = \dim Y$.

By Theorems 1.3 and 1.4, we propose a new conjecture similar to Conjecture 1.2.

Conjecture 1.5. *Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties with $\dim Y = n$. Let $\mathcal{L}$ be any ample invertible sheaf on Y. Then, for every positive integer k, the sheaf*

$$f_* \omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{L}^{\otimes l}$$

is generically generated by global sections for $l \geq n + 1$. More precisely,

$$f_* \omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{L}^{\otimes l}$$

is generated by global section on U for $l \geq n + 1$, where U is the largest Zariski open set of Y such that f is smooth over U.

Even when f is the identity in Conjecture 1.5, the generically generation, which is a special case of Fujita's conjecture (see Conjecture 1.1), looks like a difficult open problem. We note that an example (see Example 10.1) shows that $f_* \omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{L}^{\otimes l}$ is not always generated by global sections in Conjecture 1.5. We only expect that it is generically generated by global sections. The best known result on Conjecture 1.5 is the following theorem. The author learned it from Masataka Iwai.

Theorem 1.6 (Masataka Iwai, see Theorem 12.1). *Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers and let $\mathcal{L}$ be an ample invertible sheaf on Y. Let U be the largest Zariski open set of Y such that f is smooth over U. We put $\dim Y = n$. Then*

$$f_* \omega_{X/Y}^{\otimes a} \otimes \omega_Y \otimes \mathcal{L}^{\otimes b}$$

is generated by global sections on U for all integers $a \geq 1$ and $b \geq \frac{n(n+1)}{2} + 1$.

The proof of Theorem 1.6 is essentially analytic. We will give a sketch of the proof of Theorem 1.6 in Section 12. On the other hand, by Nakayama's theory of ω-sheaves (see [N, Chapter V]), we can prove:

Theorem 1.7. *Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties and let H be an ample divisor on Y such that $|H|$ is free. We put $\dim Y = n$. Then*

$$\left(\bigotimes^{s} f_* \omega_{X/Y}^{\otimes k} \right)^{**} \otimes \omega_Y \otimes \mathcal{O}_Y(lH)$$

is generically generated by global sections for all integers $k \geq 1$, $s \geq 1$, and $l \geq n + 1$.

Let $H^\dagger$ be an ample divisor on Y such that $|H^\dagger|$ is not necessarily free. Then the sheaf

$$\left(\bigotimes^s f_* \omega_{X/Y}^{\otimes k}\right)^{**} \otimes \omega_Y \otimes \mathcal{O}_Y(lH^\dagger)$$

is generically generated by global sections for all integers $k \geq 1$, $s \geq 1$, and $l \geq n^2 + \min\{2, k\}$.

In this paper, we will introduce the notion of mixed-ω-sheaves, which is a generalization of that of Nakayama's ω-sheaves, and establish Theorems 1.8 and 1.9. Then we will obtain Theorem 1.7 as a special case of Theorems 1.8 and 1.9.

Theorem 1.8. *Let $f\colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y with $\dim Y = n$. Let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and that (X, Δ) is log canonical over a nonempty Zariski open set of Y. Let L be a Cartier divisor on X with $L \sim_{\mathbb{R}} k(K_{X/Y} + \Delta)$ for some positive integer k. Let H be a big Cartier divisor on Y such that $|H|$ is free. Then*

$$\left(\bigotimes^s f_* \mathcal{O}_X(L)\right)^{**} \otimes \mathcal{O}_Y(K_Y + lH)$$

is generically generated by global sections for all integers $s \geq 1$ and $l \geq n + 1$.

Theorem 1.9. *In Theorem 1.8, we assume that $H^\dagger$ is a nef and big Cartier divisor on Y such that $|H^\dagger|$ is not necessarily free. Then we have that*

$$\left(\bigotimes^s f_* \mathcal{O}_X(L)\right)^{**} \otimes \mathcal{O}_Y(K_Y + lH^\dagger)$$

is generically generated by global sections for all integers $s \geq 1$ and $l \geq n^2 + \min\{2, k\}$.

We note that Iwai's analytic method can not be applied to log canonical pairs because it depends on L^2 method. We also note that Nakayama's theory of ω-sheaves can not be directly applied to the study of log canonical pairs.

Let us quickly explain the idea of the proof of Theorem 1.7, which is mainly due to Nakayama (see [N, Chapter V]). Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties and let H be an ample Cartier divisor on Y such that $|H|$ is free. We fix a positive integer $k \geq 2$. Then we can construct a surjective morphism $g\colon Z \to Y$ from a smooth projective variety Z and a direct summand $\mathcal{F}$ of $g_* \mathcal{O}_Z(K_Z)$ such that there exists a generically isomorphic injection

$$\mathcal{F} \hookrightarrow \left(f_* \omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(H)\right)^{**}.$$

By Kollár's vanishing theorem, we have

$$H^i(Y, \mathcal{F} \otimes \mathcal{O}_Y((n + 1 - i)H)) = 0$$

for every $i > 0$, where $n = \dim Y$. Therefore, by Castelnuovo–Mumford regularity, $\mathcal{F} \otimes \mathcal{O}_Y((n+1)H)$ is generated by global sections. This implies that

$$\left(f_* \omega_{X/Y}^{\otimes k}\right)^{**} \otimes \omega_Y \otimes \mathcal{O}_Y((n + 2)H)$$

is generically generated by global sections. Note that we do not try to establish any vanishing theorems for $f_* \omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(H)$ directly. By the above observation, it is natural to consider:

Definition 1.10 (Mixed-ω-sheaf and pure-ω-sheaf, see Definition 5.1)**.** A torsion-free coherent sheaf $\mathcal{F}$ on a normal quasi-projective variety W is called a *mixed-ω-sheaf* if there exist a projective surjective morphism $f\colon V \to W$ from a smooth quasi-projective variety V and a simple normal crossing divisor D on V such that $\mathcal{F}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D)$. When $D = 0$, $\mathcal{F}$ is called a *pure-ω-sheaf* on W.

For the study of klt pairs, the notion of pure-ω-sheaves is sufficient and is essentially due to Nakayama (see [N, Chapter V]). In this paper, we study some basic properties of mixed-ω-sheaves. They are indispensable for the study of log canonical pairs. Of course, the theory of mixed-ω-sheaves (resp. pure-ω-sheaves) in this paper is based on that of mixed (resp. pure) Hodge structures. Roughly speaking, Nakayama only treats pure-ω-sheaves in [N, Chapter V]. However, his theory of ω-sheaves is more sophisticated and some of his results are much sharper than ours. We do not try to make the framework discussed in this paper supersede Nakayama's theory of ω-sheaves in [N, Chapter V]. One of the main purposes of this paper is to make Nakayama's theory of ω-sheaves more accessible and make it applicable to the study of log canonical pairs. Theorem 9.3 (and Remark 9.4) is one of the main results of this paper, which we call a fundamental theorem of the theory of mixed-ω-sheaves.

Theorem 1.11 (see [N, Chapter V, 3.35. Theorem], Theorem 9.3, and Remark 9.4)**.** *Let $f\colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y. Let L be a Cartier divisor on X and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let D be an $\mathbb{R}$-divisor on Y. Let k be a positive integer with $k \geq 2$. Assume the following conditions:*

(i) *(X, Δ) is log canonical (resp. klt) over a nonempty Zariski open set of Y, and*
(ii) *$L + f^*D - k(K_{X/Y} + \Delta) - f^*A$ is semi-ample for some big $\mathbb{R}$-divisor A on Y.*

If $f_*\mathcal{O}_Y(L) \neq 0$, then there exist a mixed-ω-big-sheaf (resp. pure-ω-big-sheaf) $\mathcal{F}$ on Y and a generically isomorphic injection

$$\mathcal{F} \hookrightarrow \mathcal{O}_Y(K_Y + \lceil D \rceil) \otimes (f_*\mathcal{O}_X(L))^{**}.$$

For the precise definition of mixed-ω-big-sheaves and pure-ω-big-sheaves, see Definition 5.3 below.

As an application of Theorem 1.11, we give a detailed proof of:

Theorem 1.12 ([N, Chapter V, 4.1. Theorem (1)], [Fn11, Section 3], and Theorem 11.3)**.** *Let $f\colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y with connected fibers. Let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and that (X, Δ) is log canonical over a nonempty Zariski open set of Y. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D - (K_{X/Y} + \Delta)$ is nef. Then, for any $\mathbb{R}$-divisor Q on Y, we have*

$$\kappa_\sigma(X, D + f^*Q) \geq \kappa_\sigma(F, D|_F) + \kappa(Y, Q)$$

and

$$\kappa_\sigma(X, D + f^*Q) \geq \kappa(F, D|_F) + \kappa_\sigma(Y, Q)$$

where F is a sufficiently general fiber of $f\colon X \to Y$.

We note that $\kappa_\sigma(X, D)$ and $\kappa(X, D)$ denote Nakayama's numerical dimension and the Iitaka dimension of D, respectively. Theorem 1.12 already played a crucial role in the theory of minimal models. We need Theorem 1.12 for the proof of the following famous and fundamental result on the existence theorem of good minimal models for klt pairs.

Theorem 1.13 ([DHP, Remark 2.6], [GL, Theorem 4.3], and [Fn11, Theorem 3.2]). *Let (X, Δ) be a projective klt pair such that Δ is a $\mathbb{Q}$-divisor. Then (X, Δ) has a good minimal model if and only if $\kappa_\sigma(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$.*

We explain the organization of this paper. In Section 2, we collect some basic definitions. In Section 3, we prepare some useful and important lemmas. They will play a crucial role in this paper. In Section 4, we quickly explain some basic properties of Viehweg's weakly positive sheaves and big sheaves. In Section 5, we introduce mixed-ω-sheaves and mixed-ω-big-sheaves. In Sections 6 and 7, we prove some basic properties of mixed-ω-sheaves based on the theory of mixed Hodge structures. In Section 8, we treat a very special but interesting case. More precisely, we treat weakly semistable morphisms $f \colon X \to Y$ in the sense of Abramovich–Karu with the assumption that the geometric generic fiber $X_{\overline{\eta}}$ of $f \colon X \to Y$ has a good minimal model. In this case, we can prove some strong results with the aid of the theory of minimal models. Section 9 is the main part of this paper. We prove Theorem 1.11. Section 10 is devoted to the proof of Theorems 1.7, 1.8, and 1.9. In Section 11, we prove Theorem 1.12, which has already played a crucial role in the theory of minimal models, and a slight generalization of the twisted weak positivity theorem, which is an extension of results of various people. In the final section: Section 12, we give a sketch of the proof of Theorem 1.6. The proof is essentially analytic and is independent of the other sections.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. We note that a *scheme* is a separated scheme of finite type over $\mathbb{C}$ and a *variety* is an integral scheme.

2. Preliminary

In this section, we collect some basic definitions. For the details, see [Fn2], [Fn6], and [Fn10].

Let us start with the definition of canonical sheaves and canonical divisors.

Definition 2.1 (Canonical sheaf and canonical divisor). Let X be an equidimensional scheme of dimension n and let $a \colon X \to \mathrm{Spec}\,\mathbb{C}$ be the structure morphism. Then the dualizing complex of X is $\omega_X^\bullet = a^! \mathbb{C}$, where $a^!$ is the functor obtained in [H, Chapter VII. Corollary 3.4 (a)] (see also [C, Section 3.3]). Then we put

$$\omega_X := h^{-n}(\omega_X^\bullet)$$

and call it the *canonical sheaf* of X.

We further assume that X is normal. Then a *canonical divisor* K_X of X is a Weil divisor on X such that

$$\mathcal{O}_{X_{\mathrm{sm}}}(K_X) \simeq \Omega^n_{X_{\mathrm{sm}}}$$

holds, where X_{sm} is the largest smooth Zariski open set of X.

It is well known that

$$\mathcal{O}_X(K_X) \simeq \omega_X$$

holds when X is normal.

If $f\colon X \to Y$ is a morphism between Gorenstein schemes, then we put

$$\omega_{X/Y} := \omega_X \otimes f^*\omega_Y^{\otimes -1}.$$

If $f\colon X \to Y$ is a morphism from a normal scheme X to a normal Gorenstein scheme Y, then we put

$$K_{X/Y} := K_X - f^*K_Y.$$

Let us quickly look at the definition of singularities of pairs.

Definition 2.2 (Singularities of pairs). Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f\colon Y \to X$ be a resolution of singularities of X such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$ has simple normal crossing support, where $\mathrm{Exc}(f)$ is the exceptional locus of f on Y and $f_*^{-1}\Delta$ is the strict transform of Δ on Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

with $f_*\left(\sum_i a_i E_i\right) = -\Delta$. We say that (X, Δ) is *log canonical* (resp. *klt*) if $a_i \geq -1$ (resp. $a_i > -1$) for every i. If $\Delta = 0$ and $a_i \geq 0$ holds for every i, then we say that X has only *canonical singularities*.

If (X, Δ) is log canonical and there exist a resolution of singularities $f\colon Y \to X$ as above and a prime divisor E_i on Y with $a_i = -1$, then $f(E_i)$ is called a *log canonical center* of (X, Δ).

Definition 2.3 (Dlt pairs). Let (X, Δ) be a log canonical pair. If there exists a resolution of singularities $f\colon Y \to X$ such that the exceptional locus $\mathrm{Exc}(f)$ of f is a divisor on Y, $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$ has simple normal crossing support, and

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

with $a_i > -1$ for every f-exceptional divisor E_i, then the pair (X, Δ) is called a *dlt* pair.

The following definitions are very useful in this paper.

Definition 2.4 (Horizontal and vertical divisors). Let $f\colon X \to Y$ be a dominant morphism between normal varieties and let D be an $\mathbb{R}$-divisor on X. We can write

$$D = D_{\mathrm{hor}} + D_{\mathrm{ver}}$$

such that every irreducible component of D_{hor} (resp. D_{ver}) is mapped (resp. not mapped) onto Y. If $D = D_{\mathrm{hor}}$ (resp. $D = D_{\mathrm{ver}}$), then D is said to be *horizontal* (resp. *vertical*).

Definition 2.5 (Operations for $\mathbb{R}$-divisors). Let $D = \sum_i d_i D_i$ be an $\mathbb{R}$-divisor on a normal variety X, where D_i is a prime divisor on X for every i, $D_i \neq D_j$ for $i \neq j$, and $d_i \in \mathbb{R}$ for every i. Then we put

$$\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i, \quad \{D\} = D - \lfloor D \rfloor, \quad \text{and} \quad \lceil D \rceil = -\lfloor -D \rfloor.$$

Note that $\lfloor d_i \rfloor$ is the integer which satisfies $d_i - 1 < \lfloor d_i \rfloor \leq d_i$. We also note that $\lfloor D \rfloor$, $\lceil D \rceil$, and $\{D\}$ are called the *round-down*, *round-up*, and *fractional part* of D respectively.

If $0 \leq d_i \leq 1$ for every i, then we say that D is a *boundary* $\mathbb{R}$-divisor on X. We note that $\sim_{\mathbb{Q}}$ (resp. $\sim_{\mathbb{R}}$) denotes the $\mathbb{Q}$-*linear* (resp. $\mathbb{R}$-*linear*) *equivalence* of $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-divisors).

In this paper, we will repeatedly use the following notation:

$$D^{=1} = \sum_{d_i=1} D_i, \quad D^{>1} = \sum_{d_i>1} d_i D_i, \quad \text{and} \quad D^{<0} = \sum_{d_i<0} d_i D_i.$$

We recall the following definition for the reader's convenience.

Definition 2.6 (Generic generation). Let $\mathcal{F}$ be a coherent sheaf on a variety X. We say that $\mathcal{F}$ is *generated by global sections on U*, where U is a Zariski open set of X, if the natural map

$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

is surjective on U. We simply say that $\mathcal{F}$ is *generated by global sections* when $U = X$. We say that $\mathcal{F}$ is *generically generated by global sections* if $\mathcal{F}$ is generated by global sections on some nonempty Zariski open set of X.

We close this section with the definition of exceptional divisors for proper surjective morphisms between normal varieties.

Definition 2.7 (Exceptional divisors). Let $f\colon X \to Y$ be a proper surjective morphism between normal varieties. Let E be a Weil divisor on X. We say that E is *f-exceptional* if $\operatorname{codim}_Y f(\operatorname{Supp} E) \geq 2$. We note that f is not always assumed to be birational.

3. Preliminary lemmas

In this section, we collect some useful and important lemmas for the reader's convenience. They are more or less well known to the experts.

Let us start with the following easy lemmas on $\mathbb{R}$-divisors. We will use them repeatedly in this paper.

Lemma 3.1. *Let A be a Cartier divisor on a normal variety V. Let B be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on V such that $B = \sum_{i \in I} b_i B_i$ where $b_i \in \mathbb{R}$ and B_i is a prime divisor on V for every i with $B_i \neq B_j$ for $i \neq j$. Assume that $A \sim_{\mathbb{R}} B$. Then we can take a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $C = \sum_{i \in I} c_i B_i$ on V such that*

  (i) *$A \sim_{\mathbb{Q}} C$,*
  (ii) *$c_i = b_i$ holds if $b_i \in \mathbb{Q}$, and*
  (iii) *$|c_i - b_i| \ll 1$ holds for $b_i \in \mathbb{R} \setminus \mathbb{Q}$.*

In particular, $\operatorname{Supp} C = \operatorname{Supp} B$, $\lfloor C \rfloor = \lfloor B \rfloor$, $\lceil C \rceil = \lceil B \rceil$, and $\operatorname{Supp}\{C\} = \operatorname{Supp}\{B\}$.

Proof. It is an easy exercise. For the details, see, for example, the proof of [Fn2, Lemma 4.15]. □

Lemma 3.2. *Let $D = \sum_{i \in I} a_i D_i$ be an $\mathbb{R}$-divisor on a smooth projective variety V, where $a_i \in \mathbb{R}$ and D_i is a prime divisor on V for every i with $D_i \neq D_j$ for $i \neq j$. Assume that D is semi-ample. Then we can construct a $\mathbb{Q}$-divisor $D^{\dagger} = \sum_{i \in I} b_i D_i$ such that*

  (i) *$D^{\dagger}$ is semi-ample,*
  (ii) *$b_i = a_i$ holds if $a_i \in \mathbb{Q}$, and*
  (iii) *$|b_i - a_i| \ll 1$ holds for $a_i \in \mathbb{R} \setminus \mathbb{Q}$.*

Proof. Since D is semi-ample, we can write $D = \sum_{j \in J} m_j M_j$ where m_j is a positive real number and M_j is a semi-ample Cartier divisor on V for every j. As usual, by perturbing m_js suitably, we get a desired semi-ample $\mathbb{Q}$-divisor $D^{\dagger}$ on V. For the details, see, for example, the proof of [Fn2, Lemma 4.15]. □

Next, we treat a very useful covering trick, which is essentially due to Yujiro Kawamata. We will use it in the proof of Theorem 9.3.

Lemma 3.3. *Let $f\colon V \to W$ be a projective surjective morphism between smooth quasi-projective varieties and let H be a Cartier divisor on W. Let d be an arbitrary positive integer. Then we can take a finite flat morphism $\tau\colon W' \to W$ from a smooth quasi-projective variety W' and a Cartier divisor H' on W' such that $\tau^*H \sim dH'$ and that $V' = V \times_W W'$ is a smooth quasi-projective variety with $\omega_{V'/W'} = \rho^*\omega_{V/W}$, where $\rho\colon V' \to V$. By construction, we may assume that $\tau\colon W' \to W$ is Galois.*

Proof. We take general very ample Cartier divisors D_1 and D_2 with the following properties.

   (i) $H \sim D_1 - D_2$,
   (ii) D_1, D_2, f^*D_1, and f^*D_2 are smooth,
   (iii) D_1 and D_2 have no common components, and
   (iv) $\mathrm{Supp}(D_1 + D_2)$ and $\mathrm{Supp}(f^*D_1 + f^*D_2)$ are simple normal crossing divisors.

We take a finite flat cover due to Kawamata with respect to W and $D_1 + D_2$ (see [EV, 3.19. Lemma] and [V3, Lemma 2.5]). Then we obtain $\tau\colon W' \to W$ and H' such that $\tau^*H \sim dH'$. By the construction of the above Kawamata cover $\tau\colon W' \to W$, we may assume that the ramification locus Σ of τ in W is a general simple normal crossing divisor. This means that f^*P is a smooth divisor for any irreducible component P of Σ and that $f^*\Sigma$ is a simple normal crossing divisor on V. In this situation, we can easily check that $V' = V \times_W W'$ is a smooth quasi-projective variety.

$$
\begin{array}{ccc}
V' & \xrightarrow{\ \rho\ } & V \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
W' & \xrightarrow[\ \tau\ ]{} & W
\end{array}
$$

By construction, we can also easily check that $\omega_{V'/W'} = \rho^*\omega_{V/W}$ by the Hurwitz formula.

Let us see the construction of $f'\colon V' \to W'$ more precisely for the reader's convenience. Let $\mathcal{A}$ be an ample invertible sheaf on W such that $\mathcal{A}^{\otimes d} \otimes \mathcal{O}_W(-D_i)$ is generated by global sections for $i = 1, 2$. We put $n = \dim W$. We take smooth divisors

$$
H_1^{(1)}, \ldots, H_n^{(1)}, H_1^{(2)}, \ldots, H_n^{(2)}
$$

on W in general position such that $\mathcal{A}^{\otimes d} = \mathcal{O}_W(D_i + H_j^{(i)})$ for $1 \le j \le n$ and $i = 1, 2$. Let $Z_j^{(i)}$ be the cyclic cover associated to $\mathcal{A}^{\otimes d} = \mathcal{O}_W(D_i + H_j^{(i)})$ for $1 \le j \le n$ and $i = 1, 2$. Then W' is the normalization of

$$
\left(Z_1^{(1)} \times_W \cdots \times_W Z_n^{(1)}\right) \times_W \left(Z_1^{(2)} \times_W \cdots \times_W Z_n^{(2)}\right).
$$

We note that W' is smooth since

$$
\bigcap_{j=1}^{n}(D_i \cap H_j^{(i)}) = \emptyset
$$

for $i = 1, 2$. For the details, see, for example, [EV, 3.19. Lemma] and [V3, Lemma 2.5]. Let $S_j^{(i)}$ be the cyclic cover of V associated to $(f^*\mathcal{A})^{\otimes d} = \mathcal{O}_V(f^*D_i + f^*H_j^{(i)})$. Then we define V' as the normalization of

$$
\left(S_1^{(1)} \times_V \cdots \times_V S_n^{(1)}\right) \times_V \left(S_1^{(2)} \times_V \cdots \times_V S_n^{(2)}\right).
$$

As before, V' is smooth since

$$\bigcap_{j=1}^{n}(f^{*}D_{i}\cap f^{*}H_{j}^{(i)})=\emptyset$$

for $i=1,2$. Note that $\rho\colon V'\to V$ is a finite flat morphism between smooth quasi-projective varieties. Since $V\times_{W}W'\to V$ is finite and flat and V is smooth, $V\times_{W}W'$ is Cohen–Macaulay (see, for example, [KM, Corollary 5.5]). We put

$$\Sigma=D_{1}+D_{2}+\sum_{i,j}H_{j}^{(i)}.$$

Then Σ is a simple normal crossing divisor on W. Let U be the largest Zariski open set of W such that f is smooth over U. We put

$$\Pi=\Sigma_{\mathrm{sing}}\cup\left((W\setminus U)\cap\Sigma\right),$$

where Σ_{sing} is the singular locus of Σ. Since

$$D_{1},D_{2},H_{1}^{(1)},\ldots,H_{n}^{(1)},H_{1}^{(2)},\ldots,H_{n}^{(2)}$$

are general, $\mathrm{codim}_{W}\Pi\geq 2$ and $\mathrm{codim}_{V}f^{-1}\Pi\geq 2$ hold. We note that $\tau\colon W'\to W$ is étale outside Σ. Let P be any closed point of $\Sigma\setminus\Pi$. Then $f\colon V\to W$ is smooth over a neighborhood of P. Hence we can check that $V\times_{W}W'$ is smooth in codimension one. Therefore, $V\times_{W}W'$ is normal. Since ρ factors through $V\times_{W}W'$ by construction, we see that $V'=V\times_{W}W'$ by Zariski's main theorem. By the above description, if $K_{W'}=\tau^{*}K_{W}+R$, then $K_{V'}=\rho^{*}K_{V}+f'^{*}R$. Therefore, $\omega_{V'/W'}=\rho^{*}\omega_{V/W}$ holds. By the above construction of $\tau\colon W'\to W$, we see that $\tau\colon W'\to W$ is Galois. $\qquad\square$

We give a very important remark on Lemma 3.3.

Remark 3.4. In the proof of Lemma 3.3, let S be any simple normal crossing divisor on V. Then we can choose the ramification locus Σ of τ such that $f^{*}P\not\subset S$ for any irreducible component P of Σ and that $f^{*}\Sigma\cup S$ is a simple normal crossing divisor on V. If we choose Σ as above, then we obtain that $\rho^{*}S$ is a simple normal crossing divisor on V'.

Lemma 3.5 is an elementary property of semi-ample $\mathbb{R}$-divisors. We give a proof for the sake of completeness.

Lemma 3.5. *Let $f\colon V\to W$ be a surjective morphism between normal projective varieties. Let D be a nef and f-semi-ample $\mathbb{R}$-divisor on V and let H be an ample $\mathbb{R}$-divisor on W. Then $aD+bf^{*}H$ is semi-ample for any positive real numbers a and b.*

Proof. Since D is f-semi-ample, there exists a surjective morphism $g\colon V\to Z$ with the following commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ \ g\ \ } & Z \\
 & {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle h} & \\
 & W &
\end{array}
$$

such that

  (i) Z is a normal projective variety, and
  (ii) $D\sim_{\mathbb{R}}g^{*}A$, where A is a nef and h-ample $\mathbb{R}$-divisor on Z.

We can take a large positive real number c such that $A + ch^*H$ is ample because H is ample and A is h-ample. If $b/a \geq c$ holds, then

$$aA + bh^*H = a(A + ch^*H) + (b - ac)h^*H$$

is ample. If $b/a < c$ holds, then

$$aA + bh^*H = (a - b/c)A + (b/c)(A + ch^*H)$$

is ample. Hence $aA + bh^*H$ is always ample. This implies that

$$aD + bf^*H \sim_\mathbb{R} g^*(aA + bh^*H)$$

is semi-ample for any positive real numbers a and b. $\qquad\qquad\square$

Finally, let us explain Viehweg's fiber product trick. We include the proof for the benefit of the reader. We will use it in the proof of Theorems 1.8 and 1.9 in Section 10.

Lemma 3.6 (see [M, (4.9) Lemma]). *Let V be a reduced Gorenstein scheme. Note that V may be reducible. We consider*

$$V' \xrightarrow{\delta} V^\nu \xrightarrow{\nu} V$$

where $\nu \colon V^\nu \to V$ is the normalization and $\delta \colon V' \to V^\nu$ is a resolution of singularities. Then, for every positive integer n, we have

$$(3.1) \qquad\qquad \nu_*\mathcal{O}_{V^\nu}(nK_{V^\nu}) \subset \omega_V^{\otimes n}$$

and

$$(3.2) \qquad\qquad \delta_*\mathcal{O}_{V'}(nK_{V'} + E) \subset \mathcal{O}_{V^\nu}(nK_{V^\nu})$$

where E is any δ-exceptional divisor on V'. In particular, we have

$$(3.3) \qquad\qquad (\nu \circ \delta)_*\mathcal{O}_{V'}(nK_{V'} + E) \subset \omega_V^{\otimes n}$$

for every positive integer n. If U is a Zariski open set of V such that $\nu \circ \delta$ is an isomorphism over U, then the inclusion (3.3) is an isomorphism over U.

Proof. In Steps 1 and 2, we will prove (3.2) and (3.1), respectively.

Step 1. By taking the double dual of $\delta_*\mathcal{O}_{V'}(nK_{V'}+E)$, we obtain $\mathcal{O}_{V^\nu}(nK_{V^\nu})$. Therefore, we have

$$\delta_*\mathcal{O}_{V'}(nK_{V'} + E) \subset \mathcal{O}_{V^\nu}(nK_{V^\nu})$$

for every integer n.

Step 2. Since ν is birational, the trace map $\nu_*\mathcal{O}_{V^\nu}(K_{V^\nu}) \to \omega_V$ is a generically isomorphic injection

$$(3.4) \qquad\qquad \nu_*\mathcal{O}_{V^\nu}(K_{V^\nu}) \hookrightarrow \omega_V.$$

More precisely, the above trace map is an isomorphism over the isomorphism locus of ν and then it is injective since $\nu_*\mathcal{O}_{V^\nu}(K_{V^\nu})$ is torsion-free. Since ν is finite,

$$(3.5) \qquad\qquad \nu^*\nu_*\mathcal{O}_{V^\nu}(K_{V^\nu}) \to \mathcal{O}_{V^\nu}(K_{V^\nu})$$

is surjective. Since $\mathcal{O}_{V^\nu}(K_{V^\nu})$ is torsion-free, the kernel of (3.5) is the torsion part of $\nu^*\nu_*\mathcal{O}_{V^\nu}(K_{V^\nu})$. Therefore, by (3.4), we get an inclusion

$$(3.6) \qquad\qquad \mathcal{O}_{V^\nu}(K_{V^\nu}) \hookrightarrow \nu^*\omega_V$$

because $\nu^*\omega_V$ is torsion-free. Let n be a positive integer with $n \geq 2$. Then we have

$$\mathcal{O}_{V^\nu}(nK_{V^\nu}) = \mathcal{O}_{V^\nu}(K_{V^\nu} + (n-1)K_{V^\nu}) \hookrightarrow \mathcal{O}_{V^\nu}(K_{V^\nu}) \otimes \nu^*\omega_V^{\otimes n-1}$$

by (3.6). Therefore, by taking ν_*, we get

$$\nu_*\mathcal{O}_{V^\nu}(nK_{V^\nu}) \hookrightarrow \nu_*\mathcal{O}_{V^\nu}(K_{V^\nu}) \otimes \omega_V^{\otimes n-1} \hookrightarrow \omega_V^{\otimes n}$$

by (3.4). This is what we wanted.

By the above construction of (3.1) and (3.2), it is obvious that the inclusion

$$(\nu \circ \delta)_*\mathcal{O}_{V'}(nK_{V'} + E) \subset \omega_V^{\otimes n}$$

is an isomorphism over U. $\hfill\square$

Lemma 3.7. *Let $f\colon X_0 \to Y_0$ be a projective surjective morphism between smooth quasi-projective varieties and let Δ_0 be an effective $\mathbb{R}$-divisor on X_0 such that $\operatorname{Supp}\Delta_0$ is a simple normal crossing divisor on X_0 and (X_0, Δ_0) is log canonical over a nonempty Zariski open set of Y_0. Let L_0 be a Cartier divisor on X_0 such that $L_0 \sim_{\mathbb{R}} k(K_{X_0/Y_0} + \Delta_0)$ for some positive integer k. Assume that f is flat. We consider the s-fold fiber product*

$$X_0^s := \underbrace{X_0 \times_{Y_0} X_0 \times_{Y_0} \cdots \times_{Y_0} X_0}_{s}$$

of X_0 over Y_0 and let $f^s\colon X_0^s \to Y_0$ be the induced morphism. We take a resolution of singularities $\rho\colon X_0^{(s)} \to X_0^s$ which is an isomorphism over a nonempty Zariski open set of Y_0. Then we can write

$$\mathcal{O}_{X_0^{(s)}}(K_{X_0^{(s)}}) = \rho^*\omega_{X_0^s} \otimes \mathcal{O}_{X_0^{(s)}}(R)$$

where R is an $(f^s \circ \rho)$-vertical Cartier divisor by construction. Let $p_i\colon X_0^s \to X_0$ be the i-th projection. We put $\pi_i = p_i \circ \rho\colon X_0^{(s)} \to X_0$. We consider

$$L_0^{(s)} := \sum_{i=1}^{s} \pi_i^*L_0 + kR.$$

We further assume that $f_*\mathcal{O}_{X_0}(L_0)$ is locally free. Then there exists a generically isomorphic injection

$$f_*^{(s)}\mathcal{O}_{X_0^{(s)}}(L_0^{(s)}) \hookrightarrow \bigotimes_{i=1}^{s} f_*\mathcal{O}_{X_0}(L_0)$$

with $f^{(s)} = f^s \circ \rho$. We have

$$L_0^{(s)} \sim_{\mathbb{R}} k\left(K_{X_0^{(s)}/Y_0} + \sum_{i=1}^{s} \pi_i^*\Delta_0\right).$$

Note that

$$\left(X_0^{(s)}, \sum_{i=1}^{s} \pi_i^*\Delta_0\right)$$

is log canonical over a nonempty Zariski open set of Y_0. We also note that $X_0^{(s)}$ may be reducible, that is, $X_0^{(s)}$ may be a disjoint union of smooth varieties.

Proof. By the flat base change theorem, we have

$$\omega_{X_0^s/Y_0} = \bigotimes_{i=1}^{s} p_i^*\omega_{X_0/Y_0}.$$

In particular, X_0^s is Gorenstein. We note that

$$
L_0^{(s)} = \sum_{i=1}^{s} \rho^* p_i^*(k K_{X_0/Y_0} + (L_0 - k K_{X_0/Y_0})) + kR
$$

(3.7)

$$
\sim k K_{X_0^{(s)}/Y_0} + \sum_{i=1}^{s} \pi_i^*(L_0 - k K_{X_0/Y_0}).
$$

Claim. *We have the following isomorphism of locally free sheaves:*

$$
f_*^s \mathcal{O}_{X_0^s}\left(\sum_{i=1}^{s} p_i^* L_0\right) \simeq \bigotimes^{s} f_* \mathcal{O}_{X_0}(L_0).
$$

Proof of Claim. We use induction on s. If $s = 1$, then the statement is obvious. So we assume that $s \geq 2$. We consider the following commutative diagram

$$
\begin{array}{ccc}
X_0^s & \xrightarrow{q} & X_0^{s-1} \\
\downarrow{\scriptstyle p_s} & \searrow{\scriptstyle f^s} & \downarrow{\scriptstyle f^{s-1}} \\
X_0 & \xrightarrow{f} & Y_0
\end{array}
$$

where $q = (p_1, \cdots, p_{s-1})$. Then we have

(3.8)
$$
\mathcal{O}_{X_0^s}\left(\sum_{i=1}^{s} p_i^* L_0\right) \simeq \mathcal{O}_{X_0^s}(p_s^* L_0) \otimes q^* \mathcal{O}_{X_0^{s-1}}\left(\sum_{i=1}^{s-1} p_i^* L_0\right).
$$

Therefore, we obtain

$$
\begin{aligned}
f_*^s \mathcal{O}_{X_0^s}\left(\sum_{i=1}^{s} p_i^* L_0\right) &\simeq f_* p_{s*}\left(\mathcal{O}_{X_0^s}(p_s^* L_0) \otimes q^* \mathcal{O}_{X_0^{s-1}}\left(\sum_{i=1}^{s-1} p_i^* L_0\right)\right) \\
&\simeq f_*\left(\mathcal{O}_{X_0}(L_0) \otimes p_{s*} q^* \mathcal{O}_{X_0^{s-1}}\left(\sum_{i=1}^{s-1} p_i^* L_0\right)\right) \\
&\simeq f_*\left(\mathcal{O}_{X_0}(L_0) \otimes f^* f_*^{s-1} \mathcal{O}_{X_0^{s-1}}\left(\sum_{i=1}^{s-1} p_i^* L_0\right)\right) \\
&\simeq f_*\left(\mathcal{O}_{X_0}(L_0) \otimes f^*\left(\bigotimes^{s-1} f_* \mathcal{O}_{X_0}(L_0)\right)\right) \\
&\simeq f_* \mathcal{O}_{X_0}(L_0) \otimes \bigotimes^{s-1} f_* \mathcal{O}_{X_0}(L_0) \\
&\simeq \bigotimes^{s} f_* \mathcal{O}_{X_0}(L_0).
\end{aligned}
$$

Note that the first isomorphism follows from (3.8), the second one is due to the projection formula, the third one is obtained by the flat base change theorem, the fourth one is due to induction on s, and the fifth one follows from the projection formula. Hence we obtain the desired isomorphism. $\square$

Let us go back to the proof of Lemma 3.7. We have an inclusion

$$\rho_* \mathcal{O}_{X_0^{(s)}}(L_0^{(s)}) \subset \omega_{X_0^s/Y_0}^{\otimes k} \otimes \mathcal{O}_{X_0^s}\left(\sum_{i=1}^s p_i^*(L_0 - kK_{X_0/Y_0})\right)$$

$$\simeq \mathcal{O}_{X_0^s}\left(\sum_{i=1}^s p_i^* L_0\right)$$

by (3.7) and Lemma 3.6, which is an isomorphism over a nonempty Zariski open set of Y_0. By taking f_*^s, the Claim yields a generically isomorphic injection

$$f_*^{(s)}\mathcal{O}_{X_0^{(s)}}(L_0^{(s)}) \subset f_*^s \mathcal{O}_{X_0^s}\left(\sum_{i=1}^s p_i^* L_0\right)$$

$$\simeq \bigotimes_{i=1}^s f_*\mathcal{O}_{X_0}(L_0),$$

where $f^{(s)} = f^s \circ \rho \colon X_0^{(s)} \to Y_0$. By assumption, $L_0 - kK_{X_0/Y_0} \sim_{\mathbb{R}} k\Delta_0$. Therefore,

$$L_0^{(s)} \sim_{\mathbb{R}} k\left(K_{X_0^{(s)}/Y_0} + \sum_{i=1}^s \pi_i^*\Delta_0\right)$$

by (3.7). We can take a nonempty Zariski open set U of Y_0 such that f is smooth over U, $\mathrm{Supp}\Delta$ is relatively simple normal crossing over U, and ρ is an isomorphism over U. Then we see that $\left(X_0^{(s)}, \sum_{i=1}^s \pi_i^*\Delta_0\right)$ is log canonical over U. $\square$

4. WEAKLY POSITIVE SHEAVES AND BIG SHEAVES

We briefly recall some basic properties of Viehweg's weakly positive sheaves and big sheaves. For the details, see [Fn10, Chapter 3], [V1], [V2], and [V3].

Definition 4.1 (Weak positivity and bigness). Let $\mathcal{F}$ be a torsion-free coherent sheaf on a smooth quasi-projective variety W. We say that $\mathcal{F}$ is *weakly positive* if, for every positive integer α and every ample invertible sheaf $\mathcal{H}$, there exists a positive integer β such that $\widehat{S}^{\alpha\beta}(\mathcal{F}) \otimes \mathcal{H}^{\otimes\beta}$ is generically generated by global sections. We say that a nonzero torsion-free coherent sheaf $\mathcal{F}$ is *big* (in the sense of Viehweg) if, for every ample invertible sheaf $\mathcal{H}$, there exists a positive integer a such that $\widehat{S}^a(\mathcal{F}) \otimes \mathcal{H}^{\otimes -1}$ is weakly positive.

Remark 4.2. In this paper and [Fn10], we adopt Viehweg's definition of weakly positive sheaves in [V2, Definition 3.1]. Note that it is slightly weaker than [V1, Definition 2.1]. The definition of weakly positive sheaves depends on the literature. Definition 4.1 seems to be sufficient and suitable for applications to the Iitaka conjecture (see [Fn10]).

Example 4.3. Let D be a Cartier divisor on a smooth projective variety W. Then $\mathcal{O}_W(D)$ is weakly positive if and only if D is pseudo-effective in the usual sense. We note that $\mathcal{O}_W(D)$ is big in the sense of Definition 4.1 if and only if D is big in the usual sense, that is, $\kappa(W, D) = \dim W$.

For the reader's convenience, let us recall the following basic properties of big sheaves without proof.

Lemma 4.4 ([V2, Lemma 3.6] and [Fn10, Lemma 3.1.15]). *Let $\mathcal{F}$ be a nonzero torsion-free coherent sheaf on a smooth quasi-projective variety W. Then the following three conditions are equivalent.*

(i) *There exist an ample invertible sheaf $\mathcal{H}$ on W, some positive integer ν, and an inclusion $\bigoplus \mathcal{H} \hookrightarrow \widehat{S}^\nu(\mathcal{F})$, which is an isomorphism over a nonempty Zariski open set of W.*

(ii) *For every invertible sheaf $\mathcal{M}$ on W, there exists some positive integer γ such that $\widehat{S}^\gamma(\mathcal{F}) \otimes \mathcal{M}^{\otimes -1}$ is weakly positive. In particular, $\mathcal{F}$ is a big sheaf.*

(iii) *There exist some positive integer γ and an ample invertible sheaf $\mathcal{M}$ on W such that $\widehat{S}^\gamma(\mathcal{F}) \otimes \mathcal{M}^{\otimes -1}$ is weakly positive.*

We will use the following three easy lemmas on big sheaves in this paper. So we explicitly state them here for the reader's convenience.

Lemma 4.5. *Let $\mathcal{F}$ be a weakly positive sheaf and let $\mathcal{H}$ be an ample invertible sheaf on a smooth quasi-projective variety W. Then $\mathcal{F} \otimes \mathcal{H}$ is big.*

We give a proof for the sake of completeness.

Proof of Lemma 4.5. Since $\mathcal{F}$ is weakly positive, $\widehat{S}^{2b}(\mathcal{F}) \otimes \mathcal{H}^{\otimes b}$ is generically generated by global sections for some positive integer b. By replacing b with a multiple, we may assume that $\mathcal{H}^{\otimes b-1}$ is generated by global sections. Then $\widehat{S}^{2b}(\mathcal{F} \otimes \mathcal{H}) \otimes \mathcal{H}^{\otimes -1} = \widehat{S}^{2b}(\mathcal{F}) \otimes \mathcal{H}^{\otimes 2b-1}$ is generically generated by global sections. In particular, $\widehat{S}^{2b}(\mathcal{F} \otimes \mathcal{H}) \otimes \mathcal{H}^{\otimes -1}$ is weakly positive. This implies that $\mathcal{F} \otimes \mathcal{H}$ is big by Lemma 4.4. $\square$

Lemma 4.6. *Let $\mathcal{G}$ be any nonzero torsion-free coherent sheaf on a smooth quasi-projective variety W and let $\mathcal{H}$ be an ample invertible sheaf on W. Then there exists a positive integer l such that $\mathcal{G} \otimes \mathcal{H}^{\otimes l}$ is big.*

Proof. Since $\mathcal{H}$ is ample, $\mathcal{G} \otimes \mathcal{H}^{\otimes m}$ is generated by global sections for some positive integer m. Therefore, there exists a surjection

$$\bigoplus_{\text{finite}} \mathcal{O}_W \to \mathcal{G} \otimes \mathcal{H}^{\otimes m}.$$

This implies that $\mathcal{G} \otimes \mathcal{H}^{\otimes m}$ is weakly positive (see [Fn10, Lemma 3.1.12 (ii)]). By Lemma 4.5, we obtain that $\mathcal{G} \otimes \mathcal{H}^{\otimes l}$ is big for every integer $l \geq m+1$. $\square$

Lemma 4.7. *Let $\mathcal{F}$ be a torsion-free coherent sheaf on a smooth quasi-projective variety W and let $\tau \colon W' \to W$ be a finite surjective morphism from a smooth quasi-projective variety W'. Assume that $\tau^* \mathcal{F}$ is big. Then $\mathcal{F}$ is a big sheaf on W.*

We include the proof for the benefit of the reader.

Proof of Lemma 4.7. We take an ample invertible sheaf $\mathcal{H}$ on W. By replacing W with $W \setminus \Sigma$ for some suitable closed subset Σ of codimension ≥ 2 (see, for example, [Fn10, Lemma 3.1.12 (i)]), we may assume that $\mathcal{F}$ is locally free. Since $\tau^* \mathcal{F}$ is big by assumption, there exists a positive integer a such that $S^a(\tau^* \mathcal{F}) \otimes \tau^* \mathcal{H}^{\otimes -1} = \tau^*(S^a(\mathcal{F}) \otimes \mathcal{H}^{\otimes -1})$ is weakly positive (see Lemma 4.4). Therefore, $S^a(\mathcal{F}) \otimes \mathcal{H}^{\otimes -1}$ is weakly positive since τ is finite (see, for example, [Fn10, Lemma 3.1.12 (v)]). This means that $\mathcal{F}$ is big by Lemma 4.4. $\square$

5. MIXED-ω-SHEAVES AND MIXED-ω-BIG SHEAVES

In this section, we introduce mixed-ω-sheaves, mixed-ω-big-sheaves, mixed-$\widehat{\omega}$-sheaves, and mixed-$\widehat{\omega}$-big-sheaves. We also treat some important examples in Lemmas 5.5, 5.9, and 5.10.

Let us start with the definition of mixed-ω-sheaves and pure-ω-sheaves.

Definition 5.1 (Mixed-ω-sheaf and pure-ω-sheaf)**.** A torsion-free coherent sheaf $\mathcal{F}$ on a normal quasi-projective variety W is called a *mixed-ω-sheaf* if there exist a projective surjective morphism $f\colon V \to W$ from a smooth quasi-projective variety V and a simple normal crossing divisor D on V such that $\mathcal{F}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D)$. When $D = 0$, $\mathcal{F}$ is called a *pure-ω-sheaf* on W.

We give a very important remark on Definition 5.1.

Remark 5.2 (Pure-ω-sheaves versus Nakayama's ω-sheaves)**.** The notion of pure-ω-sheaves is essentially the same as that of Nakayama's ω-sheaves in [N] when we treat torsion-free coherent sheaves on normal projective varieties (see the Remark after [N, Chapter V, 3.8. Definition]). However, the definition of pure-ω-sheaves in Definition 5.1 does not coincide with [N, Chapter V, 3.8. Definition]. Our definition seems to be more reasonable than Nakayama's from the mixed Hodge theoretic viewpoint.

For some geometric applications, the notion of mixed-ω-big-sheaves and pure-ω-big-sheaves is very useful.

Definition 5.3 (Mixed-ω-big-sheaf and pure-ω-big-sheaf)**.** Let $\mathcal{F}$ be a torsion-free coherent sheaf on a normal quasi-projective variety W. If there exist projective surjective morphisms $f\colon V \to W$, $p\colon V \to Z$, and an ample divisor A on Z satisfying the following conditions:

   (i) V is a smooth quasi-projective variety,
  (ii) Z is a normal quasi-projective variety,
 (iii) D is a simple normal crossing divisor on V,
 (iv) there exists a projective surjective morphism $q\colon Z \to W$ such that $f = q \circ p$, and
  (v) $\mathcal{F}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D + P)$, where P is a Cartier divisor on V such that $P \sim_{\mathbb{Q}} p^*A$,

then $\mathcal{F}$ is called a *mixed-ω-big-sheaf* on W. As in Definition 5.1, $\mathcal{F}$ is called a *pure-ω-big-sheaf* on W when $D = 0$. The relationships between V, W, Z and f, p, q can be visualized as follows.

$$
\begin{array}{ccc}
V & \xrightarrow{\ p\ } & \\
\downarrow{\scriptstyle f} & \searrow & Z \\
W & \nwarrow{\scriptstyle q} &
\end{array}
$$

Remark 5.4. Of course, we defined mixed-ω-big-sheaves and pure-ω-big-sheaves referring to [N, Chapter V, 3.16. Definition (1)]. However, Nakayama's definition of ω-bigness is different from ours. Roughly speaking, we treat only a special case where $X = Y$ in [N, Chapter V, 3.16. Definition (1)].

Lemma 5.5 gives a very basic example of mixed-ω-sheaves.

Lemma 5.5. *Let V be a smooth quasi-projective variety and let D be a simple normal crossing divisor on V. Let L be a semi-ample Cartier divisor on V. Then $\mathcal{O}_V(K_V + D + L)$ is a mixed-ω-sheaf on V and $\mathcal{O}_V(K_V + L)$ is a pure-ω-sheaf on V.*

Although this lemma is well known, we give a proof for the sake of completeness.

Proof of Lemma 5.5. Let m be a positive integer such that $|mL|$ is free. We take a general section $s \in H^0(V, \mathcal{O}_V(mL))$, whose zero divisor is B. We may assume that B is a smooth divisor, B and D have no common irreducible components, and $\mathrm{Supp}(B + D)$ is a simple normal crossing divisor on V. The dual of

$$s \colon \mathcal{O}_V \to \mathcal{O}_V(mL)$$

defines an $\mathcal{O}_V$-algebra structure on

$$\bigoplus_{i=0}^{m-1} \mathcal{O}_V(-iL).$$

We put

$$\pi \colon Z := \mathrm{Spec}_V \bigoplus_{i=0}^{m-1} \mathcal{O}_V(-iL) \to V.$$

Then Z is a smooth quasi-projective variety and π^*D is a simple normal crossing divisor on Z by construction. We can check that

$$\pi_*\mathcal{O}_Z(K_Z + \pi^*D) \simeq \bigoplus_{i=0}^{m-1} \mathcal{O}_V(K_V + D + iL)$$

since $\pi_*\mathcal{O}_V = \bigoplus_{i=0}^{m-1} \mathcal{O}_V(-iL)$. This means that $\mathcal{O}_V(K_V + D + L)$ is a mixed-ω-sheaf on V. We put $D = 0$ in the above argument. Then we see that $\mathcal{O}_V(K_V + L)$ is a pure-ω-sheaf on V. $\qquad\square$

We treat two elementary lemmas.

Lemma 5.6. *Let $\mathcal{F}$ be a mixed-ω-big-sheaf (resp. pure-ω-big-sheaf) on a normal quasi-projective variety W. Then $\mathcal{F}$ is a mixed-ω-sheaf (resp. pure-ω-sheaf) on W.*

Proof. We may assume that $\mathcal{F}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D + P)$ as in Definition 5.3. By Lemma 5.5, $\mathcal{O}_V(K_V + D + P)$ is a mixed-ω-sheaf on V. Therefore, we see that $\mathcal{F}$ is a mixed-ω-sheaf on W. If we put $D = 0$, then we see that $\mathcal{F}$ is a pure-ω-sheaf on W. $\qquad\square$

Lemma 5.7. *Let $\mathcal{F}$ be a mixed-ω-sheaf (resp. pure-ω-sheaf) on a normal quasi-projective variety W and let $\mathcal{A}$ be an ample invertible sheaf on W. Then $\mathcal{F} \otimes \mathcal{A}$ is a mixed-ω-big-sheaf (resp. pure-ω-big-sheaf) on W.*

Proof. We may assume that $\mathcal{F}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D)$ as in Definition 5.1. We put $Z = W$. Let A be an ample divisor on W such that $\mathcal{O}_W(A) = \mathcal{A}$. Then $\mathcal{F} \otimes \mathcal{A}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D + f^*A)$. Therefore, $\mathcal{F} \otimes \mathcal{A}$ is a mixed-ω-big-sheaf on W. When $D = 0$, we see that $\mathcal{F} \otimes \mathcal{A}$ is a pure-ω-big-sheaf on W. $\qquad\square$

We can not replace the assumption that $\mathcal{A}$ is ample with one that $\mathcal{A}$ is big in Lemma 5.7.

Remark 5.8. Let $\mathcal{F}$ be a pure-ω-sheaf on a normal quasi-projective variety W and let B be an effective big Cartier divisor on W. A simple example (see Example 7.2 below) shows that $\mathcal{F} \otimes \mathcal{O}_W(B)$ is not necessarily a mixed-ω-big-sheaf on W.

Lemmas 5.9 and 5.10 give many nontrivial important examples of mixed-ω-sheaves and mixed-ω-big-sheaves in the study of higher-dimensional algebraic varieties.

Lemma 5.9. *Let $f \colon V \to W$ be a projective surjective morphism from a smooth projective variety V onto a normal projective variety W. Let D be a simple normal crossing divisor on V and let M be an $\mathbb{R}$-divisor on V such that $M - f^*H$ is semi-ample for some ample $\mathbb{Q}$-divisor H on W. We assume that D and $\operatorname{Supp}\{M\}$ have no common irreducible components and $\operatorname{Supp}(D + \{M\})$ is a simple normal crossing divisor on V. Then $f_*\mathcal{O}_V(K_V + D + \lceil M \rceil)$ is a mixed-ω-big-sheaf on W and $f_*\mathcal{O}_V(K_V + \lceil M \rceil)$ is a pure-ω-big-sheaf on W.*

Proof. By Lemma 3.2, we can construct a $\mathbb{Q}$-divisor $M^\dagger$ on V such that $M^\dagger - f^*H$ is semi-ample, $\operatorname{Supp}\{M^\dagger\} = \operatorname{Supp}\{M\}$, and $\lceil M^\dagger \rceil = \lceil M \rceil$. Therefore, we may assume that M is a $\mathbb{Q}$-divisor by replacing M with $M^\dagger$. By Kawamata's covering construction, we can construct a finite Galois cover $\pi \colon V' \to V$ from a smooth projective variety V' with the following properties:

(i) π^*D is a simple normal crossing divisor on V',
(ii) $\pi^*\{M\}$ is a $\mathbb{Z}$-divisor on V',
(iii) $\operatorname{Supp}(\pi^*D + \pi^*\{M\})$ is a simple normal crossing divisor on V', and
(iv) $(\pi_*\mathcal{O}_{V'}(K_{V'} + \pi^*D + \pi^*M))^G \simeq \mathcal{O}_V(K_V + D + \lceil M \rceil)$, where G is the Galois group of $\pi \colon V' \to V$.

By assumption, π^*M is semi-ample. Let us consider the contraction morphism $p \colon V' \to Z$ associated to $|m\pi^*M|$ for some sufficiently large and divisible positive integer m. Since $\pi^*M - (f \circ \pi)^*H$ is semi-ample, we have a morphism $q \colon Z \to W$ with the following commutative diagram:

$$
\begin{array}{ccc}
 & V' & \\
 & \big\downarrow \;\; \searrow^{p} & \\
f\circ\pi & & Z \\
 & \searrow \;\; \swarrow_{q} & \\
 & W &
\end{array}
$$

such that

(a) Z is a normal projective variety, and
(b) there is an ample $\mathbb{Q}$-divisor A on Z with $\pi^*M \sim_{\mathbb{Q}} p^*A$.

Therefore, $f_*\mathcal{O}_V(K_V + D + \lceil M \rceil)$ is a mixed-ω-big-sheaf on W since it is a direct summand of $(f \circ \pi)_*\mathcal{O}_{V'}(K_{V'} + \pi^*D + \pi^*M)$. We put $D = 0$ in the above argument. Then $f_*\mathcal{O}_V(K_V + \lceil M \rceil)$ is a pure-ω-big-sheaf on W. $\qquad\square$

Lemma 5.10. *Let V be a smooth quasi-projective variety and let D be a simple normal crossing divisor on V. Let B be a $\mathbb{Q}$-divisor on V such that $rB \sim 0$ for some positive integer r, $\operatorname{Supp}\{B\}$ and D have no common irreducible components, and $\operatorname{Supp}(\{B\} + D)$ is a simple normal crossing divisor on V. Then there exist a generically finite proper morphism $\pi \colon V' \to V$ from a smooth quasi-projective variety V' and a simple normal crossing divisor D' on V' such that $\mathcal{O}_V(K_V + D + \lceil B \rceil)$ is a direct summand of $\pi_*\mathcal{O}_{V'}(K_{V'} + D')$. In particular, $\mathcal{O}_V(K_V + D + \lceil B \rceil)$ is a mixed-ω-sheaf on V. When $D = 0$, $\mathcal{O}_V(K_V + \lceil B \rceil)$ is obviously a pure-ω-sheaf on V.*

Proof. If $B \sim 0$, then there is nothing to prove. By replacing r suitably, we may assume that $iB \not\sim 0$ for $1 \leq i \leq r - 1$ and that $r \geq 2$. We consider the following $\mathcal{O}_V$-algebra

$$
\mathcal{A} = \bigoplus_{i=0}^{r-1} \mathcal{O}_V(\lfloor -iB \rfloor)
$$

defined by an isomorphism $\mathcal{O}_V(-rB) \simeq \mathcal{O}_V$. Let Z be the normalization of $\mathrm{Spec}_V \mathcal{A}$. Then we have

$$\tau_* \mathcal{O}_Z(K_Z + \tau^* D) \simeq \bigoplus_{i=0}^{r-1} \mathcal{O}_V(K_V + D + \lceil iB \rceil)$$

where $\tau \colon Z \to V$. By construction, we see that $(Z, \tau^* D)$ is dlt. We take a suitable resolution of singularities $\rho \colon V' \to Z$ and write

$$K_{V'} + D' = \rho^*(K_Z + \tau^* D) + E$$

where D' is a reduced simple normal crossing divisor on V' and E is an effective ρ-exceptional $\mathbb{Q}$-divisor on V'. We put $\pi := \tau \circ \rho \colon V' \to V$. Then

$$\pi_* \mathcal{O}_{V'}(K_{V'} + D') \simeq \tau_* \mathcal{O}_Z(K_Z + \tau^* D)$$

$$\simeq \bigoplus_{i=0}^{r-1} \mathcal{O}_V(K_V + D + \lceil iB \rceil).$$

Therefore, we have the desired statement. $\qquad\square$

We close this section with the definition of mixed-$\widehat{\omega}$-sheaves, mixed-$\widehat{\omega}$-big-sheaves, pure-$\widehat{\omega}$-sheaves, and pure-$\widehat{\omega}$-big-sheaves.

Definition 5.11 (Mixed-$\widehat{\omega}$-sheaf, mixed-$\widehat{\omega}$-big-sheaf, pure-$\widehat{\omega}$-sheaf, and pure-$\widehat{\omega}$-big-sheaf)**.** A torsion-free coherent sheaf $\mathcal{G}$ on a normal quasi-projective variety W is called a *mixed-$\widehat{\omega}$-sheaf* (resp. *mixed-$\widehat{\omega}$-big-sheaf*) if there exist a mixed-ω-sheaf (resp. mixed-ω-big-sheaf) $\mathcal{F}$ on W and a generically isomorphic injection $\mathcal{F} \hookrightarrow \mathcal{G}^{**}$ into the double dual $\mathcal{G}^{**}$ of $\mathcal{G}$. If $\mathcal{F}$ is a pure-ω-sheaf (resp. pure-ω-big-sheaf) in the above inclusion $\mathcal{F} \hookrightarrow \mathcal{G}^{**}$, then $\mathcal{G}$ is called a *pure-$\widehat{\omega}$-sheaf* (resp. *pure-$\widehat{\omega}$-big-sheaf*).

Remark 5.12. Let X be a smooth projective variety, let D be a simple normal crossing divisor on X, let H be an ample Cartier divisor on X, and let B be an effective Cartier divisor on X. Then $\mathcal{O}_X(K_X)$ is a pure-ω-sheaf, $\mathcal{O}_X(K_X + H)$ is a pure-ω-big-sheaf, $\mathcal{O}_X(K_X + B)$ is a pure-$\widehat{\omega}$-sheaf, and $\mathcal{O}_X(K_X + H + B)$ is a pure-$\widehat{\omega}$-big-sheaf. By definition, it is obvious that $\mathcal{O}_X(K_X + D)$ is a mixed-ω-sheaf, $\mathcal{O}_X(K_X + D + H)$ is a mixed-ω-big-sheaf, $\mathcal{O}_X(K_X + D + B)$ is a mixed-$\widehat{\omega}$-sheaf, and $\mathcal{O}_X(K_X + D + H + B)$ is a mixed-$\widehat{\omega}$-big-sheaf.

Let $f \colon X \to Y$ be a surjective morphism between smooth projective varieties and let Δ be a simple normal crossing divisor on X. Let k be a positive integer with $k \geq 2$ and let H be an ample Cartier divisor on Y. Then we will show that

$$\mathcal{O}_Y(K_Y + H) \otimes f_* \mathcal{O}_X(k(K_{X/Y} + \Delta))$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y when $f_* \mathcal{O}_X(k(K_{X/Y} + \Delta)) \neq 0$. This is a special case of Theorem 9.3, which we call a fundamental theorem of the theory of mixed-ω-sheaves.

6. Basic properties: Part 1

In this section, we treat the weak positivity and the bigness of mixed-ω-sheaves and mixed-ω-big-sheaves, respectively.

Let us start with the following weak positivity theorem, which follows from the theory of mixed Hodge structures.

Theorem 6.1. *Let $f \colon V \to W$ be a projective surjective morphism between smooth quasi-projective varieties. Let D be a simple normal crossing divisor on V. Then $f_* \mathcal{O}_V(K_{V/W} + D)$ is weakly positive.*

Proof. We may assume that V and W are smooth projective varieties by compactifying $f\colon V \to W$ suitably. Then this result is more or less well known. For the proof based on the theory of variations of mixed Hodge structure (see [Fn1], [FFS], [FF], [Fs], and so on), see [Fn9, Theorem 7.8 and Corollary 7.11]. For the proof based on the vanishing theorem, see [Fn3, Theorem 8.4]. $\square$

As an easy consequence of Theorem 6.1, we have:

Theorem 6.2 (Weak positivity). *Let $\mathcal{F}$ be a mixed-ω-sheaf on a smooth quasi-projective variety W. Then $\mathcal{F} \otimes \omega_W^{\otimes -1}$ is weakly positive.*

Proof. We may assume that $\mathcal{F}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D)$ as in Definition 5.1. By Theorem 6.1, $f_*\mathcal{O}_V(K_{V/W} + D)$ is weakly positive. Then $\mathcal{F} \otimes \omega_W^{\otimes -1}$ is weakly positive since it is a direct summand of $f_*\mathcal{O}_V(K_{V/W} + D)$. $\square$

When $\mathcal{F}$ is a mixed-ω-big-sheaf on W in Theorem 6.2, $\mathcal{F} \otimes \omega_W^{\otimes -1}$ is not only weakly positive but also big.

Theorem 6.3 (Bigness). *Let $\mathcal{F}$ be a mixed-ω-big-sheaf on a smooth quasi-projective variety W. Then $\mathcal{F} \otimes \omega_W^{\otimes -1}$ is big.*

Proof. Without loss of generality, we may assume that $\mathcal{F}$ is a direct summand of $f_*\mathcal{O}_V(K_V + D + P)$ as in Definition 5.3. It is sufficient to prove that $f_*\mathcal{O}_V(K_{V/W} + D + P)$ is big. Let

$$
\begin{array}{ccc}
V & \xrightarrow{\;p\;} & \\
\downarrow{f} & & Z \\
W & \xrightarrow{\;q\;} &
\end{array}
$$

and A be as in Definition 5.3. Let H be an ample Cartier divisor on W. We take a positive integer m such that $mA - q^*H$ is ample. We can take a finite surjective morphism $\tau\colon W' \to W$ from a smooth quasi-projective variety W' and get the following commutative diagram

$$
\begin{array}{ccc}
V' & \xrightarrow{\;\rho\;} & V \\
\downarrow{f'} & & \downarrow{f} \\
W' & \xrightarrow{\;\tau\;} & W
\end{array}
$$

such that $\tau^*H \sim mH'$ for some Cartier divisor H', $V' = V \times_W W'$ is a smooth quasi-projective variety, ρ^*D is a simple normal crossing divisor, and $\rho^*\omega_{V/W}^{\otimes n} = \omega_{V'/W'}^{\otimes n}$ holds for every integer n (see Lemma 3.3 and Remark 3.4). By Lemma 4.7, It is sufficient to prove that

$$
\tau^*f_*\mathcal{O}_V(K_{V/W} + D + P) \simeq f'_*\mathcal{O}_{V'}(K_{V'/W'} + \rho^*D + \rho^*P)
$$

is a big sheaf on W'. By construction, we see that $\rho^*P - f'^*H'$ is a semi-ample Cartier divisor on V' since it is $\mathbb{Q}$-linearly equivalent to $\rho^*p^*(A - (1/m)q^*H)$. Therefore, by Lemma 5.5, $\mathcal{O}_{V'}(K_{V'} + \rho^*D + \rho^*P - f'^*H')$ is a mixed-ω-sheaf on V'. Thus, $\mathcal{E} := f'_*\mathcal{O}_{V'}(K_{V'} + \rho^*D + \rho^*P - f'^*H')$ is a mixed-ω-sheaf on W'. We note that

$$
f'_*\mathcal{O}_{V'}(K_{V'/W'} + \rho^*D + \rho^*P) \simeq \mathcal{E} \otimes \omega_{W'}^{\otimes -1} \otimes \mathcal{O}_{W'}(H').
$$

By Theorem 6.2, $\mathcal{E} \otimes \omega_{W'}^{\otimes -1}$ is weakly positive. By Lemma 4.5, $\mathcal{E} \otimes \omega_{W'}^{\otimes -1} \otimes \mathcal{O}_{W'}(H')$ is big since H' is ample. This is what we wanted. $\square$

We close this section with an obvious corollary.

Corollary 6.4. *Let $\mathcal{F}$ be a mixed-$\widehat{\omega}$-sheaf (resp. mixed-$\widehat{\omega}$-big-sheaf) on a smooth quasi-projective variety W. Then $\mathcal{F} \otimes \omega_W^{\otimes -1}$ is weakly positive (resp. big).*

Proof. We note that $\mathcal{F} \otimes \omega_W^{\otimes -1}$ is weakly positive (resp. big) if and only if so is $\mathcal{F}^{**} \otimes \omega_W^{\otimes -1}$. Therefore, the desired statement follows from Theorems 6.2 and 6.3. $\square$

7. Basic properties: Part 2

In this section, we discuss some vanishing theorems for mixed-ω-sheaves and several related topics.

Lemma 7.1 (Vanishing theorem for mixed-ω-big-sheaf). *Let $\mathcal{F}$ be a mixed-ω-big-sheaf on a normal projective variety W. Then $H^i(W, \mathcal{F} \otimes \mathcal{N}) = 0$ for every $i > 0$ and every nef invertible sheaf $\mathcal{N}$ on W.*

Proof. We may assume that $\mathcal{F}$ is a direct summand of $f_* \mathcal{O}_V(K_V + D + P)$ as in Definition 5.3. Let N be a Cartier divisor on W such that $\mathcal{N} \simeq \mathcal{O}_W(N)$. It is sufficient to prove that $H^i(W, f_* \mathcal{O}_V(K_V + D + P + f^*N)) = 0$ for every $i > 0$. We take an ample $\mathbb{Q}$-divisor H on W such that $A - q^*H$ is an ample $\mathbb{Q}$-divisor on Z, where A and $q \colon Z \to W$ are as in Definition 5.3. Then we can take a boundary $\mathbb{Q}$-divisor Δ on V such that $\Delta \sim_{\mathbb{Q}} D + P - f^*H$ and that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on V. Then we have

$$K_V + D + P + f^*N - (K_V + \Delta) \sim_{\mathbb{Q}} f^*(H + N).$$

We note that $H + N$ is ample. Therefore, by [Fn2, Theorem 6.3 (ii)] (see also [Fn6, Theorem 3.16.3 (ii) and Theorem 5.6.2 (ii)], and so on), we obtain that $H^i(W, f_* \mathcal{O}_V(K_V + D + P + f^*N)) = 0$ for every $i > 0$. $\square$

Example 7.2 shows that the vanishing theorem does not necessarily hold for mixed-$\widehat{\omega}$-big-sheaves.

Example 7.2. We put $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Let C be the unique (-1)-curve on X. It is not difficult to see that there exists an ample Cartier divisor H on X such that $C \cdot H = 1$. By definition, $\mathcal{O}_X(K_X + C)$ is a mixed-ω-sheaf on X. Then, by Lemma 5.7, $\mathcal{O}_X(K_X + C + H)$ is a mixed-ω-big-sheaf on X. By definition, $\mathcal{O}_X(K_X + C + H + C)$ is a mixed-$\widehat{\omega}$-big-sheaf on X. Let us consider the following short exact sequence

$$0 \to \mathcal{O}_X(K_X + C + H) \to \mathcal{O}_X(K_X + C + H + C) \to \mathcal{O}_C(K_C + H|_C + C|_C) \to 0.$$

Since $C \simeq \mathbb{P}^1$ and $H \cdot C = 1$, we obtain

$$\mathcal{O}_C(K_C + H|_C + C|_C) \simeq \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1}).$$

Since $H^i(X, \mathcal{O}_X(K_X + C + H)) = 0$ for $i = 1$ and 2, we have

$$H^1(X, \mathcal{O}_X(K_X + C + H + C)) \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1})) = \mathbb{C}.$$

In particular, $\mathcal{O}_X(K_X + C + H + C)$ is not a mixed-ω-big-sheaf on X by Lemma 7.1. We note that $\mathcal{O}_X(K_X)$ is a pure-ω-sheaf and that $H + 2C$ is an effective big Cartier divisor. However, $\mathcal{O}_X(K_X + H + 2C)$ is not a mixed-ω-big-sheaf.

As an easy consequence of Lemma 7.1, we have:

Lemma 7.3. *Let $\mathcal{F}$ be a mixed-ω-sheaf (resp. mixed-ω-big-sheaf) on a normal projective variety W with $\dim W = n$. Let $\mathcal{A}$ be an ample invertible sheaf on W such that $|\mathcal{A}|$ is free. Then $\mathcal{F} \otimes \mathcal{A}^{\otimes n+1}$ (resp. $\mathcal{F} \otimes \mathcal{A}^{\otimes n}$) is generated by global sections.*

Proof. This is a direct consequence of Lemmas 5.7, 7.1, and Castelnuovo–Mumford regularity. □

Let us recall a vanishing theorem for dlt pairs.

Lemma 7.4. *Let $f\colon V \to W$ be a surjective morphism from a smooth projective variety V onto a normal projective variety W. Let Δ be an effective $\mathbb{R}$-divisor on V such that (V, Δ) is dlt and that every log canonical center of (V, Δ) is dominant onto W. Let L be a Cartier divisor on V such that $L - (K_V + \Delta) \sim_{\mathbb{R}} f^*H$ for some nef and big $\mathbb{R}$-divisor H on W. Then $H^i(W, R^j f_* \mathcal{O}_V(L) \otimes \mathcal{N}) = 0$ for $i > 0$, $j \geq 0$, and every nef invertible sheaf $\mathcal{N}$ on W.*

Sketch of Proof. By Kodaira's lemma, we can write $H \sim_{\mathbb{R}} A + E$ such that A is an ample $\mathbb{R}$-divisor on W and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on W. Since every log canonical center of (V, Δ) is dominant onto W, $(V, \Delta + \varepsilon f^*E)$ is dlt for $0 < \varepsilon \ll 1$. Let N be a Cartier divisor on W such that $\mathcal{N} \simeq \mathcal{O}_W(N)$. We note that

$$L + f^*N - (K_V + \Delta + \varepsilon f^*E) \sim_{\mathbb{R}} f^*(N + (1-\varepsilon)H + \varepsilon A)$$

and that $N + (1-\varepsilon)H + \varepsilon A$ is ample for $0 < \varepsilon \ll 1$. By [Fn9, Lemma 7.14],

$$H^i(W, R^j f_* \mathcal{O}_V(L) \otimes \mathcal{N}) = 0$$

for $i > 0$ and $j \geq 0$. □

In Lemmas 7.5 and 7.6, we treat mixed-$\widehat{\omega}$-big-sheaves on smooth projective curves.

Lemma 7.5. *Let $\mathcal{G}$ be a mixed-$\widehat{\omega}$-big-sheaf on a smooth projective curve C. Then $H^1(C, \mathcal{G} \otimes \mathcal{N}) = 0$ holds for every nef invertible sheaf $\mathcal{N}$ on C.*

Proof. We note that $\mathcal{G}$ is locally free since C is a smooth curve. By definition, we have a mixed-ω-big-sheaf $\mathcal{F}$ on C and a generically isomorphic injection $\iota\colon \mathcal{F} \hookrightarrow \mathcal{G}$. Note that the cokernel of ι is a skyscraper sheaf on C. By Lemma 7.1, $H^1(C, \mathcal{F} \otimes \mathcal{N}) = 0$ holds. Therefore, we have $H^1(C, \mathcal{G} \otimes \mathcal{N}) = 0$ by the surjection $H^1(C, \mathcal{F} \otimes \mathcal{N}) \to H^1(C, \mathcal{G} \otimes \mathcal{N})$. □

Lemma 7.6. *Let $\mathcal{E}$ be a locally free sheaf on a smooth projective curve C and let P be a closed point of C. If $\mathcal{E} \otimes \mathcal{O}_C(-P) \otimes \mathcal{N}^{\otimes -1}$ is a mixed-$\widehat{\omega}$-big-sheaf on C for some nef invertible sheaf $\mathcal{N}$ on C, then $\mathcal{E}$ is generated by global sections at P.*

Proof. By Lemma 7.5, $H^1(C, \mathcal{E} \otimes \mathcal{O}_C(-P)) = 0$. This means that the natural restriction map

$$H^0(C, \mathcal{E}) \to \mathcal{E} \otimes \mathbb{C}(P)$$

is surjective. Therefore, $\mathcal{E}$ is generated by global sections at P. □

Let us discuss generically global generations of mixed-ω-big-sheaves.

Lemma 7.7. *Let $\mathcal{F}$ be a mixed-ω-sheaf on a normal projective variety W with $\dim W = n$. Let H be a big Cartier divisor on W such that $|H|$ is free. Then $\mathcal{F} \otimes \mathcal{O}_W((n+1)H)$ is generically generated by global sections.*

Proof. If W is a curve, then H is ample. Therefore, the statement follows from Lemma 7.3 when $n = 1$. We will use induction on n. We may assume that $\mathcal{F} = f_* \mathcal{O}_V(K_V + D)$ as in Definition 5.1.

Step 1. Let $\mu\colon \widetilde{V} \to V$ be a projective birational morphism from a smooth projective variety $\widetilde{V}$ such that

$$K_{\widetilde{V}} + \widetilde{D} = \mu^*(K_V + D) + E$$

where $\widetilde{D}$ and E are effective divisors and have no common irreducible components. Since $\mu_*\mathcal{O}_{\widetilde{V}}(K_{\widetilde{V}} + \widetilde{D}) \simeq \mathcal{O}_V(K_V + D)$, we may replace (V, D) and $f\colon V \to W$ with $(\widetilde{V}, \widetilde{D})$ and $f \circ \mu\colon \widetilde{V} \to W$, respectively. By taking $\mu\colon \widetilde{V} \to V$ suitably, we may assume that all the log canonical centers of (V, D_{hor}) are dominant onto W, where D_{hor} is the horizontal part of D. Since $f_*\mathcal{O}_V(K_V + D_{\mathrm{hor}}) \hookrightarrow f_*\mathcal{O}_V(K_V + D)$ is a generically isomorphic injection, we may replace D with D_{hor}.

Step 2. We will prove that $f_*\mathcal{O}_V(K_V + D) \otimes \mathcal{O}_V((n+1)H)$ is generically generated by global sections by induction on $n = \dim W$. We take a general member W' of $|H|$. We put $f^{-1}(W') = V'$. Then we have a short exact sequence

$$0 \to \mathcal{O}_V(K_V + D) \to \mathcal{O}_V(K_V + V' + D) \to \mathcal{O}_{V'}(K_{V'} + D|_{V'}) \to 0$$

by adjunction. Since W' is a general member of $|H|$,

$$R^1 f_*\mathcal{O}_V(K_V + D) \otimes \mathcal{O}_W(nH) \to R^1 f_*\mathcal{O}_V(K_V + V' + D) \otimes \mathcal{O}_W(nH)$$

is injective. Hence we get a short exact sequence

(7.1)
$$\begin{aligned}
0 \to {}& f_*\mathcal{O}_V(K_V + D) \otimes \mathcal{O}_W(nH) \to f_*\mathcal{O}_V(K_V + V' + D) \otimes \mathcal{O}_W(nH) \\
&\to f_*\mathcal{O}_{V'}(K_{V'} + D|_{V'}) \otimes \mathcal{O}_{W'}(nH|_{W'}) \to 0.
\end{aligned}$$

By the vanishing theorem (see Lemma 7.4), we have

(7.2)
$$H^1(W, f_*\mathcal{O}_V(K_V + D) \otimes \mathcal{O}_W(nH)) = 0.$$

Therefore, the restriction map

$$H^0(W, f_*\mathcal{O}_V(K_V + D) \otimes \mathcal{O}_W((n+1)H)) \to H^0(W', f_*\mathcal{O}_{V'}(K_{V'} + D|_{V'}) \otimes \mathcal{O}_{W'}(nH|_{W'}))$$

is surjective by (7.1) and (7.2). By induction on n, $f_*\mathcal{O}_{V'}(K_{V'} + D|_{V'}) \otimes \mathcal{O}_{W'}(nH|_{W'})$ is generically generated by global sections. This implies that so is $f_*\mathcal{O}_V(K_V + D) \otimes \mathcal{O}_W((n+1)H)$.

We obtain the desired statement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 7.8 is similar to Lemma 7.7.

Lemma 7.8. *Let $\mathcal{F}$ be a mixed-ω-big-sheaf on a normal projective variety W with $\dim W = n$. Let H be a big Cartier divisor on W such that $|H|$ is free. Then $\mathcal{F} \otimes \mathcal{O}_W(nH)$ is generically generated by global sections.*

The proof of Lemma 7.8 is essentially the same as that of Lemma 7.7.

Sketch of Proof of Lemma 7.8. If $n = 0$, then the statement is trivial. If $n = 1$, then it follows from Lemma 7.6. Therefore, we assume that $n \geq 2$. As in the proof of Lemma 7.7, we may assume that $\mathcal{F} = f_*\mathcal{O}_V(K_V + D + P)$ as in Definition 5.3. Moreover, we may further assume that $D = D_{\mathrm{hor}}$ and that every log canonical center of (V, D) is dominant onto W (see Step 1 in the proof of Lemma 7.7). We take a general member W' of $|H|$ and put $V' = f^{-1}(W')$. Then the natural restriction map

$$\begin{aligned}
&H^0(W, f_*\mathcal{O}_V(K_V + D + P) \otimes \mathcal{O}_W(nH)) \\
&\to H^0(W', f_*\mathcal{O}_{V'}(K_{V'} + D|_{V'} + P|_{V'}) \otimes \mathcal{O}_{W'}((n-1)H|_{W'}))
\end{aligned}$$

is surjective as in Step 2 in the proof of Lemma 7.7. By induction on dimension, we see that

$$f_* \mathcal{O}_V(K_V + D + P) \otimes \mathcal{O}_W(nH)$$

is generically generated by global sections.   □

We close this section with the following result, which is due to [DuM]. We will use it in the proof of Theorem 1.9.

Lemma 7.9. *Let $\mathcal{F}$ be a mixed-ω-sheaf on a normal projective variety W and let H be a nef and big Cartier divisor on W. We put $\dim W = n$. Then $\mathcal{F} \otimes \mathcal{O}_W(lH)$ is generically generated by global sections for $l \geq n^2 + 1$.*

Proof. We may assume that $\mathcal{F} = f_* \mathcal{O}_V(K_V + D)$ as in Definition 5.1. Then, by [EKL, Theorem 1] and [DuM, Theorems C and 2.20], $\mathcal{F} \otimes \mathcal{O}_W(lH)$ is generically generated by global sections for $l \geq n^2 + 1$.   □

For the details of Lemma 7.9, we recommend the reader to see [DuM].

8. A SPECIAL CASE

In this section, we freely use the standard notation and some basic results in the theory of minimal models (see, for example, [Fn2], [Fn6], and [Fn10]). We treat weakly semistable morphisms $f \colon X \to Y$ in the sense of Abramovich–Karu with the assumption that the geometric generic fiber $X_{\overline{\eta}}$ of $f \colon X \to Y$ has a good minimal model. In this case, we can prove some strong results with the aid of the theory of minimal models. We do not need the results of this section in the subsequent sections. Hence the reader can skip this section.

Here we adopt the following definition of weakly semistable morphisms in the sense of Abramovich–Karu (see [AK]).

Definition 8.1 (Weakly semistable morphisms)**.** Let $f \colon X \to Y$ be a projective surjective morphism between normal quasi-projective varieties with connected fibers. We say that $f \colon X \to Y$ is *weakly semistable* if

  (i) the varieties X and Y admit toroidal structures $(U_X \subset X)$ and $(U_Y \subset Y)$ with $U_X = f^{-1}(U_Y)$,
 (ii) with this structure, the morphism f is toroidal,
(iii) the morphism f is equidimensional,
 (iv) all the fibers of the morphism f are reduced, and
  (v) Y is smooth.

The following result is the main theorem of this section.

Theorem 8.2. *Let $f \colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y with connected fibers. Assume that f is weakly semistable in the sense of Abramovich–Karu and that the geometric generic fiber $X_{\overline{\eta}}$ of $f \colon X \to Y$ has a good minimal model. Let H be an ample Cartier divisor on Y. Let k be a positive integer such that $k \geq 2$ and $f_* \omega_{X/Y}^{\otimes k} \neq 0$. Then*

$$f_* \omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(H)$$

is locally free and is a pure-ω-big-sheaf on Y. More generally, we obtain that

$$\left(\bigotimes^s f_* \omega_{X/Y}^{\otimes k} \right) \otimes \omega_Y \otimes \mathcal{O}_Y(H)$$

is a pure-ω-big-sheaf on Y for every positive integer s. Therefore, if A is an ample Cartier divisor on Y such that $|A|$ is free, then

$$\left(\bigotimes^{s} f_* \omega_{X/Y}^{\otimes k} \right) \otimes \omega_Y \otimes \mathcal{O}_Y(H + nA)$$

is generated by global sections, where $n = \dim Y$.

Proof. As mentioned above, we will freely use some basic results in the theory of minimal models. We note that X has only rational Gorenstein singularities by [AK, Lemma 6.1] since $f : X \to Y$ is weakly semistable by assumption. Hence X has only canonical Gorenstein singularities.

Step 1. By the proof of [Fn4, Theorem 1.6] (see also [Fn5]), we have already known that $f_* \omega_{X/Y}^{\otimes m}$ is a nef locally free sheaf on Y for every $m \geq 1$.

Step 2. We consider a relative good minimal model $f' : X' \to Y$ of $f : X \to Y$ (see [Fn4, Theorem 3.3]).

$$X \dashrightarrow^{\phi} X'$$

with f and f' mapping to Y.

Since

$$f_* \omega_{X/Y}^{\otimes m} \simeq f'_* \mathcal{O}_{X'}(m K_{X'/Y})$$

holds for every $m \geq 1$, it is sufficient to prove that

$$f'_* \mathcal{O}_{X'}(K_{X'} + (k-1)K_{X'/Y} + f'^* H)$$

is a pure-ω-big-sheaf on Y.

Step 3. In this step, we will prove:

Claim. $K_{X'/Y}$ *is nef and* f'-*semi-ample.*

Proof of Claim. Since $f' : X' \to Y$ is a relative good minimal model of $f : X \to Y$, $K_{X'/Y}$ is f'-semi-ample. Therefore,

$$f'^* f'_* \mathcal{O}_{X'}(l K_{X'/Y}) \to \mathcal{O}_{X'}(l K_{X'/Y})$$

is surjective for a sufficiently large and divisible positive integer l. Since $f'_* \mathcal{O}_{X'}(l K_{X'/Y}) \simeq f_* \omega_{X/Y}^{\otimes l}$ is a nef locally free sheaf, $K_{X'/Y}$ is nef by the above surjection. $\square$

Step 4. Since $K_{X'/Y}$ is nef and f'-semi-ample, $(k-1)K_{X'/Y} + a f'^* H$ is semi-ample for every positive rational number a by Lemma 3.5. Since X' is a relative minimal model of $f : X \to Y$ and X has only canonical singularities, X' also has only canonical singularities. We take a birational morphism $\rho : \widetilde{X} \to X'$ from a smooth projective variety $\widetilde{X}$ such that the exceptional locus $\mathrm{Exc}(\rho)$ of ρ is a simple normal crossing divisor on $\widetilde{X}$. Since X' has only canonical singularities, we see that

$$\rho_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \lceil (k-1)\rho^* K_{X'/Y} + \rho^* f'^* H \rceil) \simeq \mathcal{O}_{X'}(K_{X'} + (k-1)K_{X'/Y} + f'^* H)$$

holds. By Lemma 5.9,

$$(f' \circ \rho)_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \lceil (k-1)\rho^* K_{X'/Y} + \rho^* f'^* H \rceil) \simeq f'_* \mathcal{O}_{X'}(K_{X'} + (k-1)K_{X'/Y} + f'^* H)$$

is a pure-ω-big-sheaf on Y since

$$(k-1)\rho^* K_{X'/Y} + \rho^* f'^* H - \frac{1}{2}\rho^* f'^* H$$

is semi-ample. This means that

$$f_* \omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(H)$$

is locally free and a pure-ω-big-sheaf on Y.

Step 5. In this step, we will briefly explain how to prove that

$$\left(\bigotimes^{s} f_* \omega_{X/Y}^{\otimes k}\right) \otimes \omega_Y \otimes \mathcal{O}_Y(H)$$

is a pure-ω-big-sheaf on Y for every positive integer s.

Let

$$X^s = \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{s}$$

be the s-fold fiber product of $f\colon X \to Y$ and let $f^s\colon X^s \to Y$ be the induced morphism. Then we can check that X^s has only rational Gorenstein singularities since $f\colon X \to Y$ is weakly semistable (see [Fn4, Section 5] and Step 4 in the proof of [Fn10, Theorem 4.3.1]). Therefore, X^s has only canonical Gorenstein singularities. Let F be a general fiber of $f\colon X \to Y$. Then F has a good minimal model by assumption and [Fn4, Theorem 3.3]. This implies that

$$F^s = \underbrace{F \times F \times \cdots \times F}_{s}$$

also has a good minimal model. We note that F^s is a general fiber of $f^s\colon X^s \to Y$. Hence $f^s\colon X^s \to Y$ has a relative good minimal model over Y by [Fn4, Theorem 3.3]. By the flat base change theorem and the projection formula, we can check that

$$(8.1) \qquad \left(\bigotimes^{s} f_* \omega_{X/Y}^{\otimes k}\right) \simeq f_*^s \omega_{X^s/Y}^{\otimes k}$$

holds (see [Fn4, Section 5] and Step 4 in the proof of [Fn10, Theorem 4.3.1]). In particular, $f_*^s \omega_{X^s/Y}^{\otimes k}$ is a nef locally free sheaf on Y. By applying the arguments in Steps 2, 3, and 4 to $f^s\colon X^s \to Y$, we obtain that

$$(8.2) \qquad f_*^s \omega_{X^s/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(H)$$

is a pure-ω-big-sheaf on Y. Therefore, by (8.1) and (8.2), we have that

$$\left(\bigotimes^{s} f_* \omega_{X/Y}^{\otimes k}\right) \otimes \omega_Y \otimes \mathcal{O}_Y(H)$$

is a pure-ω-big-sheaf on Y for every positive integer s.

By Lemma 7.3,

$$\left(\bigotimes^{s} f_* \omega_{X/Y}^{\otimes k}\right) \otimes \omega_Y \otimes \mathcal{O}_Y(H + nA)$$

is generated by global sections. $\qquad\qquad\square$

Remark 8.3. In the proof of [Fn4, Theorem 3.3], which was used in the proof of Theorem 8.2, we need Theorem 1.13 (see [Fn4, Theorem 3.4]). We note that we will not use Theorem 8.2 in the subsequent sections. Hence there is no problem.

We note that the global generation of

$$\left(\bigotimes^{s} f_*\omega_{X/Y}\right) \otimes \omega_Y \otimes \mathcal{O}_Y(H + nA)$$

was already treated in [Ko, Theorem 3.6]. In some sense, Theorem 8.2 generalizes [SY, Theorem 1.8].

Theorem 8.2 predicts that $f_*\omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(H)$ has good properties. Of course, we strongly hope to prove Theorem 8.2 without using the assumption that $X_{\overline{\eta}}$ has a good minimal model.

9. Fundamental theorem

This section is the main part of this paper. The main result of this section is Theorem 9.3, which we call a fundamental theorem of the theory of mixed-ω-sheaves.

Let us start with the following lemma.

Lemma 9.1 ([N, Chapter V, 3.34. Lemma])**.** *Let $f: X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y. Let L be a Cartier divisor on X and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let k be a positive integer with $k \geq 2$. We assume the following conditions:*

(i) (X, Δ) is log canonical over a nonempty Zariski open set of Y, and
(ii) $L - k(K_{X/Y} + \Delta)$ is nef and f-semi-ample.

Let H be an ample divisor on Y. We assume that $f_*\mathcal{O}_Y(L) \neq 0$. We take a positive integer l such that

$$\mathcal{O}_Y(lH) \otimes f_*\mathcal{O}_X(L)$$

is big. Then

$$\mathcal{O}_Y(K_Y + (l - \lfloor l/k \rfloor)H) \otimes f_*\mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y. Hence we obtain that

$$\mathcal{O}_Y(K_Y + (k - 1)H) \otimes f_*\mathcal{O}_X(L)$$

is always a mixed-$\widehat{\omega}$-big-sheaf on Y.

We include all the details although Lemma 9.1 is essentially the same as [N, Chapter V, 3.34. Lemma].

Proof of Lemma 9.1. We divide the proof into several small steps.

Step 1 (Resolution of singularities)**.** Let $\mu: \widetilde{X} \to X$ be a projective birational morphism from a smooth projective variety $\widetilde{X}$ such that $K_{\widetilde{X}} + \widetilde{\Delta} = \mu^*(K_X + \Delta)$ and that $\mathrm{Supp}\widetilde{\Delta}$ is a simple normal crossing divisor on $\widetilde{X}$. We put $E = \lceil -(\widetilde{\Delta}^{<0}) \rceil$. Then E is an effective μ-exceptional divisor on $\widetilde{X}$, $\widetilde{\Delta} + E$ is effective, and $(\widetilde{X}, \widetilde{\Delta} + E)$ is log canonical over a nonempty Zariski open set of Y by construction. We note that

$$\mu^*L + kE - k(K_{\widetilde{X}/Y} + \widetilde{\Delta} + E) = \mu^*(L - k(K_{X/Y} + \Delta))$$

and that $\mu_* \mathcal{O}_{\widetilde{X}}(\mu^* L + kE) \simeq \mathcal{O}_X(L)$. Therefore, by replacing $f \colon X \to Y$, L, and Δ with $f \circ \mu \colon \widetilde{X} \to Y$, $\mu^* L + kE$, and $\widetilde{\Delta} + E$ respectively, we may assume that X is smooth and $\operatorname{Supp}\Delta$ is a simple normal crossing divisor on X.

Step 2. We note that by Lemma 4.6 we can always take a positive integer l such that $\mathcal{O}_Y(lH) \otimes f_* \mathcal{O}_X(L)$ is big since H is an ample divisor on Y. Since $\mathcal{O}_Y(lH) \otimes f_* \mathcal{O}_X(L)$ is big, we can take a positive integer a such that

$$\widehat{S}^a(\mathcal{O}_Y(lH) \otimes f_* \mathcal{O}_X(L)) \otimes \mathcal{O}_Y(-H) = \mathcal{O}_Y((al-1)H) \otimes \widehat{S}^a(f_* \mathcal{O}_X(L))$$

is generically generated by global sections by Lemma 4.4.

Step 3. We take an effective f-exceptional divisor E on X such that

$$(f_* \mathcal{O}_X(bL))^{**} \simeq f_* \mathcal{O}_X(b(L + E))$$

holds for every $1 \leq b \leq a$. For a related result, see [Fn12, Theorem 1.2]. By taking a resolution of singularities as in Step 1, we may assume that $\operatorname{Supp}(\Delta + E)$ is a simple normal crossing divisor on X. Since

$$(L + E) - k(K_{X/Y} + \Delta + (1/k)E) = L - k(K_{X/Y} + \Delta),$$

we may replace L and Δ with $L + E$ and $\Delta + (1/k)E$, respectively. This is because

$$\mathcal{O}_Y(K_Y + (l - \lfloor l/k \rfloor)H) \otimes f_* \mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y if and only if so is

$$\mathcal{O}_Y(K_Y + (l - \lfloor l/k \rfloor)H) \otimes (f_* \mathcal{O}_X(L))^{**}.$$

Hence we may assume that $f_* \mathcal{O}_X(bL)$ is reflexive for every $1 \leq b \leq a$.

Step 4. By taking a suitable birational modification of X again (see Step 1), we may further assume that the image of the natural map

$$f^* f_* \mathcal{O}_X(L) \to \mathcal{O}_X(L)$$

is invertible and can be written as $\mathcal{O}_X(L - B)$ such that $\operatorname{Supp}(\Delta + B)$ is a simple normal crossing divisor on X. By the definition of B, we have $f_* \mathcal{O}_X(L - B) = f_* \mathcal{O}_X(L)$.

Step 5. We note that we can take an effective f-exceptional divisor E on X such that the map $f^* f_* \mathcal{O}_X(L) \to \mathcal{O}_X(L - B)$ induces

$$f^* \widehat{S}^a(f_* \mathcal{O}_X(L)) \to \mathcal{O}_X(a(L - B) + E).$$

Then we have the following map

$$(9.1) \quad H^0(Y, \mathcal{O}_Y((al-1)H) \otimes \widehat{S}^a(f_* \mathcal{O}_X(L))) \otimes \mathcal{O}_X \to \mathcal{O}_X(a(L - B) + E + (al-1)f^* H).$$

By taking a suitable birational modification of X again (see Step 1), we may assume that the image of (9.1) is

$$\mathcal{O}_X(a(L - B) + E - F + (al-1)f^* H)$$

for some effective f-vertical divisor F on X. We may further assume that $\operatorname{Supp}(\Delta + B + E + F)$ is a simple normal crossing divisor on X. We put

$$N := a(L - B) + E - F + (al-1)f^* H.$$

Then $|N|$ is free by (9.1) and the definition of F.

Step 6. We take a positive number ε. Then, by Lemma 3.5, we obtain that $L - k(K_{X/Y} + \Delta) + \varepsilon f^* H$ is semi-ample because $L - k(K_{X/Y} + \Delta)$ is nef and f-semi-ample by assumption. We put

$$M := L - (K_{X/Y} + \Delta) - \frac{k-1}{k} B + \frac{k-1}{ak}(E - F) + \left(l - \left\lfloor \frac{l}{k} \right\rfloor\right) f^* H.$$

We note that

$$\frac{(al-1)(k-1)}{ak} < \left\lceil \frac{l(k-1)}{k} \right\rceil = l - \left\lfloor \frac{l}{k} \right\rfloor.$$

Then

$$M - \frac{k-1}{ak} N - \frac{1}{k}(L - k(K_{X/Y} + \Delta) + \varepsilon f^* H) = \alpha f^* H$$

for some $\alpha > 0$ if ε is sufficiently small. Thus M and $M - \alpha f^* H$ are semi-ample. Without loss of generality, we may assume that ε and α are rational numbers since

$$\left(l - \left\lfloor \frac{l}{k} \right\rfloor\right) - \frac{(al-1)(k-1)}{k} - \frac{\varepsilon}{k} = \alpha.$$

Step 7. We consider

$$\left\lfloor \frac{k-1}{k} B + \Delta \right\rfloor.$$

We put

$$B_0 = \max \left\{ T \,\middle|\, T \text{ is a Weil divisor with } 0 \leq T \leq B \text{ and } T \leq \left\lfloor \frac{k-1}{k} B + \Delta \right\rfloor \right\}.$$

We write

$$\left\lfloor \frac{k-1}{k} B + \Delta \right\rfloor - B_0 = \Delta_1 + \Delta_2$$

where Δ_1 is the horizontal part and Δ_2 is the vertical part. We note the following easy Claim.

Claim. *Let r be a real number with $0 \leq r \leq 1$, let k be a positive integer with $k \geq 2$, and let b be a nonnegative integer. Then*

$$\left\lfloor \frac{k-1}{k} b + r \right\rfloor - \min\left\{ b, \left\lfloor \frac{k-1}{k} b + r \right\rfloor \right\} = \begin{cases} 1 & \text{if } r = 1 \text{ and } b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Claim. We assume that

$$b \leq \left\lfloor \frac{k-1}{k} b + r \right\rfloor - 1$$

holds. Then

$$b \leq \frac{k-1}{k} b + r - 1.$$

Hence we obtain $0 \leq \frac{b}{k} \leq r - 1 \leq 0$. Thus we get $r = 1$ and $b = 0$. By this observation, we obtain the desired result. $\qquad\square$

By assumption (i) and the Claim, $\Delta_1 = \Delta_1^{=1}$. By construction and the Claim, we see that $\Delta_1 \subset \operatorname{Supp}\Delta^{=1}$ and that Δ_1 and $\operatorname{Supp}\{M\}$ have no common irreducible components.

Step 8. We have the following generically isomorphic injections:

$$f_*\mathcal{O}_X(K_X + \Delta_1 + \lceil M \rceil) \hookrightarrow \omega_Y((l - \lfloor l/k \rfloor)H) \otimes \left(f_*\mathcal{O}_X \left(L - \left\lfloor \frac{k-1}{k}B + \Delta \right\rfloor + \Delta_1 \right) \right)^{**}$$

$$= \omega_Y((l - \lfloor l/k \rfloor)H) \otimes (f_*\mathcal{O}_X(L - B_0 - \Delta_2))^{**}$$

$$\hookrightarrow \omega_Y((l - \lfloor l/k \rfloor)H) \otimes f_*\mathcal{O}_X(L).$$

We note that

$$f_*\mathcal{O}_X(L) = f_*\mathcal{O}_X(L - B) \subset f_*\mathcal{O}_X(L - B_0) \subset f_*\mathcal{O}_X(L).$$

This implies that

$$\mathcal{O}_Y(K_Y + (l - \lfloor l/k \rfloor)H) \otimes f_*\mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y because $f_*\mathcal{O}_X(K_X + \Delta_1 + \lceil M \rceil)$ is a mixed-ω-big-sheaf by Lemma 5.9.

Step 9. Let l_0 be the minimum positive integer such that

$$\mathcal{O}_Y(K_Y + l_0 H) \otimes f_*\mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y. By Theorem 6.3,

$$\mathcal{O}_Y(l_0 H) \otimes f_*\mathcal{O}_X(L)$$

is big. By the result obtained above,

$$\mathcal{O}_Y(K_Y + (l_0 - \lfloor l_0/k \rfloor)H) \otimes f_*\mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y. This implies that $l_0 - \lfloor l_0/k \rfloor \geq l_0$. Thus we get $l_0 \leq k - 1$. Hence we have

$$\mathcal{O}_Y(K_Y + (k - 1)H) \otimes f_*\mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y.

Thus we get the desired statements. $\qquad\square$

Remark 9.2. In Lemma 9.1, we further assume that (X, Δ) is klt over a nonempty Zariski open set of Y. Then we can easily see that $\Delta_1 = 0$ in Step 7 in the proof of Lemma 9.1. Therefore, we obtain that

$$\mathcal{O}_Y(K_Y + (l - \lfloor l/k \rfloor)H) \otimes f_*\mathcal{O}_X(L)$$

and

$$\mathcal{O}_Y(K_Y + (k - 1)H) \otimes f_*\mathcal{O}_X(L)$$

are pure-$\widehat{\omega}$-big-sheaves on Y.

Theorem 9.3 is the most important result in the theory of mixed-ω-sheaves. So we call it a fundamental theorem of the theory of mixed-ω-sheaves.

Theorem 9.3 ([N, Chapter V, 3.35. Theorem]). *Let $f: X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y. Let L be a Cartier divisor on X and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let D be an $\mathbb{R}$-divisor on Y. Let k be a positive integer with $k \geq 2$. Assume the following conditions:*

    (i) *(X, Δ) is log canonical over a nonempty Zariski open set of Y, and*
    (ii) *$L + f^*D - k(K_{X/Y} + \Delta) - f^*A$ is semi-ample for some big $\mathbb{R}$-divisor A on Y.*

If $f_* \mathcal{O}_Y(L) \neq 0$, then

$$\mathcal{O}_Y(K_Y + \lceil D \rceil) \otimes f_* \mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y.

Proof. We divide the proof into several small steps.

Step 1 (Reductions)**.** By taking a resolution as in Step 1 in the proof of Lemma 9.1, we may assume that X is a smooth projective variety and that $\mathrm{Supp}\,\Delta$ is a simple normal crossing divisor on X. We note that

$$L + f^*\lceil D \rceil - k\left(K_{X/Y} + \Delta + \frac{1}{k}f^*\{-D\}\right) - f^*A = L + f^*D - k(K_{X/Y} + \Delta) - f^*A.$$

Therefore, by replacing L and Δ with $L + f^*\lceil D \rceil$ and $\Delta + \frac{1}{k}f^*\{-D\}$, respectively, we may assume that $D = 0$. By Kodaira's lemma, we have $A \sim_{\mathbb{R}} A_1 + A_2$ such that A_1 is an ample $\mathbb{R}$-divisor and A_2 is an effective $\mathbb{R}$-divisor. By replacing A and Δ with A_1 and $\Delta + \frac{1}{k}f^*A_2$ respectively, we may further assume that A is an ample $\mathbb{R}$-divisor on Y. We take an ample Cartier divisor H on Y and a positive integer m such that $A - \frac{k-1}{m}H$ is ample. Then

$$L - k(K_{X/Y} + \Delta) - \frac{k-1}{m}f^*H$$

is semi-ample. Therefore, we may replace A with $\frac{k-1}{m}H$. By taking a resolution as in Step 1 in the proof of Lemma 9.1 again, we may assume that X is a smooth projective variety and that $\mathrm{Supp}\,\Delta$ is a simple normal crossing divisor on X. By Lemma 3.2, we may further assume that Δ is a $\mathbb{Q}$-divisor. We take an effective f-exceptional divisor E and replace L and Δ with $L + E$ and $\Delta + (1/k)E$ respectively. Then we may assume that $f_* \mathcal{O}_X(L)$ is reflexive. By taking a birational modification of X, we may assume that the image of

$$f^* f_* \mathcal{O}_X(L) \to \mathcal{O}_X(L)$$

is $\mathcal{O}_X(L - B)$ for some effective divisor B such that $\mathrm{Supp}(\Delta + B)$ is a simple normal crossing divisor on X. Let S denote the union of all f-exceptional divisors on X. We may assume that $\mathrm{Supp}(\Delta + B + S)$ is a simple normal crossing divisor on X by taking a suitable birational modification of X again (see Step 1 in the proof of Lemma 9.1).

Step 2. By Lemma 3.3 and Remark 3.4, we take a finite flat Galois cover $\tau \colon Y' \to Y$ from a smooth projective variety Y' and get the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\rho} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\tau]{} & Y \end{array}$$

such that $X' = X \times_Y Y'$ is a smooth projective variety, $\tau^*H \sim mH'$ for some ample Cartier divisor H' on Y', and $\rho^* \omega_{X/Y}^{\otimes n} = \omega_{X'/Y'}^{\otimes n}$ for every integer n. Let G denote the Galois group of $\tau \colon Y' \to Y$. By construction (see the proof of Lemma 3.3), we may assume that H' is G-invariant. We put $L' = \rho^*L$, $B' = \rho^*B$, $\Delta' = \rho^*\Delta$, and $S' = \rho^*S$. Without loss of generality, we may assume that $\mathrm{Supp}(\Delta' + B' + S')$ is a simple normal crossing divisor on X' and that $\rho^*(K_{X/Y} + \Delta) = K_{X'/Y'} + \Delta'$ holds (see Remark 3.4). We note that

$$L' - (k-1)f'^*H' - k(K_{X'/Y'} + \Delta') \sim_{\mathbb{Q}} \rho^*\left(L - k(K_{X/Y} + \Delta) - \frac{k-1}{m}f^*H\right)$$

by construction. This implies that $(L' - (k-1)f'^*H') - k(K_{X'/Y'} + \Delta')$ is semi-ample.

Step 3. We apply Lemma 9.1 to

$$(L' - (k-1)f'^*H') - k(K_{X'/Y'} + \Delta').$$

Then we obtain that $\mathcal{O}_{Y'}(K_{Y'}) \otimes f'_*\mathcal{O}_{X'}(L')$ is a mixed-$\widehat{\omega}$-big-sheaf on Y'. Therefore, $f'_*\mathcal{O}_{X'}(L')$ is a big sheaf on Y' by Theorem 6.3. Thus we can take a positive integer a such that $\widehat{S}^a(f'_*\mathcal{O}_{X'}(L'))$ is generically generated by global sections (see Lemma 4.4). Then we take an effective G-invariant f'-exceptional divisor E' on X' such that

$$(f'_*\mathcal{O}_{X'}(bL'))^{**} \simeq f'_*\mathcal{O}_{X'}(b(L' + E'))$$

holds for every $1 \leq b \leq a$. By replacing L', Δ', and B' with $L' + E'$, $\Delta' + (1/k)E'$, and $B' + E'$ respectively, we may assume that $f'_*\mathcal{O}_{X'}(bL')$ is reflexive for every $1 \leq b \leq a$.

Step 4. We can take an effective G-invariant f'-exceptional divisor E' on X' such that the surjective map

$$f'^*f'_*\mathcal{O}_{X'}(L') \to \mathcal{O}_{X'}(L' - B')$$

induces

$$f'^*\widehat{S}^a(f'_*\mathcal{O}_{X'}(L')) \to \mathcal{O}_{X'}(a(L' - B') + E').$$

Then we have the following map

$$(9.2) \qquad H^0(Y', \widehat{S}^a(f'_*\mathcal{O}_{X'}(L'))) \otimes \mathcal{O}_{X'} \to \mathcal{O}_{X'}(a(L' - B') + E').$$

By taking an equivariant resolution of singularities of X', we may assume that the image of (9.2) is

$$\mathcal{O}_{X'}(a(L' - B') + E' - F')$$

for some effective G-invariant f'-vertical divisor F' on X'. Of course, we may assume that $\mathrm{Supp}(\Delta' + B' + E' + F')$ is a simple normal crossing divisor on X'. We put

$$N' := a(L' - B') + E' - F'.$$

Then $|N'|$ is free by (9.2) and the definition of F'. We put

$$M' := L' - (K_{X'/Y'} + \Delta') - \frac{k-1}{k}B' + \frac{k-1}{ak}(E' - F').$$

Then

$$M' - \frac{k-1}{ak}N' - \frac{1}{k}(L' - k(K_{X'/Y'} + \Delta') - (k-1)f'^*H') = \frac{k-1}{k}f'^*H'.$$

In particular, M' and $M' - \frac{k-1}{k}f'^*H'$ are semi-ample.

Step 5. We put

$$\left\lfloor \frac{k-1}{k}B' + \Delta' \right\rfloor = B'_0 + \Delta'_1 + \Delta'_2$$

as in Step 7 in the proof of Lemma 9.1. Then Δ'_1 is a G-invariant f'-horizontal simple normal crossing divisor on X'. As before, $\mathrm{Supp}\{M'\}$ and Δ'_1 have no common irreducible components. Thus, by Lemma 5.9, $f'_*\mathcal{O}_{X'}(K_{X'} + \Delta'_1 + \lceil M' \rceil)$ is a mixed-ω-big-sheaf on Y'. Note that the Galois group G acts on $f'_*\mathcal{O}_{X'}(K_{X'} + \Delta'_1 + \lceil M' \rceil)$.

Step 6. Therefore, we get the following generically isomorphic G-equivariant embedding:

$$(9.3) \qquad f'_*\mathcal{O}_{X'}(K_{X'} + \Delta'_1 + \lceil M' \rceil) \hookrightarrow \mathcal{O}_{Y'}(K_{Y'}) \otimes f'_*\mathcal{O}_{X'}(L')$$

as in Step 8 in the proof of Lemma 9.1. We note that $f'_*\mathcal{O}_{X'}(L') \simeq \tau^* f_*\mathcal{O}_X(L)$ by the flat base change theorem. We take τ_* of (9.3) and then take the G-invariant parts. Thus, we get a mixed-ω-big-sheaf

$$\mathcal{F} := (\tau_* f'_*\mathcal{O}_{X'}(K_{X'} + \Delta'_1 + \lceil M' \rceil))^G$$

on Y and a generically isomorphic injection

$$\mathcal{F} \hookrightarrow \mathcal{O}_Y(K_Y) \otimes f_*\mathcal{O}_X(L).$$

This means that $\mathcal{O}_Y(K_Y) \otimes f_*\mathcal{O}_X(L)$ is a mixed-$\widehat{\omega}$-big-sheaf on Y.

Hence we obtain that $\mathcal{O}_Y(K_Y + \lceil D \rceil) \otimes f_*\mathcal{O}_X(L)$ is a mixed-$\widehat{\omega}$-big-sheaf on Y. $\qquad\square$

Remark 9.4. As in Remark 9.2, we further assume that (X, Δ) is klt over a nonempty Zariski open set of Y in Theorem 9.3. Then we see that $\Delta'_1 = 0$ in Step 5 in the proof of Theorem 9.3. Hence we obtain that

$$\mathcal{O}_Y(K_Y + \lceil D \rceil) \otimes f_*\mathcal{O}_X(L)$$

is a pure-$\widehat{\omega}$-big-sheaf on Y.

As a corollary of Theorem 9.3, we have:

Corollary 9.5 ([N, Chapter V. 3.37. Corollary]). *Let* $f \colon X \to Y$ *be a surjective morphism from a normal projective variety* X *onto a smooth projective variety* Y *with* $\dim Y = n$. *Let* L *be a Cartier divisor on* X *and let* Δ *be an effective* $\mathbb{R}$-*divisor on* X *such that* $K_X + \Delta$ *is* $\mathbb{R}$-*Cartier. Let* D *be an* $\mathbb{R}$-*divisor on* Y. *Let* k *be a positive integer with* $k \geq 2$. *Assume the following conditions:*

    (i) (X, Δ) *is log canonical over a nonempty Zariski open set of* Y, *and*
    (ii) $L + f^*D - k(K_{X/Y} + \Delta)$ *is nef and* f-*semi-ample.*

Then the following properties hold.

    (1) *There exists a Cartier divisor* G *on* Y, *which depends only on* $f \colon X \to Y$, *such that*

$$\mathcal{O}_Y(G + \lceil D \rceil) \otimes f_*\mathcal{O}_X(L)$$

      is generically generated by global sections.
    (2) *Let* H *be a big Cartier divisor on* Y *such that* $|H|$ *is free. Then*

$$\mathcal{O}_Y(K_Y + \lceil D \rceil + (n+1)H) \otimes (f_*\mathcal{O}_X(L))^{**}$$

      is generically generated by global sections.
    (3) *Let* $H^\dagger$ *be a nef and big Cartier divisor on* Y *such that* $|H^\dagger|$ *is not necessarily free. Then the sheaf*

$$\mathcal{O}_Y(K_Y + \lceil D \rceil + lH^\dagger) \otimes (f_*\mathcal{O}_X(L))^{**}$$

      is generically generated by global sections for $l \geq n^2 + 2$.

Proof. In Step 1, we will treat (2) and (3), which are direct consequences of Theorem 9.3. In Step 2, we will prove (1), which is much more difficult than (2) and (3).

Step 1. By Kodaira's lemma, we have $H \sim_{\mathbb{Q}} A + B$ such that A is an ample $\mathbb{Q}$-divisor and B is an effective $\mathbb{Q}$-divisor on Y. Let us consider

$$L + f^* H + f^* D - k(K_{X/Y} + \Delta) - f^* \left(\frac{1}{2} A + B \right).$$

Note that it is semi-ample by Lemma 3.5. We also note that $\frac{1}{2} A + B$ is big. Therefore, by Theorem 9.3,

$$\mathcal{O}_Y(K_Y + \lceil D \rceil + H) \otimes f_* \mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y. Thus, by Lemma 7.8,

$$\mathcal{O}_Y(K_Y + \lceil D \rceil + (n+1)H) \otimes (f_* \mathcal{O}_X(L))^{**}$$

is generically generated by global sections. By the same argument, we see that

$$\mathcal{O}_Y(K_Y + \lceil D \rceil + H^\dagger) \otimes f_* \mathcal{O}_X(L)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y. Thus, by Lemma 7.9, the sheaf

$$\mathcal{O}_Y(K_Y + \lceil D \rceil + l H^\dagger) \otimes (f_* \mathcal{O}_X(L))^{**}$$

is generically generated by global sections for $l \geq n^2 + 2$.

Step 2. By taking a resolution of singularities as in Step 1 in the proof of Lemma 9.1, we may assume that X is smooth. By replacing L and Δ with $L + f^* \lceil D \rceil$ and $\Delta + \frac{1}{k} f^* \{-D\}$ respectively, we may assume that $D = 0$. By the flattening theorem, there is a birational morphism $\tau \colon Y' \to Y$ from a smooth projective variety Y' such that the main component of $X \times_Y Y'$ is flat over Y'. Let X' be a resolution of the main component of $X \times_Y Y'$. Then we get the following commutative diagram.

$$\begin{array}{ccc} X' & \xrightarrow{\rho} & X \\ {\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f} \\ Y' & \xrightarrow{\tau} & Y \end{array}$$

By construction, any f'-exceptional divisor is ρ-exceptional. We put $K_{X'} + B = \rho^*(K_X + \Delta)$. We may assume that $\mathrm{Supp}\, B$ is a simple normal crossing divisor on X'. We write

$$(9.4) \qquad\qquad K_{Y'} = \tau^* K_Y + R$$

where R is an effective τ-exceptional divisor on Y'. We put

$$L' := \rho^* L + k \lceil -(B^{<0}) \rceil - k f'^* R$$

and $\Delta' = B + \lceil -(B^{<0}) \rceil$. Note that $\lceil -(B^{<0}) \rceil$ is effective and ρ-exceptional. Then we have

$$L' - k(K_{X'/Y'} + \Delta') = \rho^*(L - k(K_{X/Y} + \Delta)).$$

We take an effective f'-exceptional divisor E on X' such that

$$(f'_* \mathcal{O}_{X'}(L'))^{**} \simeq f'_* \mathcal{O}_{X'}(L' + E).$$

Note that E is ρ-exceptional and that there is a generically isomorphic injection

$$\tau_* f'_* \mathcal{O}_{X'}(L' + E) = f_* \rho_* \mathcal{O}_{X'}(L' + E) \subset f_* \mathcal{O}_X(L).$$

Therefore, we have a generically isomorphic injection

$$(9.5) \qquad\qquad \tau_* \left((f'_* \mathcal{O}_{X'}(L'))^{**} \right) \subset f_* \mathcal{O}_X(L).$$

By Kodaira's lemma, we have $\tau^*H \sim_{\mathbb{Q}} A + B$ such that A is an ample $\mathbb{Q}$-divisor and B is an effective $\mathbb{Q}$-divisor. Note that

$$L' + E + f'^*\tau^*H - k\left(K_{X'/Y'} + \Delta' + \frac{1}{k}E + \frac{1}{k}f'^*B\right) - \frac{1}{2}f'^*A$$

$$= L' - k(K_{X'/Y'} + \Delta') + \frac{1}{2}f'^*A$$

is semi-ample by Lemma 3.5. Therefore, by Theorem 9.3,

$$\mathcal{O}_{Y'}(K_{Y'} + \tau^*H) \otimes f'_*\mathcal{O}_{X'}(L' + E)$$

is a mixed-$\widehat{\omega}$-big-sheaf on Y'. Thus, by Lemma 7.8,

$$\mathcal{O}_{Y'}(K_{Y'} + (n+1)\tau^*H) \otimes (f'_*\mathcal{O}_{X'}(L'))^{**}$$

is generically generated by global sections. If we take a Cartier divisor G on Y such that $|\tau^*G - (K_{Y'} + (n+1)\tau^*H)| \neq \emptyset$, then

$$\mathcal{O}_{Y'}(\tau^*G) \otimes (f'_*\mathcal{O}_{X'}(L'))^{**}$$

is generically generated by global sections. By (9.5), we obtain that so is $\mathcal{O}_Y(G) \otimes f_*\mathcal{O}_X(L)$.

We complete the proof of Corollary 9.5. $\qquad\qquad\square$

We note that [N, Chapter V, 3.37. Corollary] needs the assumption that (X, Δ) is klt over a nonempty Zariski open set of Y. On the other hand, Corollary 9.5 can be applied to log canonical pairs. This is the main difference between [N, Chapter V, 3.37. Corollary] and Corollary 9.5.

Sho Ejiri pointed out the following example, which was constructed by Hiroshi Sato. For some related example, see Example 10.1 below.

Example 9.6 ([FG, Example 4.6]). There exists a flat toric morphism $f\colon X \to Y$ from a smooth projective toric threefold X onto $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$. Let Δ be the union of all torus invariant divisors on X. Then it is well known that (X, Δ) is log canonical with $K_X + \Delta \sim 0$. In this case,

$$f_*\mathcal{O}_X(k(K_{X/Y} + \Delta)) \otimes \mathcal{O}_Y(K_Y) \simeq \mathcal{O}_Y(-(k-1)K_Y)$$

holds for every integer k. Note that $-K_Y$ is not nef by $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$. Hence there are no ample Cartier divisors A on Y such that

$$f_*\mathcal{O}_X(k(K_{X/Y} + \Delta)) \otimes \mathcal{O}_Y(K_Y + A) \simeq \mathcal{O}_Y(-(k-1)K_Y + A)$$

is generated by global sections for every positive integer $k \geq 2$. We note that if $\mathcal{O}_Y(-(k-1)K_Y + A)$ were generated by global sections for every integer $k \geq 2$ then $-K_Y$ would be nef. This is a contradiction. Therefore, we can not replace generic generations with global generations in Corollary 9.5.

10. Proof of Theorems 1.7, 1.8, and 1.9

In this section, we prove Theorems 1.7, 1.8, and 1.9 in Section 1.

Let us first prove Theorem 1.8.

Proof of Theorem 1.8. We divide the proof into small steps.

Step 1. By taking a suitable resolution of singularities of X, we may assume that X is a smooth projective variety and $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X (see Step 1 in the proof of Lemma 9.1). We may further assume that every log canonical center of $(X, \Delta_{\mathrm{hor}})$ is dominant onto Y.

Step 2. In this step, we will prove the generic generation of $f_*\mathcal{O}_X(L) \otimes \mathcal{O}_Y(K_Y + lH)$ when $k = 1$.

By replacing L and Δ with $L - \lfloor\Delta_{\mathrm{ver}}\rfloor$ and $\Delta - \lfloor\Delta_{\mathrm{ver}}\rfloor$ respectively, we may further assume that (X, Δ) is dlt and that every log canonical center of (X, Δ) is dominant onto Y. By the arguments in Step 2 in the Proof of Lemma 7.7, we see that $f_*\mathcal{O}_X(L) \otimes \mathcal{O}_Y(K_Y + lH)$ is generically generated by global sections.

Step 3. In this step, we will see that $(f_*\mathcal{O}_X(L))^{**} \otimes \mathcal{O}_Y(K_Y + lH)$ is generically generated by global sections when $k \geq 2$.

This follows directly from Corollary 9.5. More precisely, we put $D = 0$ and apply Corollary 9.5 (2).

Step 4. In this final step, we treat the case when $s \geq 2$. We take the s-fold fiber product

$$X^s := \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{s}$$

of X over Y. Let $f^s \colon X^s \to Y$ be the induced morphism. Let $\rho \colon X^{(s)} \to X^s$ be a resolution of singularities of the dominant components of X^s such that ρ is an isomorphism over a nonempty Zariski open set of Y. We put $f^{(s)} = f^s \circ \rho \colon X^{(s)} \to Y$. We note that $X^{(s)}$ may be reducible, that is, a disjoint union of some smooth projective varieties. We can take a Zariski open set U of Y such that $\mathrm{codim}_Y(Y \setminus U) \geq 2$, $f_*\mathcal{O}_X(L)$ is locally free on U, and f is flat over U. By applying Lemma 3.7 to $f^{-1}(U) \to U$, we can construct a Cartier divisor $L^{(s)}$ on $X^{(s)}$ and an effective $\mathbb{R}$-divisor $\Delta^{(s)}$ on $X^{(s)}$ such that

$$L^{(s)} \sim_{\mathbb{R}} k(K_{X^{(s)}/Y} + \Delta^{(s)}),$$

$(X^{(s)}, \Delta^{(s)})$ is log canonical over a nonempty Zariski open set of Y, and there exists a generically isomorphic injection

$$\left(f_*^{(s)}\mathcal{O}_{X^{(s)}}(L^{(s)})\right)^{**} \subset \left(\bigotimes^{s} f_*\mathcal{O}_X(L)\right)^{**}.$$

By Theorem 9.3,

$$\mathcal{O}_Y(K_Y + H) \otimes f_*^{(s)}\mathcal{O}_{X^{(s)}}(L^{(s)})$$

is a finite direct sum of mixed-$\widehat{\omega}$-big-sheaves when $k \geq 2$. Note that $X^{(s)}$ may be reducible. Therefore,

$$\mathcal{O}_Y(K_Y + H) \otimes \left(\bigotimes^{s} f_*\mathcal{O}_X(L)\right)^{**}$$

is also a finite direct sum of mixed-$\widehat{\omega}$-big-sheaves. Thus, by Lemma 7.8,

$$\mathcal{O}_Y(K_Y + lH) \otimes \left(\bigotimes^{s} f_*\mathcal{O}_X(L)\right)^{**}$$

is generically generated by global sections for $l \geq n + 1$ when $k \geq 2$ (see also Corollary 9.5 (2)).

If $k = 1$, then we can check that $\mathcal{O}_Y(K_Y + lH) \otimes f_*^{(s)} \mathcal{O}_{X^{(s)}}(L^{(s)})$ is generically generated by global sections for $l \geq n + 1$ by the arguments in Steps 1 and 2. Therefore,

$$\mathcal{O}_Y(K_Y + lH) \otimes \left(\bigotimes^s f_* \mathcal{O}_X(L) \right)^{**}$$

is generically generated by global sections for $l \geq n + 1$ when $k = 1$.

Hence we have obtained the desired statements. $\qquad\square$

Next we prove Theorem 1.9.

Sketch of Proof of Theorem 1.9. It is not difficult to modify the proof of Theorem 1.8.

Step 1. In this step, we will treat the case when $k = 1$.

As usual, by taking a suitable birational modification of X, we may assume that X is smooth and $\text{Supp}\Delta$ is a simple normal crossing divisor on X. By replacing L and Δ with $L - \lfloor \Delta^{>1} \rfloor$ and $\Delta - \lfloor \Delta^{>1} \rfloor$ respectively, we may assume that Δ is a boundary $\mathbb{R}$-divisor on X. Note that $\Delta^{>1}$ is f-vertical. By perturbing the coefficients of Δ, we may further assume that Δ is a $\mathbb{Q}$-divisor with $L \sim_{\mathbb{Q}} K_{X/Y} + \Delta$. By Lemma 5.10,

$$\mathcal{O}_X(K_X + \lfloor \Delta \rfloor + \lceil L - K_{X/Y} - \Delta \rceil) \simeq \mathcal{O}_X(L) \otimes f^* \mathcal{O}_Y(K_Y)$$

is a mixed-ω-sheaf on X. Therefore, $f_* \mathcal{O}_X(L) \otimes \mathcal{O}_Y(K_Y)$ is a mixed-ω-sheaf on Y. Thus, by Lemma 7.9, $f_* \mathcal{O}_X(L) \otimes \mathcal{O}_Y(K_Y + lH^\dagger)$ is generically generated by global sections for $l \geq n^2 + 1$. Similarly, we may assume that the sheaf $f_*^{(s)} \mathcal{O}_{X^{(s)}}(L^{(s)}) \otimes \mathcal{O}_Y(K_Y)$ in the proof of Theorem 1.8 is a finite direct sum of mixed-ω-sheaves on Y when $k = 1$. Therefore,

$$\left(\bigotimes^s f_* \mathcal{O}_X(L) \right)^{**} \otimes \mathcal{O}_Y(K_Y + lH^\dagger)$$

is generically generated by global sections for $l \geq n^2 + 1$.

Step 2. In this step, we will treat the case when $k \geq 2$.

If we use Lemma 7.9 instead of Lemma 7.8, then the proof of Theorem 1.8 implies that

$$\mathcal{O}_Y(K_Y + lH^\dagger) \otimes \left(\bigotimes^s f_* \mathcal{O}_X(L) \right)^{**}$$

is generically generated by global sections for $l \geq n^2 + 2$ (see also Corollary 9.5 (3)).

Thus we get the desired statements. $\qquad\square$

Finally, we prove Theorem 1.7.

Proof of Theorem 1.7. We put $L = kK_{X/Y}$. Then this theorem directly follows from Theorems 1.8 and 1.9. $\qquad\square$

In [FF, Section 8], we constructed the following example, which shows that we can not replace the generic generation with the global generation in Conjecture 1.5.

Example 10.1. There exists a surjective morphism $f \colon X \to Y$ between smooth projective varieties with the following properties.

   (i) Y is a Kummer surface. In particular, $\omega_Y \simeq \mathcal{O}_Y$ holds.
   (ii) C_i is a (-2)-curve on Y for $1 \leq i \leq 16$.
   (iii) f is smooth over $U = Y \setminus \sum_{i=1}^{16} C_i$.

(iv) $L_{X/Y}$ is a Weil divisor on Y which is numerically equivalent to $\frac{1}{2}\sum_{i=1}^{16} C_i$.

(v) We have

$$f_*\omega_{X/Y}^{\otimes k} \simeq \begin{cases} \mathcal{O}_Y(\sum_{i=1}^{16} lC_i) & k = 2l, \\ \mathcal{O}_Y(L_{X/Y} + \sum_{i=1}^{16} lC_i) & k = 2l+1. \end{cases}$$

Let H be an ample Cartier divisor on Y. By Reider's theorem (see, for example, [BHPV, Chapter IV, (11.4) Theorem]), $|3H|$ is free. We note that $L_{X/Y} + 3H$ is ample by Nakai's ampleness criterion. Therefore, by Reider's theorem again, $|L_{X/Y} + 3H|$ is free. This means that

$$f_*\omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(3H)$$

is generated by global sections on U. On the other hand, we have

$$\left(f_*\omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(3H)\right) \cdot C_i = -k + 3H \cdot C_i.$$

Therefore, if $k > 3H \cdot C_{i_0}$ holds for some $1 \le i_0 \le 16$, then

$$f_*\omega_{X/Y}^{\otimes k} \otimes \omega_Y \otimes \mathcal{O}_Y(3H)$$

is not generated by global sections.

We close this section with an easy remark.

Remark 10.2. Let Y be a smooth projective variety and let H be an ample Cartier divisor on Y. Let m be any positive integer. Then we can construct a finite cover $f: X \to Y$ from a smooth projective variety X such that $\mathcal{O}_Y(-mH)$ is a direct summand of $f_*\mathcal{O}_X$. Therefore, we need the condition $k \ge 1$ in Theorems 1.7, 1.8, and 1.9.

11. Some other applications

In this section, we treat Nakayama's inequality on κ_σ and a slight generalization of the twisted weak positivity theorem. Theorem 11.3 and a special case of Theorem 11.7 have already played a crucial role in the theory of minimal models.

Let us first recall the definition of κ_σ for the reader's convenience.

Definition 11.1 (Nakayama's numerical dimension, see [N, Chapter V.2.5. Definition]). Let D be a pseudo-effective $\mathbb{R}$-Cartier divisor on a normal projective variety X and let A be a Cartier divisor on X. If $H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) \ne 0$ for infinitely many positive integers m, then we set

$$\sigma(D; A) = \max\left\{k \in \mathbb{Z}_{\ge 0} \;\middle|\; \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A))}{m^k} > 0\right\}.$$

If $H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) \ne 0$ only for finitely many $m \in \mathbb{Z}_{\ge 0}$, then we set $\sigma(D; A) = -\infty$. We define *Nakayama's numerical dimension* κ_σ by

$$\kappa_\sigma(X, D) = \max\{\sigma(D; A) \,|\, A \text{ is a Cartier divisor on } X\}.$$

It is well known that $\kappa_\sigma(X, D) \ge 0$ (see, for example, [N, Chapter V. 2.7. Proposition]). If D is not pseudo-effective, then we put $\kappa_\sigma(X, D) = -\infty$. By this convention, we can define $\kappa_\sigma(X, D)$ for every $\mathbb{R}$-Cartier divisor D on X. It is obvious that

$$\kappa_\sigma(X, D) \ge \kappa(X, D)$$

always holds for every $\mathbb{R}$-Cartier divisor D on X by definition, where $\kappa(X, D)$ denotes the *Iitaka dimension* of D.

For the details of $\kappa_\sigma(X, D)$ and $\kappa(X, D)$, we recommend the reader to see [N]. The following remark is easy but very useful.

Remark 11.2 ([N, Chapter V, 2.6. Remark (6)])**.** Let X be a smooth projective variety and let D be an $\mathbb{R}$-divisor on X. We put

$$\sigma(D; A)' = \max \left\{ k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \;\middle|\; \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lceil mD \rceil + A))}{m^k} > 0 \right\},$$

where A is a divisor on X. Then we have the following equality

$$\kappa_\sigma(X, D) = \max\{\sigma(D; A)' \mid A \text{ is a divisor}\}.$$

We will use this characterization of κ_σ in the proof of Theorem 11.3 below.

We note the following easy but important fact that $\kappa_\sigma(X, lD) = \kappa_\sigma(X, D)$ holds for every positive integer l (see [Fn11, Remark 2.2]), which will be useful in the proof of Theorem 11.3 below.

The inequalities in Theorem 11.3 are indispensable in the theory of minimal models (see Remarks 11.4 and 11.5).

Theorem 11.3 ([N, Chapter V, 4.1. Theorem (1)] and [Fn11, Section 3])**.** *Let $f \colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y with connected fibers. Let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and that (X, Δ) is log canonical over a nonempty Zariski open set of Y. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D - (K_{X/Y} + \Delta)$ is nef. Then, for any $\mathbb{R}$-divisor Q on Y, we have*

$$\kappa_\sigma(X, D + f^*Q) \geq \kappa_\sigma(F, D|_F) + \kappa(Y, Q)$$

and

$$\kappa_\sigma(X, D + f^*Q) \geq \kappa(F, D|_F) + \kappa_\sigma(Y, Q)$$

where F is a sufficiently general fiber of $f \colon X \to Y$.

Before we prove Theorem 11.3, we give two important remarks.

Remark 11.4. We think that one of the most important results of Nakayama's theory of ω-sheaves is the inequality on κ_σ in [N, Chapter V, 4.1. Theorem (1)]. However, as we explained in [Fn7, Remark 3.8] and [Fn11, Section 3], the proof of [N, Chapter V, 4.1. Theorem (1)] is incomplete. For the details, see, for example, [Fn11, Section 1]. So, in Theorem 11.3, we claim two weaker inequalities than Nakayama's original one (see [Fn11, (3.3) and (3.4)]). The first inequality in Theorem 11.3 is still sufficiently powerful for some geometric applications (see [Fn11, Section 3]).

Remark 11.5 (see [Fn11, Section 3])**.** The troubles in the proof of [DHP, Remark 2.6] and [GL, Theorem 4.3] caused by the incompleteness of [N, Chapter V, 4.1. Theorem (1)] can be corrected by using the first inequality in Theorem 11.3. For the details, we recommend the reader to see [HH, Lemma 2.11].

Let us prove Theorem 11.3.

Proof of Theorem 11.3. If Q is not pseudo-effective, then the desired inequalities are obviously true. So we may assume that Q is pseudo-effective. Similarly, we may further assume that $D|_F$ is pseudo-effective. As usual (see Step 1 in the proof of Lemma 9.1), we may assume that X is smooth and $\mathrm{Supp}\,\Delta$ is a simple normal crossing divisor on X

by the basic properties of κ_σ and κ. We take a sufficiently ample Cartier divisor A on X such that $A + \{-mD\}$ is ample for every integer m. Then

$$\lceil mD \rceil + A - m(K_{X/Y} + \Delta) = m(D - (K_{X/Y} + \Delta)) + A + \{-mD\}$$

is ample for every positive integer m. Then we can take an ample Cartier divisor H on Y such that $\mathcal{O}_Y(H) \otimes f_*\mathcal{O}_X(\lceil mD \rceil + A)$ is generically generated by global sections for every positive integer m by Corollary 9.5 (1). Thus there exists a generically isomorphic injection

$$\mathcal{O}_Y^{\oplus r(mD;A)} \hookrightarrow \mathcal{O}_Y(H) \otimes f_*\mathcal{O}_X(\lceil mD \rceil + A),$$

where $r(mD; A) := \mathrm{rank} f_*\mathcal{O}_X(\lceil mD \rceil + A)$. This induces the following injection

$$\mathcal{O}_Y(\lfloor mQ \rfloor + H)^{\oplus r(mD;A)} \hookrightarrow \mathcal{O}_Y(\lfloor mQ \rfloor + 2H) \otimes f_*\mathcal{O}_X(\lceil mD \rceil + A).$$

Therefore, we have

$$
\begin{aligned}
&\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(\lceil m(D + f^*Q) \rceil + A + 2f^*H) \\
(11.1) \quad &\geq \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(\lceil mD \rceil + f^*(\lfloor mQ \rfloor) + A + 2f^*H)) \\
&\geq r(mD; A) \cdot \dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(\lfloor mQ \rfloor + H))
\end{aligned}
$$

for every positive integer m. We can take a positive integer m_0 and a positive real number C_0 such that

$$(11.2) \qquad\qquad C_0 m^{\kappa(F, D|_F)} \leq r(mm_0 D; A)$$

for every large positive integer m (see, for example, [N, Chapter II, 3.7. Theorem]). Thus we have

$$
\begin{aligned}
(11.3) \quad &\dim H^0(X, \mathcal{O}_X(\lceil mm_0(D + f^*Q) \rceil + A + 2f^*H)) \\
&\geq C_0 m^{\kappa(F, D|_F)} \cdot \dim H^0(Y, \mathcal{O}_Y(\lfloor mm_0 Q \rfloor + H))
\end{aligned}
$$

for every large positive integer m by (11.1) and (11.2). We may assume that H is sufficiently ample. Then we get

$$(11.4) \qquad \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lceil mm_0(D + f^*Q) \rceil + A + 2f^*H))}{m^{\kappa(F, D|_F) + \kappa_\sigma(Y, Q)}} > 0$$

by (11.3) and the definition of $\kappa_\sigma(Y, Q)$. This means that the following inequality

$$(11.5) \qquad\qquad \kappa_\sigma(X, D + f^*Q) \geq \kappa(F, D|_F) + \kappa_\sigma(Y, Q)$$

holds.

Similarly, we can take a positive integer m_1 and a positive real number C_1 such that

$$
\begin{aligned}
(11.6) \quad C_1 m^{\kappa(Y, Q)} &\leq \dim H^0(Y, \mathcal{O}_Y(\lfloor mm_1 Q \rfloor)) \\
&\leq \dim H^0(Y, \mathcal{O}_Y(\lfloor mm_1 Q \rfloor + H))
\end{aligned}
$$

for every large positive integer m (see, for example, [N, Chapter II, 3.7. Theorem]) if H is a sufficiently ample Cartier divisor. Then, by (11.1) and (11.6), we have

$$
\begin{aligned}
(11.7) \quad &\dim H^0(X, \mathcal{O}_X(\lceil mm_1(D + f^*Q) \rceil + A + 2f^*H)) \\
&\geq C_1 m^{\kappa(Y, Q)} \cdot r(mm_1 D; A)
\end{aligned}
$$

for every large positive integer m. Therefore, we get

$$(11.8) \qquad \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lceil mm_1(D + f^*Q) \rceil + A + 2f^*H))}{m^{\kappa_\sigma(F, D|_F) + \kappa(Y, Q)}} > 0$$

when A is sufficiently ample. Note that

$$(11.9) \qquad \sigma(m_1 D|_F; A|_F)' = \max\left\{ k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \;\middle|\; \limsup_{m \to \infty} \frac{r(mm_1 D; A)}{m^k} > 0 \right\}$$

for a sufficiently general fiber F of $f\colon X \to Y$ and that

$$(11.10) \qquad \begin{aligned} \kappa_\sigma(F, D|_F) &= \kappa_\sigma(F, m_1 D|_F) \\ &= \max\{\sigma(m_1 D|_F; A|_F)' \mid A \text{ is very ample}\}. \end{aligned}$$

Hence we have the inequality

$$(11.11) \qquad \kappa_\sigma(X, D + f^* Q) \geq \kappa_\sigma(F, D|_F) + \kappa(Y, Q)$$

by (11.8). $\qquad \square$

It is highly desirable to solve the following conjecture. As we explained in [Fn11], Nakayama's original inequality on κ_σ (see [N, Chapter V, 4.1. Theorem (1)]) follows from Conjecture 11.6 and the argument in the proof of Theorem 11.3.

Conjecture 11.6 ([Fn11, Conjecture 1.4]). *Let X be a smooth projective variety and let D be a pseudo-effective $\mathbb{R}$-divisor on X. Then there exist a positive integer m_0, a positive rational number C, and an ample Cartier divisor A on X such that*

$$Cm^{\kappa_\sigma(X,D)} \leq \dim H^0(X, \mathcal{O}_X(\lfloor mm_0 D \rfloor + A))$$

holds for every large positive integer m.

Finally, we treat a slight generalization of the twisted weak positivity theorem.

Theorem 11.7 (Twisted weak positivity theorem). *Let $f\colon X \to Y$ be a surjective morphism from a normal projective variety X onto a smooth projective variety Y. Let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and that (X, Δ) is log canonical over a nonempty Zariski open set of Y. Let L be a Cartier divisor on X with $L \sim_{\mathbb{R}} k(K_{X/Y} + \Delta)$ for some positive integer k. Then the sheaf $f_* \mathcal{O}_X(L)$ is weakly positive.*

Proof. Let α be a positive integer and let $\mathcal{H}$ be an ample invertible sheaf on Y. By Theorem 1.8 or Theorem 1.9, we can take a positive integer β which depends only on Y such that

$$\left(\bigotimes^s f_* \mathcal{O}_X(L) \right)^{**} \otimes \mathcal{H}^{\otimes \beta}$$

is generically generated by global sections for every positive integer s. This implies that

$$\widehat{S}^{\alpha\beta}(f_* \mathcal{O}_X(L)) \otimes \mathcal{H}^{\otimes \beta}$$

is generically generated by global sections. This means that $f_* \mathcal{O}_X(L)$ is weakly positive. $\qquad \square$

12. On Iwai's theorem: Theorem 1.6

This section is independent of the other sections. Here we explain the following result due to Masataka Iwai. The proof of Theorem 12.1 is analytic and is completely different from the arguments in this paper.

Theorem 12.1 (Masataka Iwai). *Let $f: X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers and let $\mathcal{L}$ be an ample invertible sheaf on Y. Let U be the largest Zariski open set of Y such that f is smooth over U. We put $\dim Y = n$. Then*

$$f_* \omega_{X/Y}^{\otimes a} \otimes \omega_Y \otimes \mathcal{L}^{\otimes b}$$

is generated by global sections on U for all integers $a \geq 1$ and $b \geq \frac{n(n+1)}{2} + 1$.

Sketch of Proof. Here we will only explain how to modify the proof of [I, Theorem 1.4]. As in [I, Theorem 2.3], we take a smooth hermitian metric $h_{\mathcal{L}}$ on $\mathcal{L}$, a Kähler metric ω on Y, and a quasi-plurisubharmonic function φ on Y. Let h_a be the singular hermitian metric on $\omega_{X/Y}^{\otimes a}$ in [I, Theorem 2.4]. We put

$$\mathcal{L}^\dagger = \omega_{X/Y}^{\otimes(a-1)} \otimes f^* \mathcal{L}^{\otimes(N+1+b^\dagger)} \quad \text{and} \quad h_{\mathcal{L}^\dagger} = h_a^{\frac{a-1}{a}} f^* h_{\mathcal{L}}^{N+1+b^\dagger},$$

where $N = \frac{n(n+1)}{2}$ and $b^\dagger = b - (N+1) \geq 0$. Then we consider the adjoint bundle

$$\omega_X \otimes \mathcal{L}^\dagger \simeq \omega_{X/Y}^{\otimes a} \otimes f^*(\omega_Y \otimes \mathcal{L}^{\otimes b}).$$

In this situation, the proof of [I, Theorem 1.4] implies that

$$H^0(Y, f_*(\omega_X \otimes \mathcal{L}^\dagger)) \otimes \mathcal{O}_Y \to f_*(\omega_X \otimes \mathcal{L}^\dagger)$$

is surjective on U, equivalently,

$$H^0(Y, f_* \omega_{X/Y}^{\otimes a} \otimes \omega_Y \otimes \mathcal{L}^{\otimes b}) \otimes \mathcal{O}_Y \to f_* \omega_{X/Y}^{\otimes a} \otimes \omega_Y \otimes \mathcal{L}^{\otimes b}$$

is surjective on U. This is what we wanted. $\qquad\square$

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