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Kollár–Nadel Type Vanishing Theorem*

Osamu Fujino
Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
Email: fujino@math.sci.osaka-u.ac.jp

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Abstract. We prove an analytic generalization of Kollár's vanishing theorem, which contains the Nadel vanishing theorem as a special case.

Keywords: Injectivity theorem; Nadel vanishing theorem; Kollár vanishing theorem; Multiplier ideal sheaves.

1. Introduction

In this paper, I discussed about the Hodge theoretic aspect of injectivity and vanishing theorems (see [2, 3, 4]). Here, I will explain some analytic generalizations. In [6], Shin-ichi Matsumura and I established the following theorems.

Theorem 1.1. [6, Theorem A] *Let F be a holomorphic line bundle on a compact Kähler manifold X and let h be a singular hermitian metric on F. Let M be a holomorphic line bundle on X equipped with a smooth hermitian metric h_M. We assume that $\sqrt{-1}\Theta_{h_M}(M) \geq 0$ and $\sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_M}(M) \geq 0$ for some $a > 0$. Let s be a nonzero global section of M. Then the map*

$$\times s : H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \to H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes M)$$

induced by $\otimes s$ is injective for every i, where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

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Theorem 1.1 is a generalization of Enoki's injectivity theorem (see [1, Theorem 0.2]). Although the formulation of Theorem 1.1 may look artificial, it has many interesting applications (see [6]). Theorem 1.2 below is a Bertini-type theorem for multiplier ideal sheaves.

Theorem 1.2. [6, Theorem 1.10] *Let X be a compact complex manifold, let Λ be a free linear system on X with $\dim \Lambda \geq 1$, and let φ be a quasi-plurisubharmonic function on X. We put $\mathcal{G} = \{H \in \Lambda \,|\, H \text{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$. Then $\mathcal{G}$ is dense in Λ in the classical topology. Note that $\mathcal{J}(\varphi)$ is the multiplier ideal sheaf of φ.*

The main purpose of this paper is to prove the following theorem, which is a slight generalization of [6, Theorem D], as an application of Theorem 1.1 and Theorem 1.2.

Theorem 1.3. (Vanishing Theorem of Kollár–Nadel Type) *Let $f : X \to Y$ be a holomorphic map from a compact Kähler manifold X to a projective variety Y. Let F be a holomorphic line bundle on X equipped with a singular hermitian metric h. Let H be an ample line bundle on Y. Assume that there exists a smooth hermitian metric g on f^*H such that*

$$\sqrt{-1}\Theta_g(f^*H) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_h(F) - \varepsilon\sqrt{-1}\Theta_g(f^*H) \geq 0$$

for some $\varepsilon > 0$. Then we have $H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0$ for every $i > 0$ and j, where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf associated to the singular hermitian metric h.

We can easily see that Theorem 1.3 contains Demailly's original formulation of the Nadel vanishing theorem (see [6, Theorem 1.4]) and Kollár's vanishing theorem (see [7, Theorem 2.1 (iii)]) as special cases. Therefore, we call Theorem 1.3 the vanishing theorem of Kollár–Nadel type. For a related vanishing theorem, see [8, Theorem 1.3]. We note that we can find some relative generalizations of Theorems 1.1, 1.2, and 1.3 in [5] and [9].

In this paper, we will freely use the same notation as in [6].

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 as an application of Theorem 1.1 and Theorem 1.2. I hope that the following proof will show the reader how to use Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.3. We use the induction on $\dim Y$. If $\dim Y = 0$, then the statement is obvious. We take a sufficiently large positive integer m and a general member $B \in |H^{\otimes m}|$ such that $D = f^{-1}(B)$ is smooth, contains no associated primes of $\mathcal{O}_X/\mathcal{J}(h)$, and satisfies $\mathcal{J}(h|_D) = \mathcal{J}(h)|_D$ by Theorem 1.2. By the Serre vanishing theorem, we may further assume that

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) = 0 \tag{1}$$

for every $i > 0$ and j. We have the following big commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{J}(h) \otimes \mathcal{O}_X(-D) & \xrightarrow{\alpha} & \mathcal{J}(h) & \longrightarrow & \mathrm{Coker}\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \beta & & \\
0 & \longrightarrow & \mathcal{O}_X(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & (\mathcal{O}_X/\mathcal{J}(h)) \otimes \mathcal{O}_X(-D) & \xrightarrow{\gamma} & \mathcal{O}_X/\mathcal{J}(h) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since D contains no associated primes of $\mathcal{O}_X/\mathcal{J}(h)$, γ is injective. This implies that β is injective by the snake lemma and that $\mathrm{Coker}\alpha = \mathcal{J}(h)|_D = \mathcal{J}(h|_D)$. Thus we obtain the following short exact sequence:

$$0 \to \mathcal{J}(h) \otimes \mathcal{O}_X(-D) \to \mathcal{J}(h) \to \mathcal{J}(h|_D) \to 0.$$

By taking $\otimes \omega_X \otimes F \otimes \mathcal{O}_X(D)$ and using adjunction, we obtain the short exact sequence:

$$0 \to \omega_X \otimes F \otimes \mathcal{J}(h) \to \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* H^{\otimes m} \to \omega_D \otimes F|_D \otimes \mathcal{J}(h|_D) \to 0.$$

Therefore, we see that

$$\begin{aligned} 0 &\to R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \to R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m} \\ &\to R^j f_*(\omega_D \otimes F|_D \otimes \mathcal{J}(h|_D)) \to 0 \end{aligned} \tag{2}$$

is exact for every j since B is a general member of $|H^{\otimes m}|$. By induction on $\dim Y$, we have

$$H^i(B, R^j f_*(\omega_D \otimes F|_D \otimes \mathcal{J}(h|_D))) = 0 \tag{3}$$

for every $i > 0$ and j. By taking the long exact sequence associated to (2), we obtain

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m})$$

for every $i \geq 2$ and j by (3). Thus we have

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0 \tag{4}$$

for every $i \geq 2$ and j by (1). By Leray's spectral sequence and (1) and (4), we have the following commutative diagram:

$$\begin{array}{ccc} H^1(Y, \mathcal{F}^j) & \hookrightarrow & H^{j+1}(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ H^1(Y, \mathcal{F}^j \otimes H^{\otimes m}) & \hookrightarrow & H^{j+1}(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^*H^{\otimes m}) \end{array}$$

for every j, where $\mathcal{F}^j = R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$. Note that the horizontal arrows are injective. Since b is injective by Theorem 1.1, we obtain that a is also injective. By (1), we have

$$H^1(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) = 0$$

for every j. Therefore, we see that $H^1(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0$ for every j. Thus we obtain the desired vanishing theorem: Theorem 1.3. ■

We close this section with a remark on Nakano semipositive vector bundles.

Remark 2.1. Let E be a Nakano semipositive vector bundle on X. We can easily see that Theorem 1.3 holds even when ω_X is replaced by $\omega_X \otimes E$. We leave the details as an exercise for the reader (see [6, Section 6]).

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