Kodaira vanishing theorem for log-canonical and semi-log-canonical pairs

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Abstract: We prove the Kodaira vanishing theorem for log-canonical and semi-log-canonical pairs. We also give a relative vanishing theorem of Reid–Fukuda type for semi-log-canonical pairs.

Key words: semi-log-canonical pairs; log-canonical pairs; Kodaira vanishing theorem; vanishing theorem of Reid–Fukuda type.

1. Introduction The main purpose of this short paper is to establish:

Theorem 1.1 (Kodaira vanishing theorem for semi-log-canonical pairs)**.** *Let* (X, Δ) *be a projective semi-log-canonical pair and let* L *be an ample Cartier divisor on* X*. Then* $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ *for every* $i > 0$*.*

Theorem 1.1 is a naive generalization of the Kodaira vanishing theorem for semi-log-canonical pairs. As a special case of Theorem 1.1, we have:

Theorem 1.2 (Kodaira vanishing theorem for log-canonical pairs)**.** *Let* (X, Δ) *be a projective log-canonical pair and let* L *be an ample Cartier divisor on* X*. Then* $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ *for every* $i > 0$*.*

Precisely speaking, we prove the following theorem in this paper. Theorem 1.3 is a relative version of Theorem 1.1 and obviously contains Theorem 1.1 as a special case.

Theorem 1.3 (Main theorem)**.** *Let* (X, Δ) *be a semi-log-canonical pair and let* $f : X \to Y$ *be a projective morphism between quasi-projective varieties. Let* L *be an* f*-ample Cartier divisor on* X*. Then* $R^i f_* \mathcal{O}_X(K_X + L) = 0$ *for every* $i > 0$*.*

Although Theorem 1.3 has not been stated explicitly in the literature, it easily follows from [7], [8], [12], and so on. In our framework, Theorem 1.1 can be seen as a generalization of Kollár's vanishing theorem by the theory of mixed Hodge structures. The statement of Theorem 1.1 is a naive generalization of the Kodaira vanishing theorem. However, Theorem 1.1 is not a simple generalization of the Kodaira vanishing theorem from the Hodge-theoretic viewpoint.

We note the dual form of the Kodaira vanishing theorem for Cohen–Macaulay projective semi-log-canonical pairs.

Corollary 1.4 (cf. [17, Corollary 6.6])**.** *Let* (X, Δ) *be a projective semi-log-canonical pair and let* L *be an ample Cartier divisor on* X*. Assume that* X *is Cohen–Macaulay. Then* $H^i(X, \mathcal{O}_X(-L)) = 0$ *for every* $i < \dim X$*.*

Remark 1.5. The dual form of the Kodaira vanishing theorem, that is, $H^i(X, \mathcal{O}_X(-L)) = 0$ for every ample Cartier divisor L and every $i < \dim X$, implies that X is Cohen–Macaulay (see, for example, [16, Corollary 5.72]). Therefore, the assumption that X is Cohen–Macaulay in Corollary 1.4 is indispensable.

Remark 1.6. In [17, Corollary 6.6], Corollary 1.4 was obtained for *weakly* semi-log-canonical pairs (see [17, Definition 4.6]). Therefore, [17, Corollary 6.6] is stronger than Corollary 1.4. The arguments in [17] depend on the theory of Du Bois singularities. Our approach (see [3], [5], [7], [8], [9], [11], [12], and so on) to various vanishing theorems for reducible varieties uses the theory of mixed Hodge structures for cohomology with compact support and is different from [17].

Finally, we note that we can easily generalize Theorem 1.3 as follows.

Theorem 1.7 (Main theorem II)**.** *Let* (X, Δ) *be a semi-log-canonical pair and let* $f : X \to Y$ *be a projective morphism between quasi-projective varieties. Let* L *be a Cartier divisor on* X *such that* L *is nef and log big over* Y *with respect to* (X, Δ)*. Then* $R^i f_* \mathcal{O}_X(K_X + L) = 0$ *for every* $i > 0$*.*

For the definition of nef and log big divisors on semi-log-canonical pairs, see Definition 2.3. Theorem

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1.7 is a relative vanishing theorem of Reid–Fukuda type for semi-log-canonical pairs. It is obvious that Theorem 1.1, Theorem 1.2, and Corollary 1.4 hold true under the weaker assumption that L is nef and log big with respect to (X, Δ) by Theorem 1.7.

Throughout this paper, we will work over $\mathbb{C}$, the field of complex numbers. We will use the basic definitions and the standard notation of the minimal model program and semi-log-canonical pairs in [6], [7], [12], and so on.

2. Preliminaries In this section, we quickly recall some basic definitions and results for semi-log-canonical pairs for the reader's convenience. Throughout this paper, a variety means a reduced separated scheme of finite type over $\mathbb{C}$.

2.1 ($\mathbb{R}$-divisors)**.** Let D be an $\mathbb{R}$-divisor on an equidimensional variety X, that is, D is a finite formal $\mathbb{R}$-linear combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one. Note that $D_i \neq D_j$ for $i \neq j$ and that $d_i \in \mathbb{R}$ for every i. For every real number x, $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x + 1$. We put $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$, $D^{<1} = \sum_{d_i<1} d_i D_i$, and $D^{=1} = \sum_{d_i=1} D_i$. We call D a boundary (resp. subboundary) $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i.

Let us recall the definition of semi-log-canonical pairs.

Definition 2.2 (Semi-log-canonical pairs)**.** Let X be an equidimensional variety that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor such that no irreducible components of Δ are contained in the singular locus of X. The pair (X, Δ) is called a semi-log-canonical pair if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) (X^ν, Θ) is log-canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$.

A subvariety W of X is called an slc stratum with respect to (X, Δ) if there exist a resolution of singularities $\rho : Z \to X^\nu$ and a prime divisor E on Z such that $a(E, X^\nu, \Theta) = -1$ and $\nu \circ \rho(E) = W$ or if W is an irreducible component of X.

For the basic definitions and properties of log-canonical pairs, see [6]. For the details of semi-log-canonical pairs, see [7]. We need the notion of nef and log big divisors on semi-log-canonical pairs for Theorem 1.7

Definition 2.3 (Nef and log big divisors on semi-log-canonical pairs)**.** Let (X, Δ) be a semi-log-canonical pair and let $f : X \to Y$ be a projective morphism between quasi-projective varieties. Let L be a Cartier divisor on X. Then L is nef and log big over Y with respect to (X, Δ) if L is f-nef and $\mathcal{O}_X(L)|_W$ is big over Y for every slc stratum W of (X, Δ). We simply say that L is nef and log big with respect to (X, Δ) when $Y = \operatorname{Spec}\mathbb{C}$.

Roughly speaking, in [7], we proved the following theorem.

Theorem 2.4 (see [7, Theorem 1.2 and Remark 1.5])**.** *Let (X, Δ) be a quasi-projective semi-log-canonical pair. Then we can construct a smooth quasi-projective variety M with $\dim M = \dim X + 1$, a simple normal crossing divisor Z on M, a subboundary $\mathbb{R}$-divisor B on M, and a projective surjective morphism $h : Z \to X$ with the following properties.*

(1) *B and Z have no common irreducible components.*
(2) $\operatorname{Supp}(Z+B)$ *is a simple normal crossing divisor on M.*
(3) $K_Z + \Delta_Z \sim_{\mathbb{R}} h^*(K_X + \Delta)$ *such that $\Delta_Z = B|_Z$.*
(4) $h_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_X$.

By the properties (1), (2), (3), *and* (4), $[X, K_X + \Delta]$ *has a quasi-log structure with only qlc singularities. Furthermore, if the irreducible components of X have no self-intersection in codimension one, then we can make $h : Z \to X$ birational.*

For the details of Theorem 2.4, see [7]. In this paper, we do not discuss quasi-log schemes. For the theory of quasi-log schemes, see [5], [10], [12], and so on.

Remark 2.5. The morphism $h : (Z, \Delta_Z) \to X$ in Theorem 2.4 is called a quasi-log resolution. Note that the quasi-log structure of $[X, K_X + \Delta]$ obtained in Theorem 2.4 is compatible with the original semi-log-canonical structure of (X, Δ). For the details, see [7]. We also note that we have to know how to construct $h : Z \to X$ in [7, Section 4] for the proof of Theorem 1.3.

We note the notion of simple normal crossing pairs. It is useful for our purposes in this paper.

Definition 2.6 (Simple normal crossing pairs)**.** Let Z be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that $\operatorname{Supp}(B+Z)$ is a simple normal crossing divisor and that B and Z have no common irreducible components. We put $\Delta_Z = B|_Z$ and consider the pair (Z, Δ_Z). We call (Z, Δ_Z) a globally embedded simple normal crossing pair. A pair (Y, Δ_Y) is called a simple normal crossing pair if it is Zariski locally isomorphic to a globally embedded simple normal crossing pair.

If $(X, 0)$ is a simple normal crossing pair, then X is called a simple normal crossing variety. Let X be a simple normal crossing variety and let D be a Cartier divisor on X. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a simple normal crossing divisor on X.

Remark 2.7. Let X be a simple normal crossing variety and let D be a simple normal crossing divisor on X. Let D' be a Weil divisor on X such that $0 \lneqq D' \lneqq D$. Then D' is not necessarily a simple normal crossing divisor on X. However, if we further assume that D' is the support of some Cartier divisor, then D' is a simple normal crossing divisor on X.

For the details of simple normal crossing pairs, see [7, Definition 2.8], [8, Definition 2.6], [9, Definition 2.6], [10, Definition 2.4], [12, 5.2. Simple normal crossing pairs], and so on. We note that a simple normal crossing pair is called *semi-snc* in [15, Definition 1.10] (see also [1, Definition 1.1]) and that a globally embedded simple normal crossing pair is called an *embedded semi-snc pair* in [15, Definition 1.10].

3. Proof of Theorem 1.3 In this section, we prove Theorem 1.3 and discuss some related results.

Let us start with an easy lemma. The following lemma is more or less well-known to the experts.

Lemma 3.1 ([17, Lemma 3.15]). *Let X be a normal irreducible variety and let Δ be an effective $\mathbb{R}$-divisor on X such that (X, Δ) is log-canonical. Let $\rho : Z \to X$ be a proper birational morphism from a smooth variety Z such that $E = \mathrm{Exc}(\rho)$ and $\mathrm{Exc}(\rho) \cup \mathrm{Supp} f_*^{-1}\Delta$ are simple normal crossing divisors on Z. Let S be an integral divisor on X such that $0 \leq S \leq \Delta$ and let T be the strict transform of S. Then we have $\rho_*\mathcal{O}_Z(K_Z + T + E) \simeq \mathcal{O}_X(K_X + S)$.*

We give a proof of Lemma 3.1 here for the reader's convenience. The following proof is in [17].

Proof. We choose K_Z and K_X satisfying $\rho_* K_Z = K_X$. It is obvious that $\rho_*\mathcal{O}_Z(K_Z+T+E) \subset \mathcal{O}_X(K_X+S)$ since E is ρ-exceptional and $\mathcal{O}_X(K_X + S)$ satisfies Serre's S_2 condition. Therefore, it is sufficient to prove that $\mathcal{O}_X(K_X + S) \subset \rho_*\mathcal{O}_Z(K_Z + T + E)$. Note that we may assume that Δ is an effective $\mathbb{Q}$-divisor by perturbing the coefficients of Δ slightly. Let U be any nonempty Zariski open set of X. We will see that $\Gamma(U, \mathcal{O}_X(K_X + S)) \subset \Gamma(U, \rho_*\mathcal{O}_Z(K_Z + T + E))$. We take a nonzero rational function g of U such that $((g) + K_X + S)|_U \geq 0$, that is, $g \in \Gamma(U, \mathcal{O}_X(K_X + S))$, where (g) is the principal divisor associated to g. We assume that $U = X$ by shrinking X for simplicity. Let a be a positive integer such that $a(K_X + \Delta)$ is Cartier. We have $\rho^*(a(K_X + \Delta)) = aK_Z + a\Delta' + \Xi$, where Δ' is the strict transform of Δ and Ξ is a ρ-exceptional integral divisor on Z. By assumption, we have $0 \leq (g) + K_X + S \leq (g) + K_X + \Delta$. Then we obtain that

$$\begin{aligned} 0 &\leq (\rho^* g^a) + \rho^*(aK_X + a\Delta) \\ &\leq a\left((\rho^* g) + K_Z + \Delta' + E\right) \end{aligned}$$

since $\Xi \leq aE$. Thus we obtain $(\rho^* g) + K_Z + \Delta' + E \geq 0$.

Claim. $(\rho^* g) + K_Z + T + E \geq 0$.

Proof of Claim. By construction,

$$(\rho^* g) + K_Z + T + E = \rho_*^{-1}\left((g) + K_X + S\right) + F + E,$$

where every irreducible component of $F + E$ is ρ-exceptional. We also have

$$(\rho^* g) + K_Z + T + E = (\rho^* g) + K_Z + \Delta' + E - (\Delta' - T),$$

where $\Delta' - T$ is effective and no irreducible components of $\Delta' - T$ are ρ-exceptional. Note that $\rho_*^{-1}\left((g) + K_X + S\right) \geq 0$ and $(\rho^* g) + K_Z + \Delta' + E \geq 0$. Therefore, we have $(\rho^* g) + K_Z + T + E \geq 0$. □

This means that $\Gamma(U, \mathcal{O}_X(K_X + S)) \subset \Gamma(U, \rho_*\mathcal{O}_Z(K_Z + T + E))$ for any nonempty Zariski open set U. Thus, we have $\mathcal{O}_X(K_X + S) = \rho_*\mathcal{O}_Z(K_Z + T + E)$. □

We need the following remark for the proof of Theorem 1.7 in Section 4.

Remark 3.2. In Lemma 3.1, we put $E' = \sum E_i$ where E_i's are the ρ-exceptional divisors with $a(E_i, X, \Delta) = -1$. Then we see that $\rho_*\mathcal{O}_Z(K_Z+T+E') \simeq \mathcal{O}_X(K_X + S)$ by the proof of Lemma 3.1.

Although Theorem 1.2 is a special case of Theorem 1.1 and Theorem 1.3, we give a simple proof of Theorem 1.2 for the reader's convenience. For this purpose, let us recall an easy generalization of Kollár's vanishing theorem.

Theorem 3.3 ([2, Theorem 2.6]). *Let $f : V \to W$ be a morphism from a smooth projective variety V onto a projective variety W. Let D be a simple normal crossing divisor on V. Let H be an* *ample Cartier divisor on W. Then $H^i(W, \mathcal{O}_W(H) \otimes R^j f_* \mathcal{O}_V(K_V + D)) = 0$ for $i > 0$ and $j \geq 0$.*

For the proof, see [2, Theorem 2.6] (see also [4], [6, Sections 5 and 6], and so on). If $D = 0$ in Theorem 3.3, then Theorem 3.3 is nothing but Kollár's vanishing theorem. For more general results, see [4], [6], and so on (see also Theorem 3.7 below, [8], [12, Chapter 5], and so on, for vanishing theorems for reducible varieties).

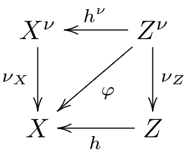
Let us start the proof of Theorem 1.2 (see [5, Corollary 2.9] when $\Delta = 0$).

Proof of Theorem 1.2. We take a projective birational morphism $\rho : Z \to X$ from a smooth projective variety Z such that $E = \mathrm{Exc}(\rho)$ and $\mathrm{Exc}(\rho) \cup \mathrm{Supp}\rho_*^{-1}\Delta$ are simple normal crossing divisors on Z. By Theorem 3.3, we obtain that $H^i(X, \mathcal{O}_X(L) \otimes \rho_* \mathcal{O}_Z(K_Z + E)) = 0$ for every $i > 0$. By Lemma 3.1, $\rho_* \mathcal{O}_Z(K_Z + E) \simeq \mathcal{O}_X(K_X)$. Therefore, we have $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ for every $i > 0$. □

The following key proposition for the proof of Theorem 1.3 is a generalization of Lemma 3.1.

Proposition 3.4. *Let (X, Δ) be a quasi-projective semi-log-canonical pair such that the irreducible components of X have no self-intersection in codimension one. Then there exist a birational quasi-log resolution $h : (Z, \Delta_Z) \to X$ from a globally embedded simple normal crossing pair (Z, Δ_Z) and a simple normal crossing divisor E on Z such that $h_* \mathcal{O}_Z(K_Z + E) \simeq \mathcal{O}_X(K_X)$.*

Proof. Since X is quasi-projective and the irreducible components of X have no self-intersection in codimension one, we can construct a birational quasi-log resolution $h : (Z, \Delta_Z) \to X$ by [7, Theorem 1.2 and Remark 1.5] (see Theorem 2.4), where (Z, Δ_Z) is a globally embedded simple normal crossing pair and the ambient space M of (Z, Δ_Z) is a smooth quasi-projective variety. By the construction of $h : Z \to X$ in [7, Section 4], $\mathrm{Sing}Z$, the singular locus of Z, maps birationally onto the closure of $\mathrm{Sing}X^{\mathrm{snc2}}$, where X^{snc2} is the open subset of X which has only smooth points and simple normal crossing points of multiplicity ≤ 2. We put $E = \mathrm{Exc}(h)$. Note that E contains no irreducible components of $\mathrm{Sing}Z$ by construction. If necessary, by taking a blow-up of Z along E and a suitable birational modification (see [1, Theorem 1.4]), we may assume that E is the support of some Cartier divisor, which is pure codimension one in Z. By taking a suitable birational modification again (see [1, Theorem 1.4]), we finally may assume that $E \cup \mathrm{Supp}h_*^{-1}\Delta$ and E are simple normal crossing divisors on Z (see Remark 2.7). In particular, (Z, E) is a simple normal crossing pair (see Definition 2.6). Note that [10, Section 8] may help us understand how to make (Z, Δ_Z) a globally embedded simple normal crossing pair. We may assume that the support of K_Z does not contain any irreducible components of $\mathrm{Sing}Z$ since Z is quasi-projective. We may also assume that $h_* K_Z = K_X$. Then we have $h_* \mathcal{O}_Z(K_Z + E) \subset \mathcal{O}_X(K_X)$ since $\mathcal{O}_X(K_X)$ satisfies Serre's S_2 condition and E is h-exceptional. We fix an embedding $\mathcal{O}_Z(K_Z + E) \subset \mathcal{K}_Z$, where $\mathcal{K}_Z$ is the sheaf of total quotient rings of $\mathcal{O}_Z$. Note that $h : Z \setminus E \to X \setminus h(E)$ is an isomorphism. We put $U = X \setminus h(E)$ and consider the natural open immersion $\iota : U \hookrightarrow X$. Then we have an embedding $\mathcal{O}_X(K_X) \subset \mathcal{K}_X$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$, by $\mathcal{O}_X(K_X) = \iota_*\big(h_* \mathcal{O}_Z(K_Z + E)|_U\big) \subset \iota_* \mathcal{K}_U = \mathcal{K}_X \big(= h_* \mathcal{K}_Z\big)$. Let $\nu_X : X^\nu \to X$ be the normalization and let $\mathcal{C}_{X^\nu}$ be the divisor on X^ν defined by the conductor ideal $\mathfrak{cond}_X$ of X (see, for example, [7, Definition 2.1]). Then we have $\mathcal{O}_X(K_X) \subset (\nu_X)_* \mathcal{O}_{X^\nu}(K_{X^\nu} + \mathcal{C}_{X^\nu})$. We put $K_{X^\nu} + \Theta = \nu_X^*(K_X + \Delta)$. Then $0 \leq \mathcal{C}_{X^\nu} \leq \Theta$ and (X^ν, Θ) is log-canonical by definition. Let $\nu_Z : Z^\nu \to Z$ be the normalization. Thus we have $K_{Z^\nu} + \mathcal{C}_{Z^\nu} = \nu_Z^* K_Z$, where $\mathcal{C}_{Z^\nu}$ is the simple normal crossing divisor on Z^ν defined by the conductor ideal $\mathfrak{cond}_Z$ of Z. Now we have the following commutative diagram.



By Lemma 3.1 and its proof, we see that $\mathcal{O}_{X^\nu}(K_{X^\nu} + \mathcal{C}_{X^\nu}) = h^\nu_* \mathcal{O}_{Z^\nu}(K_{Z^\nu} + \mathcal{C}_{Z^\nu} + \nu_Z^* E)$. Therefore, we obtain

$$(\spadesuit) \qquad \begin{aligned} \mathcal{O}_X(K_X) &\subset \varphi_* \mathcal{O}_{Z^\nu}(K_{Z^\nu} + \mathcal{C}_{Z^\nu} + \nu_Z^* E) \\ &= \varphi_* \mathcal{O}_{Z^\nu}(\nu_Z^*(K_Z + E)). \end{aligned}$$

We pick $s \in \Gamma(V, \mathcal{O}_X(K_X))$, where V is a Zariski open set of X. We can see $h^* s$ as an element of $\Gamma(h^{-1}(V), \mathcal{K}_Z)$. It is obvious that

$$h^* s|_{h^{-1}(V) \setminus E} \in \Gamma(h^{-1}(V) \setminus E, \mathcal{O}_Z(K_Z + E)).$$

Note that $h : Z \setminus E \to X \setminus h(E)$ is an isomorphism. We also note that ν_Z is an isomorphism over the generic point of any irreducible component of E. Therefore, by the inclusion ($\spadesuit$), we see that $h^* s$ is contained in $\Gamma(h^{-1}(V), \mathcal{O}_Z(K_Z+E))$. This implies that $\mathcal{O}_X(K_X) \subset h_*\mathcal{O}_Z(K_Z+E)$. Thus, we obtain $\mathcal{O}_X(K_X) = h_*\mathcal{O}_Z(K_Z+E)$ since $h_*\mathcal{O}_Z(K_Z+E) \subset \mathcal{O}_X(K_X)$. □

Remark 3.5. For the details of $\mathcal{K}_Z$ and $\mathcal{K}_X$, we recommend the reader to see the paper-back edition of [18, Section 7.1] published in 2006 (see also [14]). Note that the sheaf of total quotient rings is called the sheaf of stalks of meromorphic functions in [18].

Remark 3.6. As in Remark 3.2, in Proposition 3.4, we put $E' = \sum E_i$ where E_i's are the h-exceptional divisors with the discrepancy coefficient $a(E_i, X, \Delta)\big(= a(E_i, X^\nu, \Theta)\big) = -1$. By the usual perturbation technique, we may assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then Δ_Z is also $\mathbb{Q}$-Cartier. Thus, we see that $\Delta_Z^{=1}$ is a simple normal crossing divisor on Z. If necessary, by taking some blow-ups of Z, we may assume that $h_*^{-1}\Delta^{=1}$ is disjoint from Sing$Z$. In this case, $E' = \Delta_Z^{=1} - h_*^{-1}\Delta^{=1}$ is a simple normal crossing divisor on Z. Moreover, we have $h_*\mathcal{O}_Z(K_Z+E') \simeq \mathcal{O}_X(K_X)$ in Proposition 3.4. This easily follows from Remark 3.2 and the proof of Proposition 3.4.

For the proof of Theorem 1.3, we use the following vanishing theorem, which is obviously a generalization of Theorem 3.3. For the proof, see [8, Theorem 1.1] (see also [12, Chapter 5]).

Theorem 3.7 ([3], [8, Theorem 1.1], [12], and so on). *Let (Z, C) be a simple normal crossing pair such that C is a boundary $\mathbb{R}$-divisor on Z. Let $h : Z \to X$ be a proper morphism to a variety X and let $f : X \to Y$ be a projective morphism to a variety Y. Let D be a Cartier divisor on Z such that $D - (K_Z + C) \sim_{\mathbb{R}} h^*H$ for some f-ample $\mathbb{R}$-divisor H on X. Then we have $R^if_*R^jh_*\mathcal{O}_Z(D) = 0$ for every $i > 0$ and $j \geq 0$.*

Let us start the proof of Theorem 1.3.

Proof of Theorem 1.3. We take a natural finite double cover $p : \widetilde{X} \to X$ due to Kollár (see [7, Lemma 5.1]), which is étale in codimension one. Since $K_{\widetilde{X}} + \widetilde{\Delta} = p^*(K_X+\Delta)$ is semi-log-canonical and $\mathcal{O}_X(K_X)$ is a direct summand of $p_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$, we may assume that the irreducible components of X have no self-intersection in codimension one by replacing (X, Δ) with $(\widetilde{X}, \widetilde{\Delta})$. By Proposition 3.4, we can take a birational quasi-log resolution $h : (Z, \Delta_Z) \to X$ from a globally embedded simple normal crossing pair (Z, Δ_Z) such that there exists a simple normal crossing divisor E on Z satisfying $h_*\mathcal{O}_Z(K_Z+E) \simeq \mathcal{O}_X(K_X)$. Note that $K_Z + E + h^*L - (K_Z+E) = h^*L$. Therefore, we obtain that

$$\begin{aligned} &R^if_*\mathcal{O}_X(K_X+L) \\ &\simeq R^if_*\left(h_*\mathcal{O}_Z(K_Z+E)\otimes\mathcal{O}_X(L)\right) = 0 \end{aligned}$$

for every $i > 0$ by Theorem 3.7. □

Remark 3.8. If $\Delta = 0$ in Theorem 1.3, then Theorem 1.3 follows from [7, Theorem 1.7]. Note that the formulation of [7, Theorem 1.7] seems to be more useful for some applications than the formulation of Theorem 1.3.

Let (X, Δ) be a semi-log-canonical Fano variety, that is, (X, Δ) is a projective semi-log-canonical pair such that $-(K_X+\Delta)$ is ample (see [10, Section 6]). Then $H^i(X, \mathcal{O}_X) = 0$ for every $i > 0$ by [7, Theorem 1.7]. Unfortunately, this vanishing result for semi-log-canonical Fano varieties does not follow from Theorem 1.1. See also Remark 3.10 below.

Let us prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Theorem 1.1 is a special case of Theorem 1.3. By putting $Y = \mathrm{Spec}\mathbb{C}$ in Theorem 1.3, we obtain Theorem 1.1. □

Proof of Theorem 1.2. If (X, Δ) is log-canonical, then (X, Δ) is semi-log-canonical. Therefore, Theorem 1.2 is contained in Theorem 1.1. □

As a direct easy application of Theorem 1.1, we have:

Corollary 3.9. *Let X be a stable variety, that is, X is a projective semi-log-canonical variety such that K_X is ample. Then $H^i(X, \mathcal{O}_X((1+ma)K_X)) = 0$ for every $i > 0$ and every positive integer m, where a is a positive integer such that aK_X is Cartier.*

Remark 3.10. Let X be a stable variety as in Corollary 3.9. By [7, Corollary 1.9], we have already known that $H^i(X, \mathcal{O}_X(mK_X)) = 0$ for every $i > 0$ and every positive integer $m \geq 2$. This is an easy consequence of [7, Theorem 1.7].

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4. Since X is Cohen–Macaulay, we see that the vector space $H^i(X, \mathcal{O}_X(-L))$ is dual to $H^{\dim X - i}(X, \mathcal{O}_X(K_X + L))$ by Serre duality. Therefore, we have $H^i(X, \mathcal{O}_X(-L)) = 0$ for every $i < \dim X$ by Theorem 1.1. □

Remark 3.11. The approach to the Kodaira vanishing theorem explained in [17, Section 6] can not be directly applied to non-Cohen–Macaulay varieties. The above proof of Corollary 1.4 is different from the strategy in [17, Section 6].

4. Proof of Theorem 1.7 In this final section, we just explain how to modify the proof of Theorem 1.3 in order to obtain Theorem 1.7. We do not explain a generalization of Theorem 3.7 for nef and log big divisors (see [12, Theorem 5.7.3]), which is a main ingredient of the proof of Theorem 1.7 below.

Let us start the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $p : \widetilde{X} \to X$ be a natural finite double cover as in the proof of Theorem 1.3. Note that p^*L is nef and log big over Y with respect to $(\widetilde{X}, \widetilde{\Delta})$. Therefore, we may assume that the irreducible components of X have no self-intersection in codimension one by replacing (X, Δ) with $(\widetilde{X}, \widetilde{\Delta})$. We take a birational quasi-log resolution $h : (Z, \Delta_Z) \to X$ as in Proposition 3.4. Let E' be the divisor defined in Remark 3.5. In this case, L is nef and log big over Y with respect to $h : (Z, E') \to X$ (see [12, Definition 5.7.1]). Then we obtain that

$$\begin{aligned} &R^i f_* \mathcal{O}_X(K_X + L) \\ &\simeq R^i f_* \left(h_* \mathcal{O}_Z(K_Z + E') \otimes \mathcal{O}_X(L)\right) = 0 \end{aligned}$$

for every $i > 0$ by [12, Theorem 5.7.3] (see also [3, Theorem 2.47 (ii)] and [13, Theorem 6.3 (ii)]). Note that $K_Z + E' + h^*L - (K_Z + E') = h^*L$ and that the h-image of any stratum of (Z, E') is an slc stratum of (X, Δ) by construction (see Definition 2.2). □

Remark 4.1. For the details of the vanishing theorem for nef and log big divisors and some related topics, see [12, 5.7. Vanishing theorems of Reid–Fukuda type]. Note that [12] is a completely revised and expanded version of the author's unpublished manuscript [3].

Remark 4.2. We strongly recommend the reader to see Theorem 1.10, Theorem 1.11, and Theorem 1.12 in [7]. They are useful and powerful vanishing theorems for semi-log-canonical pairs related to Theorem 1.7.

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