KAWAMATA–VIEHWEG VANISHING THEOREM

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0.1. **Kawamata–Viehweg vanishing theorem.** In this subsection, we generalize Theorem z-vani ?? for the latter usage. The following theorem is well known as the Kawamata–Viehweg vanishing theorem.

kv-thm2 **Theorem 0.1** (cf. kmm [KMM, Theorem 1-2-3])**.** *Let X be a smooth variety and $\pi : X \to S$ a proper surjective morphism onto a variety S. Assume that a $\mathbb{Q}$-divisor D on X satisfies the following conditions*:

(i) *D is π-nef and π-big, and*
(ii) *$\{D\}$ has support with only normal crossings.*

Then $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$ for all $i > 0$.

Proof. We divide the proof into two steps.

stesteste1 **Step 1.** In this step, we treat a special case.

We prove the theorem under the conditions:

(1) D is π-ample, and
(2) $\{D\}$ has support with only simple normal crossings.

We can assume that S is affine since the statement is local. Then, by Lemma lem-co 0.2 below, we can assume that X and S are projective and D is ample by replacing D with $D + \pi^*A$, where A is a sufficiently ample Cartier divisor on S.

We take an ample Cartier divisor H on S and a positive integer m. Let us consider the following spectral sequence

$$\begin{aligned} E_2^{p,q} =& H^p(S, R^q\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner + m\pi^*H)) \\ \simeq& H^p(S, R^q\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) \otimes \mathcal{O}_S(mH)) \\ &\Rightarrow H^{p+q}(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner + m\pi^*H)). \end{aligned}$$

For every sufficiently large integer m, we have $E_2^{p,q} = 0$ for $p > 0$ by Serre's vanishing theorem. Therefore, $E_2^{0,q} = E_\infty^q$ holds for every q.

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This note will be contained in my book.

Thus, we obtain

$$\begin{aligned} &H^0(S, R^q\pi_*\mathcal{O}_X(K_X+\ulcorner D\urcorner+m\pi^*H)) \\ &= H^q(X, \mathcal{O}_X(K_X+\ulcorner D\urcorner+m\pi^*H))=0 \end{aligned}$$

for $q>0$ by Theorem [z-vani] ??. Since H is ample on S and m is sufficiently large,

$$\begin{aligned} &R^q\pi_*\mathcal{O}_X(K_X+\ulcorner D\urcorner+m\pi^*H) \\ &\simeq R^q\pi_*\mathcal{O}_X(K_X+\ulcorner D\urcorner)\otimes\mathcal{O}_S(mH) \end{aligned}$$

is generated by global sections. Therefore, we obtain

$$R^i\pi_*\mathcal{O}_X(K_X+\ulcorner D\urcorner)=0$$

for all $i>0$.

[stesteste2] **Step 2.** In this step, we treat the general case by using the result obtained in Step [stesteste1] 1.

Now we prove the theorem under the conditions (i) and (ii). We can assume that S is affine since the statement is local. By Kodaira's lemma and Hironaka's resolution theorem, we can construct a projective birational morphism $f:Y\to X$ from another smooth variety Y which is projective over S and divisors F_α's on Y such that $\mathrm{Supp} f^*D\cup(\underset{\alpha}{\cup}F_\alpha)$ is a simple normal crossing divisor on Y and that $f^*D-\sum\delta_\alpha F_\alpha$ is $\pi\circ f$-ample for some $\delta_\alpha\in\mathbb{Q}$ with $0<\delta_\alpha\ll 1$ (cf. [kmm] [KMM, Corollary 0-3-6]). Then by applying the result proved in Step [stesteste1] 1 to f, we obtain

$$0=R^if_*\mathcal{O}_Y(K_Y+\ulcorner f^*D-\sum\delta_\alpha F_\alpha\urcorner)=R^if_*\mathcal{O}_Y(K_Y+\ulcorner f^*D\urcorner)$$

for all $i>0$. We can also see that $f_*\mathcal{O}_Y(K_Y+\ulcorner f^*D\urcorner)\simeq\mathcal{O}_X(K_X+\ulcorner D\urcorner)$. So, we have, by the special case treated in Step [stesteste1] 1,

$$\begin{aligned} 0&=R^i(\pi\circ f)_*\mathcal{O}_Y(K_Y+\ulcorner f^*D-\sum\delta_\alpha F_\alpha\urcorner) \\ &=R^i\pi_*(f_*\mathcal{O}_Y(K_Y+\ulcorner f^*D\urcorner)) \\ &=R^i\pi_*\mathcal{O}_X(K_X+\ulcorner D\urcorner) \end{aligned}$$

for all $i>0$. □

We note that Theorem [kvn] ?? below is a complete generalization of Theorem [kv-thm2] 0.1. It is much stronger than [kmm] [KMM, Theorem 1-2-5].

We used the following lemma in the proof of Theorem [kv-thm2] 0.1. It is an application of Szabó's resolution lemma (cf. [15-resol] ??). We give a detailed proof for the reader's convenience.

lem-co **Lemma 0.2.** *Let $\pi : X \to S$ be a projective surjective morphism from a smooth variety X to an affine variety S. Let D be a $\mathbb{Q}$-divisor on X such that D is π-ample and* $\mathrm{Supp}\{D\}$ *is a simple normal crossing divisor on X. Then there exist a completion $\overline{\pi} : \overline{X} \to \overline{S}$ of $\pi : X \to S$ where $\overline{X}$ and $\overline{S}$ are both projective with $\overline{\pi}|_X = \pi$ and a $\overline{\pi}$-ample $\mathbb{Q}$-divisor $\overline{D}$ on $\overline{X}$ with $\overline{D}|_X = D$ such that* $\mathrm{Supp}\{\overline{D}\}$ *is a simple normal crossing divisor on $\overline{X}$.*

Proof. Let m be a sufficiently large and divisible integer such that the natural surjection

$$\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$$

induces an embedding of X into $\mathbb{P}_S(\pi_*\mathcal{O}_X(mD))$ over S. Let $\pi' : X' \to \overline{S}$ be an arbitrary completion of $\pi : X \to S$ such that X' and $\overline{S}$ are both projective and X' is smooth. We can construct such $\pi' : X' \to \overline{S}$ by Hironaka's resolution theorem. Let D' be the closure of D on $\overline{X}$. We consider the natural map

$$\pi'^*\pi'_*\mathcal{O}_{X'}(mD') \to \mathcal{O}_{X'}(mD').$$

The image of the above map can be written as

$$\mathcal{J} \otimes \mathcal{O}_{X'}(mD') \subset \mathcal{O}_{X'}(mD'),$$

where $\mathcal{J}$ is an ideal sheaf on X' such that $\mathrm{Supp}\mathcal{O}_{X'}/\mathcal{J} \subset X' \setminus X$. Let X'' be the normalization of the blow-up of X' by $\mathcal{J}$ and $f : X'' \to X'$ the natural map. We note that f is an isomorphism over $X \subset X'$. We can write $f^{-1}\mathcal{J} \cdot \mathcal{O}_{X''} = \mathcal{O}_{X''}(-E)$ for some effective Cartier divisor E on X''. By replacing X' with X'' and mD' with $mf^*D' - E$, we can assume that mD' is π-very ample over S and is π-generated over $\overline{S}$. Therefore, we can consider the morphism $\varphi : X' \to X''$ over $\overline{S}$ associated to the surjection

$$\pi'^*\pi'_*\mathcal{O}_{X'}(mD') \to \mathcal{O}_{X'}(mD') \to 0.$$

We note that φ is an isomorphsim over S by the construction. By replacing X' with X'', we can assume that D' is π'-ample. By using Hironaka's resolution theorem, we can further assume that X' is smooth. By Szabó's resolution lemma (cf. 15-resol ??), we can make $\mathrm{Supp}\{D'\}$ simple normal crossing. Thus, we obtain desired completions $\overline{\pi} : \overline{X} \to \overline{S}$ and $\overline{D}$. □

Viehweg's formulation of the Kawamata–Viehweg vanishing theorem is slightly different from Theorem kv-thm2 0.1. We do not treat it in this book because the formulation of Theorem kv-thm2 0.1 is much more suitable than Viehweg's for various applications in the log minimal model program. For the details of Viehweg's formulation, see fuji99 [F99, Section 3]. We contain the statement for the reader's convenience. We note that the condition (i′) in the following theorem is slightly weaker than (i) in Theorem 0.1.

vie-vani **Theorem 0.3.** *Let X be a smooth variety and $\pi : X \to S$ a proper surjective morphism onto a variety S. Assume that a $\mathbb{Q}$-divisor D on X satisfies the following conditions*:

(i′) *D is π-nef and $\ulcorner D \urcorner$ is π-big, and*
(ii) *$\{D\}$ has support with only normal crossings.*

Then $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$ for all $i > 0$.

Let us generalize Theorem 0.1 for $\mathbb{R}$-divisors. We will repeatedly use it in the subsequent chapters.

kv-thm3 **Theorem 0.4** (Kawamata–Viehweg vanishing theorem for $\mathbb{R}$-divisors)**.** *Let X be a smooth variety and $\pi : X \to S$ a proper surjective morphism onto a variety S. Assume that an $\mathbb{R}$-divisor D on X satisfies the following conditions*:

(i) *D is π-nef and π-big, and*
(ii) *$\{D\}$ has support with only normal crossings.*

Then $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$ for all $i > 0$.

Proof. When D is π-ample, we perturb the coefficients of D and can assume that D is a $\mathbb{Q}$-divisor. Then, by Theorem 0.1, we obtain $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$ for all $i > 0$. By using this special case, Step 2 in the proof of Theorem 0.1 works without any changes. So, we obtain this theorem. □

As a corollary, we obtain the vanishing lemma of Reid–Fukuda type. It will play important roles in the subsequent chapters. Before we state it, we prepare the following definition.

Definition 0.5 (Nef and log big divisors)**.** Let $f : V \to W$ be a proper surjective morphism from a smooth variety and B a boundary $\mathbb{R}$-divisor on V such that Supp$B$ is a simple normal crossing divisor. We put $T = \llcorner B \lrcorner$ and $T = \sum_{i=1}^{m} T_i$ is the irreducible decomposition. Let G be an $\mathbb{R}$-divisor on V. We say that G is f-nef and f-log big if and only if G is f-nef, f-big, and $G|_C$ is $f|_C$-big for every C, where C is an irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k}$ for some $\{i_1, \cdots, i_k\} \subset \{1, \cdots, m\}$.

vani-rf-le **Lemma 0.6** (Vanishing lemma of Reid–Fukuda type)**.** *Let V be a smooth variety and B a boundary $\mathbb{R}$-divisor on V such that* Supp$B$ *is a simple normal crossing divisor. Let $f : V \to W$ be a proper morphism onto a variety W. Assume that D is a Cartier divisor on V such that*

$D-(K_V+B)$ is f-nef and f-log big. Then $R^if_*\mathcal{O}_V(D)=0$ for all $i>0$.

Proof. We use the induction on the number of irreducible components of $\llcorner B\lrcorner$ and on the dimension of V. If $\llcorner B\lrcorner=0$, then the lemma follows from the Kawamata–Viehweg vanishing theorem (cf. Theorem kv-thm3 0.4). Therefore, we can assume that there is an irreducible divisor $S\subset \llcorner B\lrcorner$. We consider the following short exact sequence

$$0\to \mathcal{O}_V(D-S)\to \mathcal{O}_V(D)\to \mathcal{O}_S(D)\to 0.$$

By induction, we see that $R^if_*\mathcal{O}_V(D-S)=0$ and $R^if_*\mathcal{O}_S(D)=0$ for all $i>0$. Thus, we have $R^if_*\mathcal{O}_V(D)=0$ for $i>0$. □

References

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kmm [KMM]

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