Notes on [FJLR26]

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2026/2/20, version 0.00

1 The Terminal Fano Case

We work over the field $\mathbb{C}$ of complex numbers.

We extend the notion of "very free" from rational curves to curves of arbitrary genus. The defining feature of very free curves is that they deform with the expected dimension.

Definition 1.1. Let X be a projective variety. A morphism $s\colon C \to X$ from a smooth projective integral curve C is called *very free* if $s(C)$ is contained in the smooth locus of X and every nonzero quotient of s^*T_X has slope at least $2g(C)+1$.

By [JLR25a, Theorem 1.3], every terminal Fano variety admits a very free curve contained in its smooth locus. Moreover, the degree of such a curve can be taken arbitrarily large relative to its genus.

The following lemma shows that, in our setting, a general very free curve of sufficiently large degree passes through many general points of X.

Lemma 1.2. *Let X be a terminal Fano variety of dimension n, and let $c>0$ be such that cK_X is Cartier. Assume that every rational curve on X has anticanonical degree greater than n. Let $g, d > 0$ be integers, and define*

$$t = \max\left\{\left\lceil \frac{d}{n}\right\rceil - (cn+1)g - cn, 0\right\}.$$

Then every genus g very free curve $f\colon C \to X$ of anticanonical degree d admits a deformation sending t general points of C to t general points of X.

Proof of Lemma 1.2. By hypothesis, since cK_X is Cartier, every rational curve on X has anticanonical degree at least $n+\frac{1}{c}$.

Step 1. Let M_0 denote the irreducible component of the moduli space $\mathrm{Mor}(C,X)$ containing our original very free curve $f\colon C \to X$. We construct inductively a sequence of families $M_i \subset \mathrm{Mor}(C,X)$ as follows. Suppose that a family M_i has been constructed sending general points $q_1,\ldots,q_i \in C$ to fixed points $p_1,\ldots,p_i \in X$. Choose a general point $q_{i+1}\in C$, and let $Y_{i+1}\subset X$ be the subvariety swept out by $f(q_{i+1})$ as f varies over M_i. Fix a general point $p_{i+1}\in Y_{i+1}$, and let $M_{i+1}\subset M_i$ denote the subfamily of curves sending q_{i+1} to p_{i+1}. By construction, we have $\dim M_{i+1} = \dim M_i - \dim Y_{i+1}$.

Define $\theta(i)$ as the number of indices $j \le i$ for which $\dim Y_j < n$. Since $\dim M_i$ eventually drops to 0 while $\theta(i)$ becomes positive, there exists a largest integer k such that $\dim M_k \ge \theta(k)$. The family M_k sends $q_1,\ldots,q_k$ to $p_1,\ldots,p_k$ and has dimension between $\theta(k)$ and $\theta(k)+n$. For notational convenience, set $k_1=\theta(k)$ and $k_0 = k-\theta(k)$.

Our goal is to show that $k_0 \ge t$. The following properties of the construction will be useful:

- Since $\dim Y_i \geq \dim Y_{i+1}$ for all $i \geq 1$, the set $\{p_1, \ldots, p_{k_0}\}$ forms a general collection of k_0 points in X.
- Moreover, each p_i lies in the smooth locus of X, as it is a general point on a curve meeting the smooth locus.

Step 2. In this step, we provide additional clarification on the choice of $(q_1, \ldots, q_k)$ and $(p_1, \ldots, p_k)$ made in Step 1. As a result, we make precise in what sense one may fix $(q_1, \ldots, q_k)$ and vary $(p_1, \ldots, p_{k_0})$ in general position.

Claim. *Consider the morphism*

$$\Phi_k \colon C^k \times M_0 \longrightarrow (C \times X)^k, \qquad (c_1, \ldots, c_k, f) \longmapsto \big(c_1, f(c_1), \ldots, c_k, f(c_k)\big).$$

Then there exists a nonempty Zariski open subset $V \subset (C \times X)^k$ such that for any $(q_1, p_1, \ldots, q_k, p_k) \in V$, the dimensions $\dim Y_i$ and $\dim M_i$ arising in the construction of Step 1 are uniquely determined and independent of the particular choices of the points q_i and p_i.

Proof of Claim. We construct inductively nonempty Zariski open subsets

$$U_i \subset (C \times X)^{i-1} \times C \quad \text{and} \quad V_i \subset (C \times X)^i.$$

For each integer $1 \leq i \leq k$, consider the morphism

$$\Phi_i \colon C^i \times M_0 \longrightarrow (C \times X)^i, \qquad (c_1, \ldots, c_i, f) \longmapsto \big(c_1, f(c_1), \ldots, c_i, f(c_i)\big).$$

Via the natural identification

$$(C \times X)^i = (C \times X)^{i-1} \times C \times X,$$

we regard $\operatorname{Im} \Phi_i$ as a subset of $(C \times X)^{i-1} \times C \times X$ and consider the projection

$$\begin{array}{ccc} \operatorname{Im} \Phi_i & \hookrightarrow & (C \times X)^{i-1} \times C \times X \\ & & \big\downarrow \pi_i \\ & & (C \times X)^{i-1} \times C. \end{array}$$

Let $U_i \subset V_{i-1} \times C$ be a nonempty Zariski open subset such that the morphism

$$\pi_i \colon \operatorname{Im} \Phi_i \longrightarrow (C \times X)^{i-1} \times C$$

is flat over U_i.

Next, let $V_i \subset U_i \times X \subset (C \times X)^i$ be a nonempty Zariski open subset such that $V_i \cap \operatorname{Im} \Phi_i \neq \emptyset$ and the morphism

$$\Phi_i \colon C^i \times M_0 \longrightarrow \operatorname{Im} \Phi_i \subset (C \times X)^i$$

is flat over $V_i \cap \operatorname{Im} \Phi_i \neq \emptyset$.

Finally, we set $V := V_k$. By construction, this set satisfies the desired property. This completes the proof. $\square$

By the construction in Step 1, we choose $(q_1, \dots, q_k)$ and $(p_1, \dots, p_k)$ such that

$$(q_1, p_1, \dots, q_k, p_k) \in V.$$

We fix this choice of $(q_1, \dots, q_k) \in C^k$.

Let $\Sigma \subsetneq X^{k_0}$ be an arbitrary proper Zariski closed subset of X^{k_0}. Then we can choose points $(\widetilde{p}_1, \dots, \widetilde{p}_k)$ such that $(\widetilde{p}_1, \dots, \widetilde{p}_{k_0}) \notin \Sigma$ and

$$(q_1, \widetilde{p}_1, \dots, q_k, \widetilde{p}_k) \in V.$$

In this sense, we may move $(p_1, \dots, p_{k_0})$ in general position.

Step 3. Let $\pi\colon X' \to X$ be a resolution of singularities which is an isomorphism over the smooth locus of X. Note that every very free curve on X is contained in the locus where π is an isomorphism; in particular, such a curve has the same intersection numbers with $-K_X$ and $-K_{X'}$.

We choose a nonempty Zariski open subset $M_0^\circ \subset M_0$ suitably. Then we may assume that $f \in M_0^\circ \subset \mathrm{Mor}(C, X')$. Applying the construction of Step 1 to M_0°, we obtain points $(q_1, \dots, q_k) \in C^k$ and $(p_1, \dots, p_k) \in (X')^k$. From now on, we fix the choice of $(q_1, \dots, q_k) \in C^k$.

Consider the following morphisms:

$$\begin{array}{ccc} C \times M_0^\circ & \xrightarrow{\mathrm{ev}} & X', \qquad (c, \tilde{f}) \longmapsto \tilde{f}(c), \\ {\scriptstyle p}\big\downarrow & & \\ M_0^\circ & & \end{array}$$

where $p\colon C \times M_0^\circ \to M_0^\circ$ denotes the projection. Viewing the marked points $(q_1, \dots, q_k)$ as sections of p, the above diagram determines a family of stable maps to X'. Consequently, we obtain a morphism

$$\iota\colon M_0^\circ \longrightarrow \overline{\mathcal{M}}_{g,k}(X', \beta).$$

We further consider the morphism

$$\Psi\colon M_0^\circ \xrightarrow{\Psi_k} (X')^k \longrightarrow (X')^{k_0}, \qquad \tilde{f} \longmapsto \big(\tilde{f}(q_1), \dots, \tilde{f}(q_k)\big) \longmapsto \big(\tilde{f}(q_1), \dots, \tilde{f}(q_{k_0})\big).$$

By the observation in Step 2, the morphism Ψ is dominant. Moreover, the general fiber of $\Psi_k\colon M_0^\circ \to \Psi_k(M_0^\circ)$ has dimension at least $k_1 = \theta(k)$.

Similarly, using the first k_0 evaluation maps

$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,k}(X', \beta) \longrightarrow X', \qquad 1 \le i \le k_0,$$

we define a morphism

$$\Psi^\dagger\colon \overline{\mathcal{M}}_{g,k}(X', \beta) \longrightarrow (X')^{k_0}.$$

By construction, we have $\Psi^\dagger \circ \iota = \Psi$.

Let $\overline{\mathrm{Im}\,\iota}$ denote the closure of the image of ι in $\overline{\mathcal{M}}_{g,k}(X', \beta)$. We note that the boundary

$$\overline{\mathcal{M}}_{g,k}(X', \beta) \setminus \mathcal{M}_{g,k}(X', \beta)$$

admits a finite stratification.

Applying [JLR25b, Lemma 2.1] to general fibers over $\Psi_k(M_0^\circ)$, we obtain a locally closed substack

$$\mathcal{M} \subset \overline{\mathrm{Im}\,\iota} \setminus \mathcal{M}_{g,k}(X', \beta)$$

such that the restriction

$$\Psi^{\dagger}\colon \mathcal{M} \longrightarrow (X')^{k_0}$$

is dominant, and every stable map $f'\colon C' \to X'$ parametrized by $\mathcal{M}$ satisfies the following properties: the stabilization of C' is C, and the preimages of the points $q_{k-k_1+1}, \ldots, q_k \in C$ under the morphism $C' \to C$ are trees of rational curves. Moreover, the combinatorial type of the rational tails of C' is independent of the choice of f'.

For the remainder of the proof, we regard C as the unique irreducible component of C' of positive genus, and view the points $q_i \in C \subset C'$ as points of C'.

Step 4. We claim that

$$-K_X \cdot \pi_* f'_* C' \geq (k-1)n + \frac{1}{c}k_1.$$

First, suppose that in C' each of the points $q_1, \ldots, q_{k_0}$ has a rational tree attached. Let $p'_i \in X'$ denote the preimage $\pi^{-1}(p_i)$. Then for each p'_i, there exists a distinct rational component of C' that is not contracted by f' and whose image under f' contains p'_i. Since these rational curves are not contracted by π, it follows that

$$-K_X \cdot \pi_* f'_* C' \geq k\left(n + \frac{1}{c}\right) \geq (k-1)n + \frac{1}{c}k_1,$$

where the first inequality uses that X is Fano.

Next, suppose that exactly $m < k_0$ of the points $q_1, \ldots, q_{k_0}$ have rational trees attached in C'. Then $f'(C)$ contains the remaining $k_0 - m$ general points of X'. The twist-down of $f'^* T_{X'}|_C$ by all but one of these $k_0 - m$ points is generically globally generated. Since a generically globally generated vector bundle on C has nonnegative degree, we obtain

$$-K_X \cdot \pi_* f'_* C \geq -K_{X'} \cdot f'_* C \geq (k_0 - m - 1)n.$$

Accounting for the rational components as above, the total $-K_X$-degree of $\pi_* f'_* C'$ satisfies

$$(k_1 + m)\left(n + \frac{1}{c}\right) + (k_0 - m - 1)n \geq (k-1)n + \frac{1}{c}k_1,$$

which proves the claim.

Step 5. The dimension of M_0 is $d-(g-1)n$, while the dimension of M_k is at most k_1+n. Moreover, the codimension of M_k in M_0 is at most $n(k-k_1) + (n-1)k_1$. It follows that

$$d \leq (g-1)n + k_1 + n + nk - k_1 = (g+k)n.$$

Combining this with the bound from Step 4,

$$(k-1)n + \frac{1}{c}k_1 \leq d,$$

we obtain

$$(k-1)n + \frac{1}{c}k_1 \leq gn + kn,$$

which implies

$$k_1 \leq (g+1)nc.$$

Using the bound $d \leq (g+k)n$ above, we conclude that

$$k_0 = k - k_1 \geq \left\lceil \frac{d}{n} \right\rceil - g - (g+1)cn,$$

as desired.

This completes the proof of Lemma 1.2. □

Using Lemma 1.2, we select a suitable degeneration of a very free curve of sufficiently large anticanonical degree. This degeneration splits the curve into two components: a very free rational curve of degree $n+1$ and a higher-genus very free curve. The existence of the very free rational component allows us to apply [CMSB02], from which we conclude that $X \cong \mathbb{P}^n$.

Theorem 1.3. *Let X be a terminal Fano variety of dimension n such that every rational curve on X has anticanonical degree greater than n. Then X is isomorphic to $\mathbb{P}^n$.*

Proof of Theorem 1.3. We first show that X contains a very free rational curve of degree $n+1$ in its smooth locus.

Step 1. Choose a positive integer c such that cK_X is Cartier. Let $\pi \colon X' \to X$ be a resolution of singularities that is an isomorphism over the smooth locus.

By [JLR25a, Theorem 1.3], there exists a genus $g \geq 2$ such that X admits a very free curve C of genus g and arbitrarily large anticanonical degree. In particular, setting $\gamma = (1+nc)(g+1)$, we may assume that the anticanonical degree d of C satisfies

$$d > n(1+nc)(3g+\gamma). \tag{1.1}$$

Define

$$s = \left\lceil \frac{d}{n} \right\rceil - \gamma, \qquad \delta = d - (g-1)n - (s+1)n.$$

By construction, $s \geq 1$ and $\delta \geq 1$.

Let $q_0, q_1, \dots, q_s, r_1, \dots, r_{\delta-1}$ be general points on C, and let $p_0, p_1, \dots, p_s$ be general points on X. Let $D_1, \dots, D_{\delta-1}$ be general divisors in a very ample complete linear system $|H|$ on X.

By Lemma 1.2, there exists a δ-parameter family of maps $f \colon C \to X$ satisfying $f(q_i) = p_i$. From this, we may extract a 1-parameter subfamily M such that, in addition, $f(r_i) \in D_i$.

Taking the strict transforms of these maps to X' and applying Bend-and-Shatter, we obtain a limiting stable map

$$f' \colon C' \to X',$$

where C' has a rational tail attached to C at q_0. Since the image of q_0 does not lie in the π-exceptional locus, the rational tail is not contracted by $\pi \circ f'$.

For the remainder of the proof, we regard C as the unique irreducible component of C' with positive genus, and consider the points $q_i, r_j \in C \subset C'$ as points of C'.

Step 2. For the reader's convenience, we provide a more detailed description of the construction of the stable map $f' \colon C' \to X'$ in Step 1.

First, we fix general points $(q_0, \dots, q_s)$ on C so that the assumptions of Lemma 1.2 are satisfied. Next, we choose general points $(r_1, \dots, r_{\delta-1})$ on C and fix them once and for all.

Let N be the irreducible component of $\mathrm{Mor}(C, X)$ containing f. For each i, let

$$D_i := (t_i = 0) \in |H| = \mathbb{P}\big(H^0(X, \mathcal{O}_X(H))^\vee\big).$$

We consider the evaluation maps

$$\mathrm{ev}_{r_i} \colon N \to X, \qquad f \longmapsto f(r_i).$$

Define

$$\mathcal{D} := \left\{ (f, t_1, \dots, t_{\delta-1}) \in N \times |H|^{\delta-1} \;\middle|\; \mathrm{ev}_{r_i}^*(t_i)(f) = 0 \text{ for every } 1 \leq i \leq \delta-1 \right\} \subset N \times |H|^{\delta-1}.$$

Note that $\mathrm{ev}_{r_i}^*(t_i)(f) = t_i(f(r_i))$. Hence the condition $\mathrm{ev}_{r_i}^*(t_i)(f) = 0$ is equivalent to requiring that $f(r_i) \in D_i$.

We further consider the morphism

$$\Psi\colon \mathcal{D} \to X^{s+1} \times |H|^{\delta-1}, \qquad (f, t_1, \ldots, t_{\delta-1}) \longmapsto \big(f(q_0), \ldots, f(q_s), t_1, \ldots, t_{\delta-1}\big).$$

Here we have written $f(q_i) = \mathrm{ev}_{q_i}(f)$ for each i. By Lemma 1.2, Ψ is dominant and its general fiber has dimension at least 1.

We use the same notation as above. We take a suitable nonempty Zariski open subset $N^\circ \subset N$. Then we may regard N° as a subscheme of $\mathrm{Mor}(C, X')$. By replacing N and X with N° and X', respectively, we can define

$$\mathcal{D}' \subset N^\circ \times |H|^{\delta-1} \quad \text{and} \quad \Psi'\colon \mathcal{D}' \to (X')^{s+1} \times |H|^{\delta-1}$$

in the same way as above. Of course, Ψ' is dominant and its general fiber has dimension at least 1.

We now pass to the corresponding construction on the moduli space of stable maps. Consider the following morphisms:

$$\begin{array}{ccc} C \times N^\circ & \xrightarrow{\mathrm{ev}} & X', \qquad (c, \tilde{f}) \longmapsto \tilde{f}(c), \\ {\scriptstyle p}\big\downarrow & & \\ N^\circ & & \end{array}$$

where $p\colon C \times N^\circ \to N^\circ$ denotes the projection. Viewing the marked points $\{q_0, \ldots, q_s, r_1, \ldots, r_{\delta-1}\}$ as sections of p, the above diagram determines a family of stable maps to X'. Consequently, we obtain a morphism

$$\iota\colon N^\circ \longrightarrow \overline{\mathcal{M}}_{g,s+\delta}(X', \beta).$$

Let $\overline{\mathrm{Im}\,\iota}$ denote the closure of the image of ι.

Using the evaluation maps

$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,s+\delta}(X', \beta) \to X', \qquad 1 \le i \le s + \delta,$$

we define

$$\mathcal{D}^\dagger \subset \overline{\mathcal{M}}_{g,s+\delta}(X', \beta) \times |H|^{\delta-1} \quad \text{and} \quad \Psi^\dagger\colon \mathcal{D}^\dagger \to (X')^{s+1} \times |H|^{\delta-1}$$

in the same manner. By construction, we have $\Psi^\dagger \circ \iota = \Psi'$.

By the finiteness of the boundary strata

$$\overline{\mathcal{M}}_{g,s+\delta}(X', \beta) \setminus \mathcal{M}_{g,s+\delta}(X', \beta)$$

together with an application of Bend-and-Shatter, we can find a locally closed substack

$$\mathcal{N} \subset \overline{\mathrm{Im}\,\iota} \setminus \mathcal{M}_{g,s+\delta}(X', \beta)$$

such that the restriction

$$\Psi^\dagger\colon \mathcal{N} \to (X')^{s+1} \times |H|^{\delta-1}$$

is dominant and every stable map parametrized by $\mathcal{N}$ satisfies all the desired properties. This means that the stable map $f'\colon C' \to X'$ parametrized by $\mathcal{N}$ has the property that the stabilization of C' is C, and that the combinatorial type of the rational tails of C' does not depend on the choice of f'.

Step 3. We analyze the map $f'\colon C' \to X'$.

Let R_i denote the fiber of the stabilization map $C' \to C$ over q_i, and let T_j denote the fiber over r_j. Set

$$I = \{i \in \{1,\dots,s\} \mid R_i \text{ is a curve}\}, \quad J = \{j \in \{1,\dots,\delta-1\} \mid T_j \text{ is a curve}\}, \quad h = |I|.$$

We claim that $s-h \geq 3g$. Indeed, if not, then $h > s-3g$, and the total $-K_X$-degree of $\pi_* f'_* C'$ would satisfy

$$\begin{aligned}(h+1)\Big(n+\frac{1}{c}\Big) > (s-3g)\Big(n+\frac{1}{c}\Big) &= sn + \frac{s}{c} - 3g\Big(n+\frac{1}{c}\Big)\\ &\geq d - \gamma n + \frac{s}{c} - 3g\Big(n+\frac{1}{c}\Big) \geq d + \frac{d}{nc} - (3g+\gamma)\Big(n+\frac{1}{c}\Big),\end{aligned}$$

which exceeds d by (1.1), a contradiction. Hence $s-h\geq 3g$, so the spine curve $C \subset C'$ is not contracted by $\pi \circ f'$.

The deformations of $f'|_C\colon C \to X'$ that fix $f(q_i) = p_i$ for all but one index $i \in \{1,\dots,s\}\setminus I$ define a dominant family of curves. Thus, the twist-down of $f'^*T_{X'}|_C$ by the corresponding points $\{q_i\}$ is generically globally generated. Since $s-h-1 \geq 3g-1 \geq 2g+1$, every positive-rank quotient of $f'^*T_{X'}|_C$ has slope at least $2g+1$. Therefore, $f'|_C\colon C \to X'$ is very free and deforms with the expected dimension.

Next, we compute $-K_{X'}$ degrees. Because $f'|_C$ maps $s-h$ chosen points of C to general points in X' and $\delta - 1 - |J|$ chosen points to general divisors, we have

$$h^0(f'^*T_{X'}|_C) \geq (s-h)n + \delta - 1 - |J|.$$

Since $f'|_C$ is very free, $h^1(f'^*T_{X'}|_C) = 0$, so by Riemann–Roch the $-K_{X'}$-degree of f'_*C satisfies

$$-K_{X'} \cdot f'_*C \geq (s-h)n + \delta - 1 - |J| + (g-1)n.$$

Applying [JLR25a, Lemma 4.3] to $S = R_i$ for $i \in I$, each such R_i meets the rest of C' only along the spine curve, which is not contained in the π-exceptional locus. Hence

$$-K_{X'} \cdot f'_*R_i \geq -\pi^*K_X \cdot f'_*R_i > n.$$

Since the degree is integral, $-K_{X'} \cdot f'_*R_i \geq n+1$. Similarly, $-K_{X'} \cdot f'_*T_j \geq 1$ for each $j \in J$.

Thus, the total $-K_{X'}$-degree of f'_*C' satisfies

$$\begin{aligned} d = -K_{X'} \cdot f'_*C' &= -K_{X'} \cdot f'_*C + \sum_{i \in I \cup \{0\}} -K_{X'} \cdot f'_*R_i + \sum_{j \in J} -K_{X'} \cdot f'_*T_j \\ &\geq -K_{X'} \cdot f'_*C + (h+1)(n+1) + |J| \\ &\geq (s-h)n + \delta - 1 - |J| + (g-1)n + (h+1)(n+1) + |J| \\ &= (s+1)n + (g-1)n + \delta + h \\ &= d + h. \end{aligned}$$

It follows that $h = 0$, so every inequality above is an equality. In particular, $-K_{X'} \cdot f'_*R_0 = n+1$ and

$$-K_{X'} \cdot f'_*C = sn + (g-1)n + \delta - 1 - |J|.$$

Finally, since $f'|_C$ is very free, the space of its deformations has dimension $sn + \delta - 1 - |J|$. Hence, for general choices of the points $\{p_i\}$ and divisors $\{D_j\}$, there are only finitely many maps $f'|_C$ satisfying the incidence conditions

$$\pi \circ f'(q_i) = p_i \quad (1 \leq i \leq s), \qquad \pi \circ f'(r_j) \in D_j \quad (j \in \{1,\dots,\delta-1\}\setminus J).$$

Moreover, these incidence conditions are independent of the choice of $q_0 \in C$.

Step 4. Since p_0 is a general smooth point of X, the morphism

$$\pi\colon X' \to X$$

is an isomorphism over an open neighborhood of p_0. Set $p_0' := \pi^{-1}(p_0)$.

By [Deb01, Proposition 4.20], there exists a countable intersection of nonempty Zariski open subsets $V_{p_0'} \subset X'$ such that the restriction

$$\pi\colon V_{p_0'} \to \pi(V_{p_0'})$$

is an isomorphism, and any morphism

$$\psi\colon \mathbb{P}^1 \to X'$$

with $p_0' \in \psi(\mathbb{P}^1)$ and $\psi(\mathbb{P}^1) \cap V_{p_0'} \neq \emptyset$ is very free.

From the outset, we choose p_1 sufficiently general so that $p_1 \in \pi(V_{p_0'})$, and set $p_1' := \pi^{-1}(p_1)$. By construction, $p_1' \in f'|_C(C)$, hence

$$f'|_C(C) \cap V_{p_0'} \neq \emptyset.$$

As shown in Step 3, there are only finitely many possibilities for $f'|_C$. Therefore, after moving the attaching point q_0 to a sufficiently general point of C, we may assume

$$f'|_C(q_0) \in V_{p_0'}.$$

We claim that $\pi \circ f'(R_0)$ is irreducible. Indeed, if it were reducible, it would contain two components each of $(-K_X)$-degree strictly greater than n. However,

$$n + 1 = -K_{X'} \cdot f'_*R_0 \geq -K_X \cdot \pi_* f'_* R_0$$

by [JLR25a, Lemma 4.3], which yields a contradiction.

Thus $\pi \circ f'(R_0)$ is irreducible. It follows that there exists an irreducible component $F_0 \subset R_0$ such that

$$p_0' \in f'(F_0) \quad \text{and} \quad f'(R_0 \setminus F_0) \subset X' \setminus V_{p_0'}.$$

Since R_0 is attached to C at q_0, we have $f'(R_0) \cap V_{p_0'} \neq \emptyset$. Hence

$$p_0' \in f'|_{F_0}(F_0) \quad \text{and} \quad f'|_{F_0}(F_0) \cap V_{p_0'} \neq \emptyset.$$

Therefore,

$$f'|_{F_0}\colon F_0 \to X'$$

is a very free rational curve. In particular,

$$-K_{X'} \cdot f'_* F_0 = n + 1.$$

If R_0 were reducible, then [JLR25a, Lemma 4.3] applied to a connected component of $R_0 \setminus F_0$ would imply

$$-K_{X'} \cdot f'_* R_0 > n + 1,$$

a contradiction. Hence R_0 is irreducible.

By [JLR25a, Lemma 4.2], the image $f'(R_0)$ is disjoint from the π-exceptional locus.

Step 5. We have therefore produced a very free rational curve of degree $n+1$ whose image lies in the smooth locus of X.

Let M denote the irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$ containing such curves. By construction, M is doubly dominant in the sense of [CMSB02]. Moreover, the degree assumption ensures that curves in M cannot break under deformation. Thus M is closed, irreducible, maximal, and everywhere unsplit.

Applying [CMSB02, Theorem 4.2], we conclude that

$$X \cong \mathbb{P}^n.$$

We finish the proof of Theorem 1.3. □

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