LETTERS OF A MORI THEORIST
—ON THE LENGTHS OF EXTREMAL RAYS—

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Abstract. In this short note, we discuss the lengths of extremal rays. We also propose a conjecture characterizing projective space in terms of these lengths.

1. Introduction

In this short note, we work over the complex number field $\mathbb{C}$, except where stated otherwise in Theorem 1.1. As the reader may have already noticed, the title of this paper is an homage to Shokurov's *Letters of a Bi-rationalist.*

In [JLR], Jovikelly, Lehmann, and Riedl introduced a new method, called *Bend-and-Shatter*, and used it to generalize the classical Miyaoka–Mori result (see [MM]), which had previously been obtained via the traditional Bend-and-Break technique, as follows.

Theorem 1.1 ([JLR, Theorem 1.1])**.** *Let X be a projective variety over an algebraically closed field of arbitrary characteristic. Let H be a nef $\mathbb{R}$-Cartier divisor on X. Suppose there exists an irreducible curve $C \subset X$ contained in the smooth locus of X such that*

$$K_X \cdot C < 0.$$

Then for every closed point $x \in C$, there exists a rational curve R containing x such that

$$H \cdot R \leq (\dim X + 1)\frac{H \cdot C}{-K_X \cdot C}.$$

By combining Theorem 1.1 with [F6], we obtain the following results without any additional difficulty. We remark that the notion of quasi-log structures was originally introduced by Ambro (see [A]). For further details on the theory of quasi-log schemes, we refer the reader to [F3, Chapter 6] and [F5].

It is clear that the classical studies on the lengths of extremal rays, for example [Ka], which rely on [MM], can be refined by replacing [MM] with Theorem 1.1. Moreover, unlike the approach in [Ka], the method of [F6] applies even in the presence of singularities worse than kawamata log terminal. We believe that generalizations such as Theorems 1.2 and 1.3 are not merely of technical interest but are genuinely useful in applications (see Section 3).

Theorem 1.2 ([F6, Theorem 1.12])**.** *Let $[X, \omega]$ be a quasi-log scheme and let $\varphi\colon X \to W$ be a projective morphism of schemes such that $-\omega$ is φ-ample. Let P be an arbitrary closed point of W. Let E be any positive-dimensional irreducible component of $\varphi^{-1}(P)$ such that $E \not\subset X_{-\infty}$. Then E is covered by (possibly singular) rational curves ℓ with*

$$0 < -\omega \cdot \ell \leq \dim E + 1.$$

In particular, E is uniruled.

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Theorem 1.3 (Lengths of extremal rational curves, [F6, Theorem 1.13])**.** *Let $[X, \omega]$ be a quasi-log scheme, and let $\pi\colon X \to S$ be a projective morphism of schemes. Suppose that $R \subset \overline{\mathrm{NE}}(X/S)$ is an ω-negative extremal ray which is rational and relatively ample at infinity. Let $\varphi_R\colon X \to W$ be the contraction morphism over S associated to R. We put*

$$d = \min_E \dim E,$$

where E runs over positive-dimensional irreducible components of $\varphi_R^{-1}(P)$ for all $P \in W$. Then R is spanned by a (possibly singular) rational curve ℓ with

$$0 < -\omega \cdot \ell \leq d + 1.$$

Corollary 1.4 ([F6, Corollary 12.3])**.** *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X \to S$ be a projective morphism of schemes. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ that is rational and relatively ample at infinity. Let $\varphi_R\colon X \to W$ be the contraction morphism over S associated to R. We put*

$$d = \min_E \dim E,$$

where E runs over positive-dimensional irreducible components of $\varphi_R^{-1}(P)$ for all $P \in W$. Then R is spanned by a (possibly singular) rational curve ℓ with

$$0 < -(K_X + \Delta) \cdot \ell \leq d + 1.$$

Furthermore, if φ_R is birational and (X, Δ) is kawamata log terminal, then R is spanned by a (possibly singular) rational curve ℓ with

$$0 < -(K_X + \Delta) \cdot \ell < d + 1.$$

Let V be an irreducible component of the degenerate locus

$$\bigl\{x \in X \mid \varphi_R \text{ is not an isomorphism at } x\bigr\}$$

of φ_R. Then V is uniruled.

The following corollary is an immediate consequence of the cone and contraction theorem for quasi-log schemes (see [F3, Theorems 6.7.3 and 6.7.4]) together with Theorem 1.3.

Corollary 1.5 ([F4, Corollary 1.8])**.** *Let $[X, \omega]$ be a quasi-log canonical pair and let $\pi\colon X \to S$ be a projective morphism onto a scheme S. Let $\mathcal{L}$ be a π-ample line bundle on X. We put*

$$d := \max_{s \in S} \dim \pi^{-1}(s).$$

Then $\omega + (d+1)\mathcal{L}$ is π-nef. In particular, $\omega + (\dim X + 1)\mathcal{L}$ is always π-nef.

In any case, Theorem 1.2 and [FH, Theorem 1.8] (see Conjectures 1.15 and 1.21 in [F5]) represent the optimal results on this topic in full generality.

Throughout this paper, we adopt the notation and results established in [F2], [F3], [F5], and [F6].

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2. Comments on Theorems 1.2, 1.3, and Corollary 1.4

In this section, we present the proofs of Theorems 1.2, 1.3, and Corollary 1.4.

Proof of Theorem 1.2. In the final step of the proof of [F6, Theorem 1.12] (see [F6, p. 676] for details), we use

$$\begin{aligned} 0 < -\nu^*\omega \cdot \Gamma &\leq 2\dim \overline{E} \cdot \frac{-\nu^*\omega \cdot C}{-K_{\overline{E}} \cdot C} \\ &\leq 2\dim \overline{E}, \end{aligned}$$

which essentially goes back to [MM]. By applying Theorem 1.1, we can replace $2\dim \overline{E}$ with $\dim \overline{E} + 1$. This yields the desired inequality in Theorem 1.2. ☐

Proof of Theorem 1.3. The proof of [F6, Theorem 1.13] applies verbatim. The only change is to replace [F6, Theorem 1.12] with Theorem 1.2 in the proof of [F6, Theorem 1.13] (see [F6, p. 676] for details). ☐

Proof of Corollary 1.4. All we need to do is to replace [F6, Theorem 1.12] and [F6, Theorem 1.13] with Theorem 1.2 and Theorem 1.3, respectively, in the proof of [F6, Corollary 12.3] (see [F6, p. 677] for details). The desired statement then follows. ☐

For the sake of completeness, we also record here some remarks on other statements in [F6].

Remark 2.1. In the proof of [F6, Proposition 9.1] (see [F6, p. 665] for details), the bound

$$0 < -(K_X + \Delta) \cdot C \leq 2\dim X$$

was obtained by using [F2, Theorem 1.1], [F6, Theorem 1.12], or [F6, Corollary 12.3]. By replacing these results with Theorem 1.2 or Corollary 1.4, we can improve the bound to

$$0 < -(K_X + \Delta) \cdot C \leq \dim X + 1$$

in the statement of [F6, Proposition 9.1].

Since [F6, Theorem 1.8] is an application of [F6, Proposition 9.1], the same improvement applies; namely, we may replace

$$0 < -(K_X + \Delta) \cdot C \leq 2\dim X$$

with

$$0 < -(K_X + \Delta) \cdot C \leq \dim X + 1$$

in [F6, Theorem 1.8] (see [F6, p. 667–668] for details).

Consequently, because [F6, Theorem 9.2] is derived from [F6, Theorem 1.8], we can similarly replace

$$0 < -\mathcal{P} \cdot C \leq 2\dim X$$

with

$$0 < -\mathcal{P} \cdot C \leq \dim X + 1$$

(see [F4, pp. 668–669] for details).

Applying this once more to [F6, Theorem 1.6 (iii)], which depends on [F6, Theorem 9.2], we obtain the improved inequality

$$0 < -\omega \cdot C_j \leq \dim U_j + 1$$

in place of

$$0 < -\omega \cdot C_j \leq 2\dim U_j$$

(see [F6, p. 670] for details).

Finally, since [F6, Theorem 1.5] is a special case of [F6, Theorem 1.6], we may also replace

$$0 < -(K_X+\Delta)\cdot C_j \leq 2\dim U_j$$

with the sharper bound

$$0 < -(K_X+\Delta)\cdot C_j \leq \dim U_j + 1.$$

Moreover, Theorem 1.2 also implies [F6, Conjecture 1.21] (see also [F6, Remark 1.22]).

3. Toward a Characterization of Projective Space

We begin with a characterization of projective space for smooth projective varieties. To the best of the author's knowledge, Theorem 3.1 is new.

Theorem 3.1. *Let X be an n-dimensional smooth projective variety. Assume that there exists a K_X-negative extremal ray R of $\overline{\mathrm{NE}}(X)$ such that*

$$l(R) := \inf_{[C]\in R}\{-K_X\cdot C\} \geq n+1.$$

Then $X \simeq \mathbb{P}^n$.

Theorem 3.1 follows immediately from Corollary 1.4 together with [CMS].

Proof of Theorem 3.1. Let $\varphi_R\colon X\to W$ be the extremal contraction associated to R. By Corollary 1.4, W must be a point; equivalently, $\rho(X)=1$ and $-K_X$ is ample. Therefore, by [CMS, Corollary 0.3 or Corollary 0.4], we obtain $X\simeq\mathbb{P}^n$. This completes the proof. $\square$

We propose the following conjecture, which characterizes projective space in terms of the lengths of extremal rays.

Conjecture 3.2. *Let X be an n-dimensional projective variety, and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Assume that there exists a $(K_X+\Delta)$-negative extremal ray*

$$R \subset \overline{\mathrm{NE}}(X)$$

which is rational, relatively ample at infinity, and satisfies

$$l(R) := \inf_{[C]\in R}\bigl\{-(K_X+\Delta)\cdot C\bigr\} > n.$$

Then $X \simeq \mathbb{P}^n$.

If (X,Δ) is toric, then Conjecture 3.2 is already known to hold (see [F1]). We include here an important remark concerning the value $l(R)$ in Conjecture 3.2.

Remark 3.3. Let (X,Δ) be a projective log canonical pair and let R be a $(K_X+\Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X)$. It is well known that

$$l(R) = \min_{[C]\in R}\bigl\{-(K_X+\Delta)\cdot C\bigr\}, \tag{3.1}$$

that is, there exists a curve C on X with $[C]\in R$ such that

$$l(R) = -(K_X+\Delta)\cdot C.$$

If $K_X+\Delta$ is $\mathbb{Q}$-Cartier, then the equality (3.1) is immediate. When $K_X+\Delta$ is only $\mathbb{R}$-Cartier, the same conclusion follows by adapting the argument in the proof of [F3, Theorem 4.7.2 (1)].

By Remark 3.3, Conjecture 3.4 reduces to Conjecture 3.2.

Conjecture 3.4. *Let (X, Δ) be an n-dimensional projective log canonical pair such that*

$$-(K_X + \Delta) \cdot C > n$$

for every curve C on X. Then $X \simeq \mathbb{P}^n$.

If X is smooth and $\Delta = 0$, then Conjecture 3.4 is a special case of [CMS, Corollary 0.4] (see also [Ke]).

Proposition 3.5 follows directly from the statements in Section 1.

Proposition 3.5. *Let X be a projective variety, and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Assume that $\overline{\mathrm{NE}}(X)$ contains a $(K_X + \Delta)$-negative extremal ray R, which is rational and relatively ample at infinity, and whose length*

$$l(R) := \inf_{[C] \in R} \{-(K_X + \Delta) \cdot C\}$$

satisfies $l(R) > \dim X$. Then X is $\mathbb{Q}$-factorial with $\rho(X) = 1$, and (X, Δ) is terminal with $\lfloor \Delta \rfloor = 0$. In particular, (X, Δ) is kawamata log terminal and $-(K_X + \Delta)$ is ample.

We now prove Proposition 3.5.

Proof of Proposition 3.5. Let $\varphi_R \colon X \to W$ be the extremal contraction associated to R (see, e.g., [F2, Theorem 1.1]). By Corollary 1.4 together with the assumption $l(R) > \dim X$, the target W is a point. Consequently, $\rho(X) = 1$, the divisor $-(K_X + \Delta)$ is ample, and the non-lc locus $\mathrm{Nlc}(X, \Delta)$ satisfies $\dim \mathrm{Nlc}(X, \Delta) \le 0$.

Suppose that (X, Δ) is not kawamata log terminal. Assume that there exists a positive-dimensional log canonical center V of (X, Δ). By adjunction (see, e.g., [F3, Theorem 6.3.5 (i)]), the pair $[V, \omega]$, where $\omega := (K_X + \Delta)|_V$, is a quasi-log scheme, and $-\omega$ is ample. By Theorem 1.2, we obtain a rational curve $\ell \subset V$ such that

$$0 < -\omega \cdot \ell \le \dim V + 1.$$

This yields a rational curve $\ell \subset X$ satisfying

$$0 < -(K_X + \Delta) \cdot \ell \le \dim X,$$

which contradicts the assumption $l(R) > \dim X$. Therefore (X, Δ) has no positive-dimensional log canonical centers, and the non-klt locus $\mathrm{Nklt}(X, \Delta)$ is zero-dimensional since $\dim \mathrm{Nlc}(X, \Delta) \le 0$. By [F6, Theorem 1.17] (see also [FH, Theorem 1.7]), there exists a rational curve $\ell \subset X$ such that

$$0 < -(K_X + \Delta) \cdot \ell \le 1,$$

which again contradicts $l(R) > \dim X$. Hence (X, Δ) is kawamata log terminal.

By the minimal model program, there exists a projective birational morphism $f \colon \widetilde{X} \to X$ such that $(\widetilde{X}, \widetilde{\Delta})$ is a $\mathbb{Q}$-factorial terminal pair and

$$K_{\widetilde{X}} + \widetilde{\Delta} = f^*(K_X + \Delta).$$

Note that $-(K_{\widetilde{X}} + \widetilde{\Delta})$ is nef and big. Thus we may take a $(K_{\widetilde{X}} + \widetilde{\Delta})$-negative extremal ray $\widetilde{R} \subset \overline{\mathrm{NE}}(\widetilde{X})$ with $f_* \widetilde{R} = R$. Then $l(\widetilde{R}) > \dim \widetilde{X}$.

Let $\varphi_{\widetilde{R}} \colon \widetilde{X} \to \widetilde{W}$ be the associated contraction. By Corollary 1.4 again, $\widetilde{W}$ is a point, and hence $\rho(\widetilde{X}) = 1$. Therefore $f \colon \widetilde{X} \to X$ is an isomorphism. Consequently, (X, Δ) is a $\mathbb{Q}$-factorial terminal pair. This completes the proof. □

Let (X, Δ) be as in Conjecture 3.2. By assumption, there exists a $(K_X + \Delta)$-negative extremal ray

$$R \subset \overline{\mathrm{NE}}(X),$$

which is rational and relatively ample at infinity, and whose length satisfies $l(R) > n$. Thus, by Proposition 3.5, the variety X is $\mathbb{Q}$-factorial with only terminal singularities, $\rho(X) = 1$, and $-K_X$ is ample. In particular, $-K_X \cdot C > n$ for every curve C on X. Therefore Conjecture 3.4 reduces to the following more restrictive statement.

Conjecture 3.6. *Let X be an n-dimensional projective variety. Assume that X has only $\mathbb{Q}$-factorial terminal singularities, $-K_X$ is ample, and $\rho(X) = 1$. Suppose further that*

$$-K_X \cdot C > n$$

for every curve C on X. Then $X \simeq \mathbb{P}^n$.

Conjecture 3.6 remains open even in dimension three. It may be regarded as the principal outstanding problem in this direction.

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