

MINIMAL MODEL THEORY FOR LOG SURFACES II

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ABSTRACT. We treat three different topics on log surfaces. The first topic is the minimal model theory for \mathbb{Q} -factorial log surfaces and log canonical surfaces in Fujiki's class \mathcal{C} . The second one is how to construct complete non-projective log canonical algebraic surfaces with several properties. The final one is a generalization of the minimal model theory for log surfaces over any algebraically closed field.

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1. INTRODUCTION

We treat three different topics on log surfaces. The first topic (Sections 2–11) is the minimal model theory for \mathbb{Q} -factorial log surfaces and log canonical surfaces in Fujiki's class \mathcal{C} . It is the main part of this paper. The second one (Section 12) is how to construct complete non-projective log canonical algebraic surfaces with several properties. The final one (Section 13) is a generalization of the minimal model theory for log surfaces over any algebraically closed field. Note that each topic can be read independently.

A log surface (X, Δ) in Fujiki's class \mathcal{C} consists of a compact normal analytic surface X that is bimeromorphically equivalent to a compact Kähler manifold and a \mathbb{Q} -divisor Δ on X whose coefficients are in $[0, 1] \cap \mathbb{Q}$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier, that is, there exists a positive integer m such that $m\Delta$ is integral and $(\omega_X^{\otimes m} \otimes \mathcal{O}_X(m\Delta))^{**}$ is locally free, where ω_X is the canonical sheaf of X .

One of the main results of the first topic is as follows, which is a generalization of the minimal model theory for projective \mathbb{Q} -factorial log surfaces established in [Fn4].

Date: 2019/7/25, version 0.28.

2010 Mathematics Subject Classification. Primary 14E30; Secondary 32J27.

Key words and phrases. log surfaces, log canonical surfaces, Fujiki's class \mathcal{C} , minimal model program, abundance theorem, complete non-projective algebraic surfaces.

Theorem 1.1 (Minimal model theory for \mathbb{Q} -factorial log surfaces in Fujiki's class \mathcal{C}). *Let (X, Δ) be a \mathbb{Q} -factorial log surface in Fujiki's class \mathcal{C} . Then we can construct a finite sequence of projective bimeromorphic morphisms starting from (X, Δ)*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} (X_k, \Delta_k) =: (X^*, \Delta^*)$$

such that (X_i, Δ_i) , where $\Delta_i := \varphi_{i-1} \Delta_{i-1}$, is a \mathbb{Q} -factorial log surface in Fujiki's class \mathcal{C} and that $\text{Exc}(\varphi_i) =: C_i \simeq \mathbb{P}^1$ and $-(K_{X_i} + \Delta_i) \cdot C_i > 0$ for every i . The final model (X^*, Δ^*) satisfies one of the following conditions.*

- (i) (Good minimal model). $K_{X^*} + \Delta^*$ is semi-ample.
- (ii) (Mori fiber space). There exists a surjective morphism $g : X^* \rightarrow W$ onto a normal projective variety W with connected fibers such that $-(K_{X^*} + \Delta^*)$ is g -ample, $\dim W < 2$, and the relative Picard number $\rho(X^*/W)$ is one.

We note that

- (1) if X_{i_0} is projective for some i_0 then X_i is automatically projective for every i , and
- (2) if X_{i_0} has only rational singularities for some i_0 then all the singularities of X_i are rational for every i .

We also note that the above sequence of contraction morphisms is nothing but the minimal model program for projective \mathbb{Q} -factorial log surfaces established in [Fn4] when X is projective and that X is automatically projective when $\kappa(X, K_X + \Delta) = -\infty$ or 2.

Theorem 1.1 is not difficult to check once we know the minimal model theory for projective \mathbb{Q} -factorial log surfaces in [Fn4], the Enriques–Kodaira classification of compact complex surfaces (see, for example, [BHPV, Chapter VI]), and some basic results on complex analytic spaces.

We note that Theorem 1.1 includes the abundance theorem for \mathbb{Q} -factorial log surfaces in Fujiki's class \mathcal{C} .

Theorem 1.2 (Abundance theorem for \mathbb{Q} -factorial log surfaces in Fujiki's class \mathcal{C} , see Theorem 7.2). *Let (X, Δ) be a \mathbb{Q} -factorial log surface in Fujiki's class \mathcal{C} . Assume that $(K_X + \Delta) \cdot C \geq 0$ for every curve C on X . Then $K_X + \Delta$ is semi-ample.*

From the minimal model theoretic viewpoint, it is very natural to treat log canonical surfaces (X, Δ) in Fujiki's class \mathcal{C} . Unfortunately, X is not necessarily \mathbb{Q} -factorial in this case. So we can not directly apply Theorem 1.1 to log canonical surfaces in Fujiki's class \mathcal{C} . In order to establish the minimal model theory for log canonical surfaces in Fujiki's class \mathcal{C} , we first prove the following theorem.

Theorem 1.3 (Projectivity of log canonical surfaces in Fujiki's class \mathcal{C} with negative Kodaira dimension, see Theorem 9.1). *Let (X, Δ) be a log canonical surface in Fujiki's class \mathcal{C} . Assume that $\kappa(X, K_X + \Delta) = -\infty$ holds. Then X is projective.*

The proof of Theorem 1.3 is much more difficult than we expected. We prove it with the aid of the classification of two-dimensional log canonical singularities. We note that there are non-projective normal complete rational surfaces (see [Ng, Section 4]). Fortunately, such surfaces do not appear under the assumption of Theorem 1.3. Since Nagata's example in [Ng, Section 4] is not log canonical, we explicitly construct some examples of complete non-projective log canonical algebraic surfaces in Section 12 for the reader's convenience. Of course, this is the second topic of this paper. Here, we explain the most interesting example.

Example 1.4 (see Example 12.3). There exists a complete non-projective log canonical algebraic surface S with $\text{Pic}(S) = \{0\}$ and $K_S \sim 0$. In particular, $\kappa(S, K_S) = 0$ holds.

For the details of Example 1.4 and some other examples of complete non-projective algebraic surfaces, see Section 12, where the reader can find some examples of complete non-projective log canonical algebraic surfaces S with $\text{Pic}(S) = \{0\}$, $\overline{NE}(S) = \mathbb{R}_{\geq 0}$, or $\overline{NE}(S) = N_1(S)$. We note that Section 12 is based on a joint work with Kento Fujita.

Anyway, by using Theorem 1.3, we have the following minimal model theory for log canonical surfaces in Fujiki's class \mathcal{C} .

Theorem 1.5 (Minimal model theory for log canonical surfaces in Fujiki's class \mathcal{C}). *Let (X, Δ) be a log canonical surface in Fujiki's class \mathcal{C} . Then we can construct a finite sequence of projective bimeromorphic morphisms starting from (X, Δ)*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} (X_k, \Delta_k) =: (X^*, \Delta^*)$$

such that (X_i, Δ_i) , where $\Delta_i := \varphi_{i-1*} \Delta_{i-1}$, is a log canonical surface in Fujiki's class \mathcal{C} and that $\text{Exc}(\varphi_i) =: C_i \simeq \mathbb{P}^1$ and $-(K_{X_i} + \Delta_i) \cdot C_i > 0$ for every i . The final model (X^*, Δ^*) satisfies one of the following conditions.

- (i) (Good minimal model). $K_{X^*} + \Delta^*$ is semi-ample.
- (ii) (Mori fiber space). There exists a surjective morphism $g : X^* \rightarrow W$ onto a normal projective variety W with connected fibers such that $-(K_{X^*} + \Delta^*)$ is g -ample, $\dim W < 2$, and the relative Picard number $\rho(X^*/W)$ is one.

We note that

- (1) if X_{i_0} is projective for some i_0 then X_i is automatically projective for every i ,
- (2) if X_{i_0} has only rational singularities for some i_0 then all the singularities of X_i are rational for every i , and
- (3) if X_{i_0} is \mathbb{Q} -factorial for some i_0 then so is X_i for every i .

We also note that the above sequence of contraction morphisms is nothing but the usual minimal model program for projective log canonical surfaces (see [Fn4]) when X is projective and that X is automatically projective when $\kappa(X, K_X + \Delta) = -\infty$ by Theorem 1.3.

In a series of papers (see [HP1], [HP2], and [CHP]), Campana, Höring, and Peternell established the minimal model program and the abundance theorem for Kähler threefolds (see also [HP3]). Their approach is essentially analytic. On the other hand, our approach is much more elementary than theirs and is not analytic. Although we mainly treat compact analytic surfaces in Fujiki's class \mathcal{C} , we do not discuss Kähler forms (nor currents) on singular surfaces (see [Fk]). Roughly speaking, the first topic is a generalization of [Fn4], which established the minimal model theory for *algebraic* log surfaces, with the aid of the Enriques–Kodaira classification of compact complex surfaces.

In Section 11, which is an appendix of the first topic, we treat some vanishing theorems for proper bimeromorphic morphisms between analytic surfaces. Although they are more or less known to the experts, we explain the details for the reader's convenience because we can find no suitable references. We think that the results are useful for other applications. The most useful formulation is Theorem 11.3 (2).

Theorem 1.6 (see Theorem 11.3). *Let X be a normal analytic surface and let Δ be an effective \mathbb{Q} -divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $f : X \rightarrow Y$ be a proper bimeromorphic morphism onto a normal analytic surface Y . Let \mathcal{L} be a line bundle on X and let D be a \mathbb{Q} -Cartier Weil divisor on X . Assume that $\mathcal{L} \cdot C + (D - (K_X + \Delta)) \cdot C \geq 0$ for every f -exceptional curve C on X and that the coefficients of Δ are less than one. Then $R^i f_*(\mathcal{L} \otimes \mathcal{O}_X(D)) = 0$ holds for every $i > 0$.*

As we already mentioned above, the second topic (see Section 12) is based on a joint work with Kento Fujita. In Section 12, we construct some examples of complete non-projective

log canonical algebraic surfaces. Our construction is arguably simpler than Nagata's original and classical one (see [Ng, Section 4]). The reader can find some interesting complete non-projective algebraic surfaces. The final topic (see Section 13) is a supplement to the minimal model theory for algebraic log surfaces established in [Fn4] and [T1]. In Section 13, we introduce the notion of generalized MR log canonical surfaces and generalize our results in [Fn4] and [T1] in order to cover Alexeev's minimal model theory for MR log canonical surfaces sketched in [A, Section 10]. Roughly speaking, we establish:

Theorem 1.7. *The minimal model theory works in full generality for generalized MR log canonical surfaces.*

For the precise statements, see Section 13. It treats the minimal model theory for *algebraic* log surfaces defined over any algebraically closed field k and is independent of the other sections.

We explain the organization of this paper. In Section 2, we collect some basic definitions and results. In Section 3, we define \mathbb{Q} -factorial log surfaces and log canonical surfaces in Fujiki's class \mathcal{C} . Then we explain a very easy version of the basepoint-free theorem for projective bimeromorphic morphisms between surfaces (see Theorem 3.11). In Section 4, we collect some useful projectivity criteria for \mathbb{Q} -factorial compact analytic surfaces. We will repeatedly use them in subsequent sections. In Section 5, we discuss the minimal model program for \mathbb{Q} -factorial log surfaces based on Sakai's contraction theorem, which is a slight generalization of Grauert's famous contraction theorem. Then we prove Theorem 1.1 except for the semi-ampleness of $K_{X^*} + \Delta^*$. In Section 6, we quickly discuss the finite generation of log canonical rings of \mathbb{Q} -factorial log surfaces, which is essentially contained in [Fn4], and some related topics for the reader's convenience. In Section 7, we prove the non-vanishing theorem (see Theorem 7.1) and the abundance theorem (see Theorem 7.2 and Corollary 7.8). Precisely speaking, we explain how to modify the arguments in [Fn4] for \mathbb{Q} -factorial log surfaces in Fujiki's class \mathcal{C} . In Section 8, we discuss a contraction theorem for log canonical surfaces, which is an easy consequence of Sakai's contraction theorem. A key point is that the exceptional curve automatically becomes \mathbb{Q} -Cartier. This simple fact plays a crucial role in our minimal model theory for log canonical surfaces. Section 9 is devoted to the proof of Theorem 1.3, that is, the projectivity of log canonical surfaces in Fujiki's class \mathcal{C} with negative Kodaira dimension. Our proof needs the classification of two-dimensional log canonical singularities and is non-trivial. In Section 10, we prove Theorem 1.5, that is, the minimal model theory for log canonical surfaces in Fujiki's class \mathcal{C} . In Section 11, which is an appendix, we quickly explain some elementary vanishing theorems for the reader's convenience. Roughly speaking, we discuss a Kawamata–Viehweg vanishing theorem for proper bimeromorphic morphisms between normal analytic surfaces. Fortunately, we need no deep analytic methods except for the theorem on formal functions for proper morphisms between analytic spaces. In Section 12, we construct some complete non-projective log canonical algebraic surfaces. This section is based on a joint work with Kento Fujita. We think that some of our examples are new. Section 13 is independent of the other sections. We give some remarks on the minimal model theory for *algebraic* log surfaces established in [Fn4] and [T1].

Acknowledgments. The author was partially supported by JSPS KAKENHI Grant Numbers JP16H03925, JP16H06337. He would like to thank Kenta Hashizume, Haidong Liu, and Hiromu Tanaka for very useful comments and pointing out some mistakes. He also would like to thank Kento Fujita very much for useful discussions and advice, and for allowing him to use his ideas on complete non-projective algebraic surfaces in Section 12. Finally, he thanks Seiko Hashimoto for her help.

We will use the minimal model theory for projective log surfaces defined over \mathbb{C} , the complex number field, established in [Fn4] (for some related topics, see [FT], [T1], [Ha1], [Li], [Mi], and [Fn7, Section 4.10]). We will freely use the basic notation of the minimal model theory as in [Fn3] and [Fn7].

2. PRELIMINARIES

In this section, we collect some basic definitions and results.

Definition 2.1 (Boundary and subboundary \mathbb{Q} -divisors). Let X be an irreducible normal analytic space and let Δ be a \mathbb{Q} -divisor on X . If the coefficients of Δ are in $[0, 1] \cap \mathbb{Q}$ (resp. $(-\infty, 1] \cap \mathbb{Q}$), then Δ is called a *boundary* (resp. *subboundary*) \mathbb{Q} -divisor on X .

Definition 2.2 (Algebraic dimensions). Let X be an irreducible compact analytic space. Let $\mathcal{M}(X)$ be the field of meromorphic functions on X . Then the transcendence degree of $\mathcal{M}(X)$ over \mathbb{C} is called the *algebraic dimension* of X and is denoted by $a(X)$. It is well known that $0 \leq a(X) \leq \dim X$ holds. If $a(X) = \dim X$ holds, then we usually say that X is *Moishezon*. We note that if X is Moishezon then X is an algebraic space which is proper over \mathbb{C} (see, for example, [U, Remark 3.7]).

For the basic definitions of $a(X)$, we recommend the reader to see [U, Section 3].

Definition 2.3. Let X be an irreducible compact normal analytic surface such that the algebraic dimension $a(X)$ of X is two. Then we can obtain the perfect pairing

$$N_1(X) \times N^1(X) \rightarrow \mathbb{R}$$

induced from the intersection pairing of curves and line bundles as in the case where X is projective. We put $\rho(X) = \dim_{\mathbb{R}} N^1(X) < \infty$ and call it the *Picard number* of X . When X is an algebraic surface, $\overline{NE}(X) (\subset N_1(X))$ denotes the *Kleiman–Mori cone* of X .

For the details of $\overline{NE}(X)$ and some related topics, we strongly recommend the reader to see any textbook on the minimal model program (see, for example, [KM], [Ma], and [Fn7]). When $0 \leq a(X) < 2$, we do not consider $N_1(X)$.

Remark 2.4. Let X be a compact smooth analytic surface whose algebraic dimension $a(X)$ is zero. Then it is well known that there are only finitely many curves on X (see, for example, [BHPV, (8.2) Theorem]).

Let us quickly recall the definition of the Iitaka dimension κ . For the details of κ , see, for example, [Nk] and [U].

Definition 2.5 (Iitaka dimensions). Let X be an irreducible compact normal analytic space and let \mathcal{L} be a line bundle on X . Then we set

$$\kappa(X, \mathcal{L}) := \limsup_{m \rightarrow \infty} \frac{\log \dim_{\mathbb{C}} H^0(X, \mathcal{L}^{\otimes m})}{\log m}$$

and call it the *Iitaka dimension* of \mathcal{L} . It is well known that

$$\kappa(X, \mathcal{L}) \in \{-\infty, 0, 1, 2, \dots, \dim X\}$$

holds. We can define $\kappa(X, D)$ for \mathbb{Q} -Cartier \mathbb{Q} -divisors D on X similarly.

We close this section with an easy lemma on rational singularities.

Lemma 2.6 (see [Fn6, Lemma 3.1]). *Let $\varphi : X \rightarrow Y$ be a proper bimeromorphic morphism between normal analytic spaces. If $R^i \varphi_* \mathcal{O}_X = 0$ for every $i > 0$, then X has only rational singularities if and only if so does Y .*

Proof. The problem is local. So we can freely shrink Y around an arbitrary given point. Let us consider a common resolution:

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ X & \xrightarrow{\varphi} & Y. \end{array}$$

By assumption and the Leray spectral sequence, we have $R^i p_* \mathcal{O}_W \simeq R^i q_* \mathcal{O}_W$ for every i . This implies the desired statement. \square

3. LOG SURFACES IN FUJIKI'S CLASS \mathcal{C}

In this section, we define \mathbb{Q} -factorial log surfaces and log canonical surfaces in Fujiki's class \mathcal{C} and discuss a very easy version of the basepoint-free theorem for proper bimeromorphic morphisms between normal analytic surfaces.

In this paper, we adopt the following definition of analytic spaces in Fujiki's class \mathcal{C} .

Definition 3.1 (Fujiki's class \mathcal{C}). Let X be an irreducible compact analytic space. If X is bimeromorphically equivalent to a compact Kähler manifold, then we say that X is in Fujiki's class \mathcal{C} .

Remark 3.2. Let X be an irreducible compact analytic space. We note that if X is Moishezon then X is automatically in Fujiki's class \mathcal{C} .

We have a useful characterization of surfaces in Fujiki's class \mathcal{C} .

Lemma 3.3. *Let X be an irreducible compact normal analytic surface. Then X is in Fujiki's class \mathcal{C} if and only if there exists a resolution of singularities $f : Y \rightarrow X$ such that Y is Kähler, that is, Y is a two-dimensional compact Kähler manifold and f is a bimeromorphic morphism.*

Proof. Note that a compact smooth analytic surface S is Kähler if and only if the first Betti number $b_1(S)$ is even (see, for example, [BHPV, Chapter IV, (3.1) Theorem]). We also note that the first Betti number is preserved under blow-ups. Thus we can easily check the statement. \square

As an easy consequence of Lemma 3.3 and its proof, we have:

Corollary 3.4. *Let X be a compact normal analytic surface in Fujiki's class \mathcal{C} . Let $f : Y \rightarrow X$ be any resolution of singularities. Then Y is a compact Kähler manifold.*

Let us define canonical sheaves.

Definition 3.5 (Canonical sheaves). Let X be a normal analytic surface and let $\text{Sing}X$ denote the singular locus of X . Then we have $\text{codim}_X \text{Sing}X \geq 2$ since X is normal. Let ω_U be the canonical bundle of $U := X \setminus \text{Sing}X$. We put $\omega_X := \iota_* \omega_U$, where $\iota : U \hookrightarrow X$ is the natural open immersion, and call ω_X the *canonical sheaf* of X .

Remark 3.6. Some normal analytic surface X does not admit any non-zero meromorphic section of ω_X . However, if there is no risk of confusion, we use the symbol K_X as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ and call it the *canonical divisor* of X .

In this paper, we adopt the following definition of log surfaces.

Definition 3.7 (Log surfaces). Let X be an irreducible compact normal analytic surface and let Δ be a boundary \mathbb{Q} -divisor on X . Assume that $K_X + \Delta$ is \mathbb{Q} -Cartier. Note that this means that there exists a positive integer m such that $m\Delta$ is integral and that $(\omega_X^{\otimes m} \otimes \mathcal{O}_X(m\Delta))^{**}$ is locally free. Then the pair (X, Δ) is called a *log surface*. Let (X, Δ) be a log surface. Then we usually call $\kappa(X, K_X + \Delta)$ the *Kodaira dimension* of (X, Δ) .

We need to define log canonical surfaces.

Definition 3.8 (Log canonical surfaces). Let (X, Δ) be a log surface and let $f : Y \rightarrow X$ be a proper bimeromorphic morphism from a smooth analytic surface Y . Then we can write $K_Y + \Delta_Y = f^*(K_X + \Delta)$ with $f_*\Delta_Y = \Delta$. If the coefficients of Δ_Y are less than or equal to one for every $f : Y \rightarrow X$, then (X, Δ) is called a *log canonical surface*.

The notion of \mathbb{Q} -factoriality plays a crucial role in this paper.

Definition 3.9 (\mathbb{Q} -factoriality). Let X be an irreducible compact normal analytic surface and let D be a \mathbb{Q} -divisor on X . Then we say that D is *\mathbb{Q} -Cartier* if there exists a positive integer m such that mD is Cartier. If every Weil divisor on X is \mathbb{Q} -Cartier, then we say that X is *\mathbb{Q} -factorial*.

Lemma 3.10 is well known.

Lemma 3.10. *Let X be an irreducible compact normal analytic surface. Assume that X has only rational singularities. Then X is \mathbb{Q} -factorial.*

Proof. This follows from, for example, [Nk, Chapter II, 2.12. Lemma]. □

We close this section with a very easy version of the basepoint-free theorem for projective bimeromorphic morphisms between normal analytic surfaces.

Theorem 3.11. *Let (X, Δ) be a log surface and let $\varphi : X \rightarrow Y$ be a projective bimeromorphic morphism onto a normal analytic surface Y . Assume that $C := \text{Exc}(\varphi)$ is \mathbb{Q} -Cartier, $C \simeq \mathbb{P}^1$, and $-C$ and $-(K_X + \Delta)$ are φ -ample. Let \mathcal{L} be a line bundle on X with $\mathcal{L} \cdot C = 0$. Then there exists a line bundle \mathcal{L}_Y on Y such that $\mathcal{L} \simeq \varphi^*\mathcal{L}_Y$ holds. In particular, X is \mathbb{Q} -factorial if and only if so is Y .*

Proof. In Step 1, we will prove the existence of \mathcal{L}_Y . In Steps 2 and 3, we will see that X is \mathbb{Q} -factorial if and only if so is Y .

Step 1. Since $C \simeq \mathbb{P}^1$ and $\mathcal{L} \cdot C = 0$, $\mathcal{L}|_C$ is trivial. Since $-C$ and $-(K_X + \Delta)$ are φ -ample, we may assume that $C \leq \Delta$ by increasing the coefficient of C in Δ . Let us consider the following short exact sequence:

$$0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0.$$

Note that

$$\mathcal{L} \cdot C + (-C - (K_X + \Delta - C)) \cdot C = \mathcal{L} \cdot C - (K_X + \Delta) \cdot C > 0$$

holds. Therefore, by Theorem 11.3 below, we get $R^1\varphi_*(\mathcal{L} \otimes \mathcal{O}_X(-C)) = 0$. Thus, we have the following short exact sequence:

$$(3.1) \quad 0 \rightarrow \varphi_*(\mathcal{L} \otimes \mathcal{O}_X(-C)) \rightarrow \varphi_*\mathcal{L} \rightarrow \varphi_*(\mathcal{L}|_C) \rightarrow 0.$$

We note that

$$\varphi_*(\mathcal{L}|_C) = H^0(C, \mathcal{L}|_C) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}).$$

By (3.1), \mathcal{L} is φ -free and $\mathcal{L}|_C$ is trivial. Thus we get a line bundle \mathcal{L}_Y on Y such that $\mathcal{L} \simeq \varphi^*\mathcal{L}_Y$ holds.

Step 2. Assume that X is \mathbb{Q} -factorial. We take a prime divisor D on Y . Let D' be the strict transform of D on X . Then we can take $a \in \mathbb{Q}$ and a divisible positive integer m such that $m(D' + aC)$ is Cartier and $m(D' + aC) \cdot C = 0$. We put $\mathcal{L} = \mathcal{O}_X(m(D' + aC))$ and apply the result obtained in Step 1 to \mathcal{L} . Then $mD = \varphi_*(m(D' + aC))$ is Cartier. This means that Y is \mathbb{Q} -factorial.

Step 3. Assume that Y is \mathbb{Q} -factorial. We take a prime divisor D on X . Then $D' := \varphi_*D$ is a \mathbb{Q} -Cartier prime divisor on Y . Since $D = \varphi^*D' - aC$ holds for some $a \in \mathbb{Q}$, D is \mathbb{Q} -Cartier. Therefore, X is \mathbb{Q} -factorial.

We complete the proof of Theorem 3.11. \square

4. PROJECTIVITY CRITERIA

Let us start with an easy but very useful projectivity criterion.

Lemma 4.1 (Projectivity of \mathbb{Q} -factorial compact analytic surfaces). *Let X be a \mathbb{Q} -factorial compact analytic surface. Assume that the algebraic dimension $a(X)$ of X is two, that is, X is Moishezon. Then X is projective.*

Proof. By the assumption $a(X) = 2$, we can construct a proper bimeromorphic morphism $f : Y \rightarrow X$ from a smooth projective surface Y . By the assumption $a(X) = 2$ again, X is an algebraic space which is proper over \mathbb{C} by Artin's GAGA (see, for example, [U, Remark 3.7]). We take a very ample effective Cartier divisor H on Y . We put $A = f_*H$. Since X is \mathbb{Q} -factorial, A is a \mathbb{Q} -Cartier divisor. Then we have $A \cdot C = H \cdot f^*C > 0$ for every curve C on X . In particular, we have $A^2 > 0$. Therefore, A is ample by Nakai–Moishezon's ampleness criterion for algebraic spaces (see [P, (1.4) Theorem]). This implies that X is projective. \square

The following corollary is obvious by Lemma 4.1.

Corollary 4.2. *Let X be a \mathbb{Q} -factorial compact analytic surface. Assume that there exists a line bundle \mathcal{L} such that $\kappa(X, \mathcal{L}) = 2$, that is, \mathcal{L} is a big line bundle. Then X is projective.*

Proof. By the assumption that \mathcal{L} is big, we see that the algebraic dimension $a(X)$ of X is two. Therefore, X is projective by Lemma 4.1. \square

By Corollary 4.2, the minimal model theory for projective \mathbb{Q} -factorial log surfaces established in [Fn4] works for (X, Δ) with $\kappa(X, K_X + \Delta) = 2$ in Theorem 1.1.

By combining Lemma 4.1 with the Enriques–Kodaira classification (see, for example, [BHPV, Chapter VI]), we obtain the following projectivity criterion.

Lemma 4.3. *Let (X, Δ) be a \mathbb{Q} -factorial log surface in Fujiki's class \mathcal{C} with $\kappa(X, K_X + \Delta) = -\infty$. Then X is projective.*

Proof. We take the minimal resolution $f : Y \rightarrow X$. We put $K_Y + \Delta_Y := f^*(K_X + \Delta)$. Then we see that Δ_Y is effective by the negativity lemma and that $\kappa(Y, K_X + \Delta_Y) = \kappa(X, K_X + \Delta) = -\infty$ holds. Therefore, we obtain $\kappa(Y, K_Y) = -\infty$ by $\kappa(Y, K_Y) \leq \kappa(Y, K_Y + \Delta_Y) = -\infty$. Since X is in Fujiki's class \mathcal{C} , the first Betti number $b_1(Y)$ of Y is even. Therefore, by the Enriques–Kodaira classification (see, for example, [BHPV, Chapter VI]), Y is a smooth projective surface. Thus, by Lemma 4.1, X is projective. \square

We will repeatedly use the above projectivity criteria throughout this paper.

We note that the statement of Theorem 1.3 looks very similar to that of Lemma 4.3. However, a log canonical surface is not necessarily \mathbb{Q} -factorial. Therefore, Theorem 1.3 is much harder to prove than Lemma 4.3 (see Section 9).

5. MINIMAL MODEL PROGRAM FOR \mathbb{Q} -FACTOREAL LOG SURFACES

By repeatedly using Grauert's contraction theorem, we can easily run a kind of the minimal model program for \mathbb{Q} -factorial log surfaces (X, Δ) . We note that X is not assumed to be in Fujiki's class \mathcal{C} in Theorem 5.1. A key point of Theorem 5.1 is the assumption that X is \mathbb{Q} -factorial.

Theorem 5.1. *Let (X, Δ) be a compact \mathbb{Q} -factorial log surface. We assume that $\kappa(X, K_X + \Delta) \geq 0$. Then we can construct a finite sequence of projective bimeromorphic morphisms*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} (X_k, \Delta_k) =: (X^*, \Delta^*)$$

with $\Delta_i := \varphi_{i-1*} \Delta_{i-1}$, $\text{Exc}(\varphi_i) =: C_i \simeq \mathbb{P}^1$, and $-(K_{X_i} + \Delta_i) \cdot C_i > 0$ for every i such that $(K_{X^*} + \Delta^*) \cdot C \geq 0$ for every curve C on X^* . Of course, (X_i, Δ_i) is a compact \mathbb{Q} -factorial log surface for every i .

Proof. Since $\kappa(X, K_X + \Delta) \geq 0$, we can take an effective Cartier divisor $D \in |m(K_X + \Delta)|$ for some divisible positive integer m . If $m(K_X + \Delta) \cdot C = D \cdot C \geq 0$ for every curve C on X , then we set $(X^*, \Delta^*) := (X_0, \Delta_0) = (X, \Delta)$. So we assume that there exists some irreducible curve C on X such that $D \cdot C < 0$. Then C is an irreducible component of $\text{Supp } D$ and $C^2 < 0$. By Sakai's contraction theorem (see [S, Theorem (1.2)]), which is a slight generalization of Grauert's famous contraction theorem, we get a bimeromorphic morphism $\varphi_0 : X = X_0 \rightarrow X_1$ that contracts C to a normal point of X_1 . We take a divisible positive integer l such that lC is Cartier. Then $\mathcal{O}_X(-lC)$ is a φ_0 -ample line bundle on X . In particular, φ_0 is a projective morphism. By construction, $-(K_X + \Delta) \cdot C > 0$. Therefore, $-(K_X + \Delta)$ is φ_0 -ample. Thus, $R^i \varphi_{0*} \mathcal{O}_X = 0$ for every $i > 0$ by Theorem 11.3 below.

Claim 1. *C is isomorphic to \mathbb{P}^1 .*

Proof of Claim 1. We consider the following exact sequence:

$$\cdots \rightarrow R^1 \varphi_{0*} \mathcal{O}_X \rightarrow R^1 \varphi_{0*} \mathcal{O}_C \rightarrow R^2 \varphi_{0*} \mathcal{I}_C \rightarrow \cdots,$$

where \mathcal{I}_C is the defining ideal sheaf of C on X . As we saw above, $R^1 \varphi_{0*} \mathcal{O}_X = 0$ holds. Since C is a curve, $R^2 \varphi_{0*} \mathcal{I}_C = 0$ holds by the theorem on formal functions for proper morphisms between analytic spaces (see, for example, [BS, Chapter IV, Corollary 4.7]). Thus we get $H^1(C, \mathcal{O}_C) = R^1 \varphi_{0*} \mathcal{O}_C = 0$ by the above exact sequence. This implies that C is isomorphic to \mathbb{P}^1 . \square

Therefore, by Theorem 3.11, we obtain that (X_1, Δ_1) is a \mathbb{Q} -factorial log surface. Since $\text{Supp } D$ has only finitely many irreducible components, we get a desired sequence of contraction morphisms and finally obtain (X^*, Δ^*) with $(K_{X^*} + \Delta^*) \cdot C \geq 0$ for every curve C on X^* . \square

We note the following well-known lemma on extremal rays of projective surfaces.

Lemma 5.2. *Let X be a normal projective surface and let C be a \mathbb{Q} -Cartier irreducible curve on X with $C^2 < 0$. Then the numerical equivalence class $[C]$ of C spans an extremal ray of the Kleiman–Mori cone $\overline{NE}(X)$ of X .*

Proof. This is obvious. For the proof, see, for example, [KM, Lemma 1.22]. \square

By Lemma 5.2, if X is projective in Theorem 5.1, then the minimal model program in Theorem 5.1 is nothing but the minimal model program for projective \mathbb{Q} -factorial log surfaces formulated and established in [Fn4]. We also note that X is projective in Theorem 5.1 if the algebraic dimension $a(X)$ of X is two by Lemma 4.1.

We note that \mathbb{Q} -factorial log surfaces (X, Δ) in Fujiki's class \mathcal{C} with $\kappa(X, K_X + \Delta) = -\infty$ are projective by Lemma 4.3.

Let us prove Theorem 1.1 except for the semi-ampleness of $K_{X^*} + \Delta^*$.

Proof of Theorem 1.1. If $(K_X + \Delta) \cdot C \geq 0$ for every curve C on X , then we put $(X^*, \Delta^*) := (X, \Delta)$. We will see that $K_{X^*} + \Delta^*$ is semi-ample (see Theorem 7.2 below). We note that X is in Fujiki's class \mathcal{C} . If $\kappa(X, K_X + \Delta) = -\infty$, then X is projective by Lemma

4.3. Therefore, we can run the minimal model program for \mathbb{Q} -factorial projective log surfaces in [Fn4] and finally get a Mori fiber space. Therefore, we may further assume that $\kappa(X, K_X + \Delta) \geq 0$. Then we can apply Theorem 5.1 and finally get a model (X^*, Δ^*) such that $(K_{X^*} + \Delta^*) \cdot C \geq 0$ for every curve C on X^* . In this case, by the abundance theorem below (see Theorem 7.2), we will see that $K_{X^*} + \Delta^*$ is semi-ample.

Since $R^1\varphi_{i*}\mathcal{O}_{X_i} = 0$, X_i has only rational singularities if and only if so does X_{i+1} by Lemma 2.6. Thus we have (2).

Since each contraction φ_i is projective, X_i is projective when so is X_{i+1} . On the other hand, if X_i is projective then so is X_{i+1} because φ_i is nothing but the usual contraction morphism associated to a $(K_{X_i} + \Delta_i)$ -negative extremal ray (see Lemma 5.2). Anyway, we have (1). \square

We obtained Theorem 1.1 except for the semi-ampleness of $K_{X^*} + \Delta^*$, which will be proved in Section 7.

6. FINITE GENERATION OF LOG CANONICAL RINGS

In this section, we quickly discuss the finite generation of log canonical rings of pairs for the reader's convenience.

The following theorem is the main result of this section, which is essentially contained in [Fn4].

Theorem 6.1 (Finite generation of log canonical rings). *Let (X, Δ) be a compact \mathbb{Q} -factorial log surface. Then the log canonical ring*

$$\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

*is a finitely generated \mathbb{C} -algebra. We note that the sheaf $\mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$ denotes $(\omega_X^{\otimes m} \otimes \mathcal{O}_X(\lfloor m\Delta \rfloor))^{**}$.*

As an easy consequence of Theorem 6.1, we have:

Corollary 6.2. *Let (X, Δ) be a compact log canonical surface. Then the log canonical ring*

$$\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated \mathbb{C} -algebra.

We note that X is not assumed to be in Fujiki's class \mathcal{C} in Theorem 6.1 and Corollary 6.2.

Proof of Corollary 6.2. Let $f : Y \rightarrow X$ be the minimal resolution. We put $K_Y + \Delta_Y := f^*(K_X + \Delta)$. Since (X, Δ) is log canonical, we see that Δ_Y is a boundary \mathbb{Q} -divisor by the negativity lemma. By Theorem 6.1, the log canonical ring of (Y, Δ_Y) is a finitely generated \mathbb{C} -algebra. This implies that the log canonical ring of (X, Δ) is a finitely generated \mathbb{C} -algebra. \square

Before we prove Theorem 6.1, let us recall the following easy well-known lemma for the reader's convenience.

Lemma 6.3. *Let X be an irreducible compact normal analytic space and let \mathcal{L} be a line bundle on X such that $\kappa(X, \mathcal{L}) \leq 1$. Then the graded ring*

$$R(X, \mathcal{L}) := \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^{\otimes m})$$

is a finitely generated \mathbb{C} -algebra.

Sketch of Proof. If $\kappa(X, \mathcal{L}) = -\infty$ or 0, then it is very easy to see that $R(X, \mathcal{L})$ is a finitely generated \mathbb{C} -algebra. If $\kappa(X, \mathcal{L}) = 1$, then we can reduce the problem to the case where X is a smooth projective curve and \mathcal{L} is an ample line bundle on X by taking the Iitaka fibration. Anyway, $R(X, \mathcal{L})$ is a finitely generated \mathbb{C} -algebra when $\kappa(X, \mathcal{L}) \leq 1$. \square

Let us prove Theorem 6.1.

Proof of Theorem 6.1. By Lemma 6.3, we may assume that $\kappa(X, K_X + \Delta) = 2$. Then, by Corollary 4.2, X is projective. In this case, the log canonical ring

$$\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

of (X, Δ) is a finitely generated \mathbb{C} -algebra by the minimal model theory for projective \mathbb{Q} -factorial log surfaces established in [Fn4]. \square

Let us quickly see some results and conjectures on log canonical rings of higher-dimensional pairs.

Theorem 6.4 ([BCHM], [FM], and [Fn5, Theorem 1.8]). *Let (X, Δ) be a kawamata log terminal pair such that Δ is a \mathbb{Q} -divisor on X and that X is in Fujiki's class \mathcal{C} . Then the log canonical ring*

$$\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated \mathbb{C} -algebra.

Conjecture 6.5. *Let (X, Δ) be a log canonical pair such that Δ is a \mathbb{Q} -divisor on X and that X is in Fujiki's class \mathcal{C} . Then the log canonical ring*

$$\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated \mathbb{C} -algebra.

Conjecture 6.5 is still widely open even when X is projective (see [Fn2], [FG], [Ha2], [FL], and so on). When X is projective in Conjecture 6.5, it is essentially equivalent to the existence problem of good minimal models for lower-dimensional varieties (for the details, see [FG]). Note that Corollary 6.2 completely settled Conjecture 6.5 in dimension two.

We close this section with a naive question.

Question 6.6. Let X be an irreducible compact normal analytic surface such that K_X is \mathbb{Q} -Cartier. Then is the canonical ring

$$\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

a finitely generated \mathbb{C} -algebra?

We do not know the answer even when X is projective.

7. ABUNDANCE THEOREM

In this section, we prove the abundance theorem for \mathbb{Q} -factorial log surfaces in Fujiki's class \mathcal{C} .

Let us start with the non-vanishing theorem.

Theorem 7.1 (Non-vanishing theorem). *Let (X, Δ) be a \mathbb{Q} -factorial log surface in Fujiki's class \mathcal{C} . Assume that $(K_X + \Delta) \cdot C \geq 0$ for every curve C on X . Then we have $\kappa(X, K_X + \Delta) \geq 0$.*

Proof. Let $f : Y \rightarrow X$ be the minimal resolution. We put $K_Y + \Delta_Y := f^*(K_X + \Delta)$. Then Δ_Y is an effective \mathbb{Q} -divisor by the negativity lemma. If $\kappa(Y, K_Y) \geq 0$, then we have

$$\kappa(X, K_X + \Delta) = \kappa(Y, K_Y + \Delta_Y) \geq \kappa(Y, K_Y) \geq 0.$$

Therefore, from now on, we assume that $\kappa(Y, K_Y) = -\infty$. Since X is in Fujiki's class \mathcal{C} , the first Betti number $b_1(Y)$ of Y is even. Thus, by the Enriques–Kodaira classification (see, for example, [BHPV, Chapter VI]), Y is projective. By Lemma 4.1, X is projective since X is \mathbb{Q} -factorial by assumption. Therefore, by [Fn4, Theorem 5.1], we get $\kappa(X, K_X + \Delta) \geq 0$. \square

The following theorem is the main result of this section, which is the abundance theorem for \mathbb{Q} -factorial log surfaces in Fujiki's class \mathcal{C} .

Theorem 7.2 (Abundance theorem for \mathbb{Q} -factorial log surfaces in Fujiki's class \mathcal{C}). *Let (X, Δ) be a \mathbb{Q} -factorial log surface in Fujiki's class \mathcal{C} . Assume that $(K_X + \Delta) \cdot C \geq 0$ for every curve C on X . Then $K_X + \Delta$ is semi-ample.*

For the proof of Theorem 7.2, we prepare some easy lemmas.

Lemma 7.3. *Let X be a compact normal analytic surface and let \mathcal{L} be a line bundle on X such that $\mathcal{L} \cdot C \geq 0$ for every curve C on X . Assume that $\kappa(X, \mathcal{L}) = 1$. Then \mathcal{L} is semi-ample.*

Proof. This is an easy consequence of Zariski's lemma (see, for example, [BHPV, (8.2) Lemma]). For the details, see, for example, [Ft, (4.1) Theorem]. \square

The next lemma is a special case of Lemma 4.3. We explicitly state it for the reader's convenience.

Lemma 7.4. *Let S be a compact smooth analytic surface in Fujiki's class \mathcal{C} with $\kappa(S, K_S) = -\infty$. Then S is projective.*

Proof. Since S is in Fujiki's class \mathcal{C} , the first Betti number $b_1(S)$ of S is even. Then the Enriques–Kodaira classification (see, for example, [BHPV, Chapter VI]) says that the algebraic dimension $a(S)$ of S is two. This implies that S is projective by Lemma 4.1. \square

Lemma 7.5. *Let S be a compact smooth analytic surface in Fujiki's class \mathcal{C} with $\kappa(S, K_S) = 0$. Assume that the algebraic dimension $a(S)$ of S is less than two. Then S is bimeromorphically equivalent to a K3 surface or a two-dimensional complex torus.*

Proof. Since S is in Fujiki's class \mathcal{C} , the first Betti number $b_1(S)$ of S is even. Then the Enriques–Kodaira classification (see, for example, [BHPV, Chapter VI]) and $\kappa(S, K_S) = 0$ give the desired statement. \square

Lemma 7.6. *Let B be a non-zero effective divisor on a two-dimensional complex torus S . Then we have $\kappa(S, B) \geq 1$.*

Proof. Without loss of generality, we may assume that B is an irreducible curve on S . If B is not an elliptic curve, then we can see that S is an Abelian surface (see, for example, [U, Lemma 10.8]). In this case, it is well known that $|2B|$ is basepoint-free. In particular, $\kappa(S, B) \geq 1$. Therefore, from now on, we assume that B is an elliptic curve. By taking a suitable translation, we may further assume that B is a complex subtorus of S . We set $A = S/B$. Let $p : S \rightarrow A$ be the canonical quotient map. Then $B = p^*P$ holds for $P = p(B) \in A$. Therefore, we obtain $\kappa(S, B) = \kappa(A, P) = 1$. Anyway, we always have $\kappa(S, B) \geq 1$. \square

Lemma 7.7. *Let S be a K3 surface and let B be a non-zero effective divisor on S such that $B^2 = 0$. Then we have $\kappa(S, B) \geq 1$.*

Proof. By the Riemann–Roch formula,

$$\dim H^0(S, \mathcal{O}_S(B)) + \dim H^2(S, \mathcal{O}_S(B)) \geq \chi(S, \mathcal{O}_S) = 2.$$

By Serre duality,

$$H^2(S, \mathcal{O}_S(B)) \simeq H^0(S, \mathcal{O}_S(-B)).$$

Since B is a non-zero effective divisor on S , $H^0(S, \mathcal{O}_S(-B)) = 0$ and $\dim H^0(S, \mathcal{O}_S(B)) \geq 2$. Therefore, we have $\kappa(S, B) \geq 1$. \square

Before we prove Theorem 7.2, we explicitly state the abundance theorem for log canonical surfaces in Fujiki’s class \mathcal{C} .

Corollary 7.8 (Abundance theorem for log canonical surfaces in Fujiki’s class \mathcal{C}). *Let (X, Δ) be a log canonical surface in Fujiki’s class \mathcal{C} . Assume that $(K_X + \Delta) \cdot C \geq 0$ for every curve C on X . Then $K_X + \Delta$ is semi-ample.*

Proof. Let $f : Y \rightarrow X$ be the minimal resolution of X . We put $K_Y + \Delta_Y := f^*(K_X + \Delta)$. Then Δ_Y is effective by the negativity lemma and is a subboundary \mathbb{Q} -divisor on Y since (X, Δ) is log canonical. Therefore, Δ_Y is a boundary \mathbb{Q} -divisor on Y . We can easily see that $(K_Y + \Delta_Y) \cdot C_Y \geq 0$ for every curve C_Y on Y . Thus, by Theorem 7.2, $K_Y + \Delta_Y$ is semi-ample. This implies that $K_X + \Delta$ is also semi-ample. \square

Let us start the proof of Theorem 7.2.

Proof of Theorem 7.2. By the non-vanishing theorem (see Theorem 7.1), we have $\kappa(X, K_X + \Delta) \geq 0$.

Step 1 ($\kappa = 2$). If $\kappa(X, K_X + \Delta) = 2$, then X is projective by Corollary 4.2. In this case, we can apply [Fn4, Theorem 4.1], which is one of the deepest results in [Fn4], and obtain that $K_X + \Delta$ is semi-ample.

Step 2 ($\kappa = 1$). If $\kappa(X, K_X + \Delta) = 1$, then we see that $K_X + \Delta$ is semi-ample by Lemma 7.3.

Therefore, from now on, we may assume that $\kappa(X, K_X + \Delta) = 0$.

Step 3 ($\kappa = 0$). In this step, we assume $\kappa(X, K_X + \Delta) = 0$. If X is projective, then $K_X + \Delta$ is semi-ample by [Fn4, Theorem 6.2]. Here, we will explain that the proof of [Fn4, Theorem 6.2] works with some minor modifications when X is not projective.

The first part of the proof of [Fn4, Theorem 6.2] works without any changes (see page 361 in [Fn4]). We note that Mumford’s arguments on *indecomposable curves of canonical type* can work on smooth analytic surfaces (see [Mu2, Definition, Lemma, and Corollary 1 in Section 2]). Anyway, we may assume that the compact smooth surface S constructed in the first part of [Fn4, Theorem 6.2] is not projective. Of course, S is in Fujiki’s class \mathcal{C} because S is bimeromorphically equivalent to X by construction.

By Lemma 7.4, we have $\kappa(S, K_S) \geq 0$ since S is not projective. Thus, all we have to do is to check Step 1 in the proof of [Fn4, Theorem 6.2] works when S is not projective.

In Step 1 in the proof of [Fn4, Theorem 6.2], S is a compact smooth analytic surface with $\kappa(S, K_S) = 0$ and there are no (-1) -curves on S . Since S is in Fujiki’s class \mathcal{C} , the first Betti number $b_1(S)$ of S is even. Therefore, by the Enriques–Kodaira classification (see, for example, [BHPV, Chapter VI]), S is a $K3$ surface or a complex torus (see Lemma 7.5). Then, by Lemmas 7.6 and 7.7, we have $\kappa(X, K_X + \Delta) = \kappa(S, K_S + \Delta_S) = \kappa(S, Z) \geq 1$ and get a contradiction. This means that Step 1 in the proof of [Fn4, Theorem 6.2] works when S is not projective.

Anyway, $K_X + \Delta$ is always semi-ample. This is what we wanted. \square

8. CONTRACTION THEOREM FOR LOG CANONICAL SURFACES

In this section, we discuss a contraction theorem for log canonical surfaces. Note that compact log canonical surfaces are not necessarily \mathbb{Q} -factorial. Therefore, we need Mumford's intersection theory (see [Mu1], [Ma, Remark 4-6-3], [S], and so on).

Definition 8.1 (Mumford's intersection theory). Let X be a normal analytic surface and let $\pi : Y \rightarrow X$ be a resolution. Let $\text{Exc}(\pi) = \sum_i E_i$ be the irreducible decomposition of the exceptional curve of π . Let D be a \mathbb{Q} -divisor on X . Then we can define the *inverse image* π^*D as

$$\pi^*D = D^\dagger + \sum_i \alpha_i E_i$$

where D^\dagger is the strict transform of D by π and the rational numbers α_i are uniquely determined by the following linear equations:

$$D^\dagger \cdot E_j + \sum_i \alpha_i E_i \cdot E_j = 0$$

for every j . We usually call π^*D the *pull-back of D in the sense of Mumford*. Of course, π^*D coincides with the usual one when D is \mathbb{Q} -Cartier.

From now on, we further assume that X is compact. The *intersection number $D \cdot D'$ (in the sense of Mumford)* is defined to be the rational number $(\pi^*D) \cdot (\pi^*D')$, where D and D' are \mathbb{Q} -divisors on X . We can easily see that $D \cdot D'$ is well-defined. We note that it coincides with the usual one when D or D' is \mathbb{Q} -Cartier.

Let us recall some definitions and basic properties of surface singularities for the reader's convenience.

Definition 8.2 (Numerically log canonical and numerically dlt, see [KM, Notation 4.1]). Let X be a normal analytic surface and let Δ be a \mathbb{Q} -divisor on X . Let $f : Y \rightarrow U \subset X$ be a proper bimeromorphic morphism from a smooth surface Y to an open set U of X . Then we can define $f^*(K_U + \Delta|_U)$ in the sense of Mumford (see Definition 8.1) without assuming that $K_U + \Delta|_U$ is \mathbb{Q} -Cartier. Thus we can always write

$$K_Y = f^*(K_U + \Delta|_U) + \sum_{E_i} a(E_i, X, \Delta) E_i$$

such that $f_*(\sum_{E_i} a(E_i, X, \Delta) E_i) = -\Delta|_U$. If Δ is effective and $a(E_i, X, \Delta) \geq -1$ for every exceptional curve E_i and $f : Y \rightarrow U \subset X$, then we say that (X, Δ) is *numerically log canonical*. We say that (X, Δ) is *numerically dlt* if (X, Δ) is numerically log canonical and there exists a finite set $Z \subset X$ such that $X \setminus Z$ is smooth, $\text{Supp } \Delta|_{X \setminus Z}$ is a simple normal crossing divisor on $X \setminus Z$, and $a(E, X, \Delta) > -1$ for every exceptional curve E which maps to Z . It is well known that if (X, Δ) is numerically log canonical then $K_X + \Delta$ is \mathbb{Q} -Cartier (see [Fn4, Proposition 3.5] and [Ma, Remark 4-6-3]). Moreover, if (X, Δ) is numerically dlt then X has only rational singularities (see [KM, Theorem 4.12]).

Remark 8.3. In Definition 8.2, we only require that $\text{Supp } \Delta|_{X \setminus Z}$ is a simple normal crossing divisor on $X \setminus Z$ in the classical topology. So it permits some irreducible component of $\text{Supp } \Delta|_{X \setminus Z}$ to have nodal singularities. Therefore, our definition does not coincide with [KM, Notation 4.1] when X is an algebraic surface. However, since we are mainly interested in local analytic properties of singularities of pairs (X, Δ) , this difference causes no subtle problems.

We need the following contraction theorem for log canonical surfaces in Sections 9 and 10. It is an easy consequence of Sakai's contraction theorem (see [S, Theorem (1.2)]), which is a slight generalization of Grauert's famous contraction theorem.

Theorem 8.4 (Contraction theorem for log canonical surfaces, see [Fn6, Theorem 4.1]). *Let (X, Δ) be a compact log canonical surface and let C be an irreducible curve on X such that $-(K_X + \Delta) \cdot C > 0$ and $C^2 < 0$, where C^2 is the self-intersection number of C in the sense of Mumford (see Definition 8.1). Then we have a projective bimeromorphic morphism $\varphi : X \rightarrow Y$ onto a normal surface Y such that $\text{Exc}(\varphi) = C \simeq \mathbb{P}^1$ and that C passes through no non-rational singular points of X , that is, X has only rational singularities in a neighborhood of C . In particular, C is \mathbb{Q} -Cartier. Moreover, (Y, Δ_Y) is log canonical with $\Delta_Y := \varphi_*\Delta$.*

Proof. By Sakai's contraction theorem (see [S, Theorem (1.2)]), we have a bimeromorphic morphism $\varphi : X \rightarrow Y$ which contracts C to a normal point $P \in Y$. Since $-(K_X + \Delta) \cdot C > 0$, (Y, Δ_Y) is numerically dlt in a neighborhood of P by the negativity lemma. Therefore, $K_Y + \Delta_Y$ is \mathbb{Q} -Cartier and Y has only rational singularities in a neighborhood of P . Of course, (Y, Δ_Y) is a compact log canonical surface. By Theorem 11.3 below, $R^i\varphi_*\mathcal{O}_X = 0$ for every $i > 0$. Thus, X has only rational singularities in a neighborhood of C by Lemma 2.6. In particular, C is \mathbb{Q} -Cartier (see, for example, [Nk, Chapter II, 2.12. Lemma]). Since $R^1\varphi_*\mathcal{O}_X = 0$, we can easily check that $C \simeq \mathbb{P}^1$ as in the proof of Theorem 5.1 (see Claim 1 in the proof of Theorem 5.1). Anyway, we obtain that φ is projective, $-(K_X + \Delta)$ and $-C$ are φ -ample. \square

We close this section with simple but very important remarks.

Remark 8.5 (Extremal rays). Theorem 8.4 says that X has only rational singularities in a neighborhood of the exceptional curve C and then C is automatically \mathbb{Q} -Cartier (see [Fn6, Theorem 4.1]). Therefore, if X is projective, then C spans a $(K_X + \Delta)$ -negative extremal ray R of $\overline{NE}(X)$ in the usual sense (see Lemma 5.2). Thus the contraction φ in Theorem 8.4 is nothing but the usual contraction morphism associated to the extremal ray R . In particular, Y is also projective when so is X .

Remark 8.6 (Termination of contractions). Assume that X is Moishezon. We consider a sequence of contraction morphisms as in Theorem 8.4

$$(X, \Delta) := (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{i-1}} (X_i, \Delta_i) \xrightarrow{\varphi_i} \cdots$$

starting from a log canonical surface $(X_0, \Delta_0) := (X, \Delta)$. Let C_i be the φ_i -exceptional curve for every i . By Theorem 8.4, C_i is \mathbb{Q} -Cartier for every i . Then we can easily see that $C_0, \varphi_0^*C_1, \dots, \varphi_0^*\cdots\varphi_{i-1}^*C_i, \dots$ are linearly independent in $N^1(X)$. Therefore, the sequence must terminate since $\rho(X) < \infty$.

9. LOG CANONICAL SURFACES IN FUJIKI'S CLASS \mathcal{C} WITH NEGATIVE KODAIRA DIMENSION

The main purpose of this section is to prove the following theorem.

Theorem 9.1 (see Theorem 1.3). *Let (X, Δ) be a log canonical surface in Fujiki's class \mathcal{C} . Assume that $\kappa(X, K_X + \Delta) = -\infty$ holds. Then X is projective.*

Let us recall the following well-known lemma for the reader's convenience (see, for example, [Ma, Remark 4-6-29]).

Lemma 9.2. *Let (X, Δ) be a log canonical surface. Assume that $P \in X$ is not a rational singularity. Then $P \notin \text{Supp } \Delta$ and X is Gorenstein at P .*

Sketch of Proof. If $P \in \text{Supp } \Delta$, then $(X, 0)$ is log terminal at P . In particular, X has only rational singularities in a neighborhood of P . Therefore, we have $P \notin \text{Supp } \Delta$. By the classification of two-dimensional log canonical singularities (see, for example, [KM, Theorem 4.7] and [Ma, Theorem 4-6-28]), $P \in X$ is a simple elliptic singularity or a

cuspidal singularity (see [KM, Note 4.8] and [Ma, Theorem 4-6-28]). We can check that all the other two-dimensional log canonical singularities are rational singularities (see [Ma, Remark 4-6-29]). Therefore, X is Gorenstein at P . \square

Let us start the proof of Theorem 9.1.

Proof of Theorem 9.1. We divide the proof into several small steps.

Step 1. Let $f : Y \rightarrow X$ be the minimal resolution of X with $K_Y + \Delta_Y := f^*(K_X + \Delta)$. By the negativity lemma, we see that Δ_Y is effective. Of course, (Y, Δ_Y) is log canonical since so is (X, Δ) by assumption. Since X is in Fujiki's class \mathcal{C} , the first Betti number $b_1(Y)$ of Y is even. We note that $\kappa(Y, K_Y) \leq \kappa(Y, K_Y + \Delta_Y) = \kappa(X, K_X + \Delta) = -\infty$ holds. By the Enriques–Kodaira classification (see, for example, [BHPV, Chapter VI]) and $\kappa(Y, K_Y) = -\infty$, Y is a smooth projective surface. In particular, X is Moishezon, that is, the algebraic dimension $a(X)$ of X is two.

Step 2. If X has only rational singularities, then X is \mathbb{Q} -factorial (see Lemma 3.10). In this case, by Lemma 4.1, X is projective since we have already known that X is Moishezon in Step 1.

Therefore, from now on, we may assume that X has at least one non-rational singular point.

Step 3. By applying Theorem 8.4 finitely many times, we may assume that if C is an irreducible curve on X with $-(K_X + \Delta) \cdot C > 0$ then $C^2 \geq 0$ holds (see Remark 8.6).

Step 4. Let $g : Z \rightarrow X$ be the minimal resolution of non-rational singularities of X . Then we get the following commutative diagram.

$$\begin{array}{ccc} Y & & \\ h \downarrow & \searrow f & \\ Z & \xrightarrow{g} & X \end{array}$$

Since Z has only rational singularities by construction, we see that Z is \mathbb{Q} -factorial (see Lemma 3.10). Therefore, Z is projective by Lemma 4.1 since Z is Moishezon (see Step 1). We put $K_Z + \Delta_Z := g^*(K_X + \Delta)$. Then Δ_Z is effective by the negativity lemma. Of course, (Z, Δ_Z) is log canonical. More precisely, by Lemma 9.2, we have $\Delta_Z = \sum_i E_i + g_*^{-1}\Delta$, where $\text{Exc}(g) = \sum_i E_i$. Since $K_Z + \Delta_Z$ never becomes nef by $\kappa(Z, K_Z + \Delta_Z) = \kappa(X, K_X + \Delta) = -\infty$ (see Theorem 7.1), we can take a $(K_Z + \Delta_Z)$ -negative extremal ray $R = \mathbb{R}_{\geq 0}[\bar{C}]$ of $\overline{NE}(Z)$, where \bar{C} is an irreducible rational curve on Z .

Step 5. In this step, we will prove the following claim.

Claim 2. *The self-intersection number \bar{C}^2 of \bar{C} is non-negative, where \bar{C} is a $(K_Z + \Delta_Z)$ -negative extremal rational curve as in Step 4.*

Proof of Claim 2. We assume that $\bar{C}^2 < 0$ holds. Since $-(K_Z + \Delta_Z) \cdot \bar{C} > 0$, \bar{C} is not g -exceptional. If $C_X := g_*\bar{C}$ is disjoint from non-rational singularities of X , then we have $(C_X)^2 < 0$ and $-(K_X + \Delta) \cdot C_X > 0$ because g is an isomorphism in a neighborhood of \bar{C} . This is a contradiction. Therefore, C_X passes through at least one non-rational singular point P of X . This implies that $C_X \not\subset \text{Supp } \Delta$ by Lemma 9.2. Let C_Y be the strict transform of \bar{C} on Y . Then we can easily see that $(C_Y)^2 < 0$ holds and that C_Y is not contained in $\text{Supp } \Delta_Y$. Thus we obtain that $-K_Y \cdot C_Y \geq -(K_Y + \Delta_Y) \cdot C_Y = -(K_Z + \Delta_Z) \cdot \bar{C} > 0$. This means that C_Y is a (-1) -curve on Y . In particular, $-K_Y \cdot C_Y = 1$. On the other hand, $\Delta_Y \cdot C_Y \geq 1$ since C_X passes through a non-rational singular point P and the reduced part of Δ_Y contains $f^{-1}(P)$. Thus we have $-(K_Y + \Delta_Y) \cdot C_Y = 1 - \Delta_Y \cdot C_Y \leq 0$. This is a contradiction. Therefore, $\bar{C}^2 \geq 0$ holds. \square

Anyway, every $(K_Z + \Delta_Z)$ -negative extremal ray is spanned by an irreducible rational curve \bar{C} with $\bar{C}^2 \geq 0$.

We will treat the case where $\bar{C}^2 > 0$ and $\bar{C}^2 = 0$ in Step 6 and Step 7, respectively.

Step 6. We assume that there exists a $(K_Z + \Delta_Z)$ -negative extremal ray R of $\overline{NE}(Z)$ spanned by an irreducible rational curve \bar{C} with $\bar{C}^2 > 0$. In this case, it is easy to see that $\rho(Z) = 1$ and $-(K_Z + \Delta_Z)$ is ample. This is a contradiction because $-(K_Z + \Delta_Z) \cdot E = 0$ for every g -exceptional curve E on Z . Therefore, this case does not happen.

Step 7. Anyway, we may assume that there exists a $(K_Z + \Delta_Z)$ -negative extremal ray R spanned by an irreducible rational curve \bar{C} with $\bar{C}^2 = 0$. Then there exists a surjective morphism $\varphi_R : Z \rightarrow W$ onto a smooth projective curve W such that $\rho(Z/W) = 1$ and $-(K_Z + \Delta_Z)$ is φ_R -ample, that is, $\varphi_R : Z \rightarrow W$ is a Mori fiber space. Without loss of generality, we may assume that \bar{C} is a general fiber of φ_R . Since the self-intersection number of any irreducible component of $\text{Exc}(g)$ is negative, $-K_Z \cdot \bar{C} = 2$, and $\Delta_Z \geq \sum_i E_i = \text{Exc}(g)$, we can easily check that there exists an elliptic curve E on Z such that $\text{Exc}(g) = E$ and that $E \rightarrow W$ is an isomorphism. In particular, P is a simple elliptic singularity.

Claim 3. $-K_X$ is ample, that is, X is Fano. In particular, X is projective.

Proof of Claim 3. Since $\varphi_R : Z \rightarrow W$ is a Mori fiber space, we can easily see that $-(K_Z + E)$ is \mathbb{Q} -linearly equivalent to $E + \varphi_R^* D$ where D is some \mathbb{Q} -divisor on W with $\deg D = -E^2 > 0$. Thus, we have $\kappa(Z, -(K_Z + E)) \geq 0$. Since $K_Z + E = g^* K_X$, we obtain $\kappa(X, -K_X) = \kappa(Z, -(K_Z + E)) \geq 0$. We take any irreducible curve C on X . Let C' be the strict transform of C on Z . Then $-K_X \cdot C = -(K_Z + E) \cdot C' = (E + \varphi_R^* D) \cdot C' > 0$ since $C' \neq E$. Thus, we obtain that $-K_X$ is ample by Nakai–Moishezon's ampleness criterion for algebraic spaces (see [P, (1.4) Theorem]). \square

Thus we see that X is projective when X has at least one non-rational singular point.

Anyway, we obtain that X is always projective if (X, Δ) is a log canonical surface in Fujiki's class \mathcal{C} with $\kappa(X, K_X + \Delta) = -\infty$. \square

Remark 9.3. Let (X, Δ) be a log canonical surface in Fujiki's class \mathcal{C} with $\kappa(X, K_X + \Delta) = -\infty$. The proof of Theorem 9.1 says that if X has non-rational singularities then X has one simple elliptic singularity and no cusp singularities.

By Theorem 9.1, we can freely apply the minimal model theory of projective log canonical surfaces established in [Fn4] to log canonical surfaces in Fujiki's class \mathcal{C} with negative Kodaira dimension.

Remark 9.4. We can construct a complete non-projective log canonical algebraic surface (X, Δ) with $\kappa(X, K_X + \Delta) \geq 0$. For some examples, see Section 12 below. Therefore, the assumption $\kappa(X, K_X + \Delta) = -\infty$ is indispensable in Theorem 9.1.

10. PROOF OF THEOREM 1.5

In this section, we prove Theorem 1.5, that is, the minimal model theory for log canonical surfaces in Fujiki's class \mathcal{C} .

Proof of Theorem 1.5. If $(K_X + \Delta) \cdot C \geq 0$ for every curve C on X , then $K_X + \Delta$ is semi-ample by Corollary 7.8. So (X, Δ) is itself a good minimal model of (X, Δ) . Therefore, we may assume that $(K_X + \Delta) \cdot C < 0$ for some curve C on X . If X is projective, then we can run the minimal model program for projective log canonical surfaces and finally get a good minimal model or a Mori fiber space (see [Fn4]). Thus we may assume that

X is not projective. By Theorem 9.1, we obtain $\kappa(X, K_X + \Delta) \geq 0$. Therefore, we have an effective Cartier divisor $D \in |m(K_X + \Delta)|$ for some positive divisible integer m . Since $(K_X + \Delta) \cdot C < 0$ for some curve C on X , C is an irreducible component of $\text{Supp } D$ such that the self-intersection number C^2 is negative. We apply the contraction theorem: Theorem 8.4. Since there are only finitely many irreducible components of $\text{Supp } D$, we finally get (X^*, Δ^*) such that $K_{X^*} + \Delta^*$ is semi-ample after finitely many contractions.

Since $R^1\varphi_{i*}\mathcal{O}_{X_i} = 0$ by Theorem 11.3 below, X_i has only rational singularities if and only if so does X_{i+1} by Lemma 2.6. Therefore, we have (2).

Since φ_i is projective by construction, X_i is projective when so is X_{i+1} . If X_i is projective, then φ_i is the usual contraction morphism associated to a $(K_{X_i} + \Delta_i)$ -negative extremal ray (see Remark 8.5). This implies that X_{i+1} is also projective. Thus we have (1).

Since $\text{Exc}(\varphi_i) \simeq \mathbb{P}^1$, $C_i = \text{Exc}(\varphi_i)$ is \mathbb{Q} -Cartier, and $-C_i$ is φ_i -ample, we can easily check (3) by Theorem 3.11. \square

11. APPENDIX: VANISHING THEOREMS

In this section, we quickly explain some vanishing theorems for the reader's convenience. Fortunately, we do not need difficult analytic methods. They follow from elementary arguments.

Let us start with the following vanishing theorem. We learned it from [KK] (see [K2, Theorem 10.4]).

Theorem 11.1 (Relative vanishing theorem). *Let $\varphi : V \rightarrow W$ be a proper bimeromorphic morphism from a smooth analytic surface to a normal analytic surface W . Assume that there exists a point $P \in W$ such that φ is an isomorphism over $W \setminus P$. Let $\text{Exc}(\varphi) = \sum_i E_i$ be the irreducible decomposition of the φ -exceptional locus $\text{Exc}(\varphi)$. Let \mathcal{L} be a line bundle on V , let N be a \mathbb{Q} -divisor on V , and let $E = \sum_i b_i E_i$ be an effective \mathbb{Q} -divisor on V . Assume that $N \cdot E_i \geq 0$ and $\mathcal{L} \cdot E_i = (K_V + E + N) \cdot E_i$ hold for every i . We further assume that one of the following conditions holds.*

- (1) $0 \leq b_i < 1$ for every i .
- (2) $0 < b_i \leq 1$ for every i and there exists some j such that $b_j \neq 1$.
- (3) $0 < b_i \leq 1$ for every i and there exists some j such that $N \cdot E_j > 0$.

Then $R^i\varphi_*\mathcal{L} = 0$ holds for every $i > 0$.

Theorem 11.1 is a very special case of the Kawamata–Viehweg vanishing theorem. Note that condition (1) is most useful.

Remark 11.2. In Theorem 11.1, it is sufficient to assume that N is a \mathbb{Q} -line bundle on V , that is, a finite \mathbb{Q} -linear combination of some line bundles on V such that $N \cdot E_i \geq 0$ and $\mathcal{L} \cdot E_i = (K_V + E) \cdot E_i + N \cdot E_i$ hold for every i .

Proof of Theorem 11.1. We note that [KK, 2.2.1 Theorem] holds for two-dimensional complex manifolds although it is formulated and proved for two-dimensional regular schemes in [KK]. Therefore, by using the theorem on formal functions for proper morphisms between analytic spaces (see, for example, [BS, Chapter VI, Corollary 4.7]), we get $R^i\varphi_*\mathcal{L} = 0$ for every $i > 0$ (see also [KK, 2.2.5 Corollary]). \square

As an application of Theorem 11.1, we can prove the following vanishing theorem, which is a Kawamata–Viehweg vanishing theorem for proper bimeromorphic morphisms between surfaces.

Theorem 11.3. *Let X be a normal analytic surface and let Δ be an effective \mathbb{Q} -divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $f : X \rightarrow Y$ be a proper bimeromorphic morphism onto a normal analytic surface Y . Let \mathcal{L} be a line bundle on X and let D be a \mathbb{Q} -Cartier Weil divisor on X . Assume that one of the following conditions holds.*

- (1) $\mathcal{L} \cdot C + (D - (K_X + \Delta)) \cdot C > 0$ for every f -exceptional curve C on X and the coefficients of Δ are less than or equal to one.
- (2) $\mathcal{L} \cdot C + (D - (K_X + \Delta)) \cdot C \geq 0$ for every f -exceptional curve C on X and the coefficients of Δ are less than one.

Then $R^i f_*(\mathcal{L} \otimes \mathcal{O}_X(D)) = 0$ holds for every $i > 0$.

In the proof of Theorem 11.3, let B be a \mathbb{Q} -divisor on a normal analytic surface. Then $\lceil B \rceil$ (resp. $\lfloor B \rfloor$) denotes the *round-up* (resp. *round-down*) of B . We put $\{B\} := B - \lfloor B \rfloor$ and call it the *fractional part* of B .

Let us prove Theorem 11.3. The following proof is essentially the same as that of [FT, Theorem 6.2].

Proof of Theorem 11.3. We divide the proof into small steps.

Step 1. Without loss of generality, we can freely shrink Y and assume that Y is a small relatively compact Stein open subset of normal analytic surface. We may further assume that f is an isomorphism outside $P \in Y$ and $f^{-1}(P)$ is one-dimensional.

Step 2. When $\lfloor \Delta \rfloor \neq 0$, we can take an f -ample Cartier divisor H (see (1)). Then we can find an effective \mathbb{Q} -divisor Δ' on X such that $\lfloor \Delta' \rfloor = 0$ and that Δ' is \mathbb{Q} -linearly equivalent to $\Delta + \varepsilon H$ for some $0 < \varepsilon \ll 1$. By replacing Δ with Δ' , we can always assume that $\lfloor \Delta \rfloor = 0$.

Step 3. Let $\varphi : Z \rightarrow X$ be the minimal resolution of X . We set $K_Z + \Delta_Z := \varphi^*(K_X + \Delta)$. We note that Δ_Z is effective by the negativity lemma. We note that

$$\varphi^* \mathcal{L} + \lceil \varphi^* D \rceil - (K_Z + \Delta_Z + \{-\varphi^* D\}) = \varphi^*(\mathcal{L} + D - (K_X + \Delta)).$$

We put $\Theta := \Delta_Z + \{-\varphi^* D\}$. Then

$$\varphi^* \mathcal{L} + \lceil \varphi^* D \rceil - \lfloor \Theta \rfloor = (K_Z + \{\Theta\}) + \varphi^*(\mathcal{L} + D - (K_X + \Delta)).$$

We note that we can write $\{\Theta\} = E + M$ where E is a φ -exceptional effective \mathbb{Q} -divisor with $\lfloor E \rfloor = 0$ and M is an effective \mathbb{Q} -divisor such that every irreducible component of M is not φ -exceptional. Let us consider

$$0 \rightarrow \varphi^* \mathcal{L} \otimes \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor) \rightarrow \varphi^* \mathcal{L} \otimes \mathcal{O}_Z(\lceil \varphi^* D \rceil) \rightarrow \varphi^* \mathcal{L} \otimes \mathcal{O}_{\lfloor \Theta \rfloor}(\lceil \varphi^* D \rceil) \rightarrow 0.$$

By Theorem 11.1, we have

$$R^1 \varphi_*(\varphi^* \mathcal{L} \otimes \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor)) = 0.$$

Therefore, we get the following short exact sequence

$$0 \rightarrow \mathcal{L} \otimes \varphi_* \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor) \rightarrow \mathcal{L} \otimes \varphi_* \mathcal{O}_Z(\lceil \varphi^* D \rceil) \rightarrow \mathcal{L} \otimes \varphi_* \mathcal{O}_{\lfloor \Theta \rfloor}(\lceil \varphi^* D \rceil) \rightarrow 0.$$

By construction, $\lfloor \Theta \rfloor$ is φ -exceptional. Therefore, $\mathcal{L} \otimes \varphi_* \mathcal{O}_{\lfloor \Theta \rfloor}(\lceil \varphi^* D \rceil)$ is a skyscraper sheaf on X . Thus we obtain the following surjection

$$(11.1) \quad R^1 f_*(\mathcal{L} \otimes \varphi_* \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor)) \rightarrow R^1 f_*(\mathcal{L} \otimes \mathcal{O}_X(D)) \rightarrow 0$$

since $\varphi_* \mathcal{O}_Z(\lceil \varphi^* D \rceil) \simeq \mathcal{O}_X(D)$. By the Leray spectral sequence, we have

$$(11.2) \quad R^1 f_*(\mathcal{L} \otimes \varphi_* \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor)) \subset R^1(f \circ \varphi)_*(\varphi^* \mathcal{L} \otimes \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor)).$$

As before, we can write $\{\Theta\} = E' + M'$ where E' is a $f \circ \varphi$ -exceptional effective \mathbb{Q} -divisor with $\lfloor E' \rfloor = 0$ and M' is an effective \mathbb{Q} -divisor such that every irreducible component of M' is not $f \circ \varphi$ -exceptional. By Theorem 11.1 (see also Remark 11.2), we know that

$$R^1(f \circ \varphi)_*(\varphi^* \mathcal{L} \otimes \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor)) = 0.$$

This implies that

$$R^1 f_*(\mathcal{L} \otimes \varphi_* \mathcal{O}_Z(\lceil \varphi^* D \rceil - \lfloor \Theta \rfloor)) = 0$$

by (11.2). By the surjection (11.1), we get $R^1 f_*(\mathcal{L} \otimes \mathcal{O}_X(D)) = 0$.

Step 4. Since $f^{-1}(P)$ is one-dimensional, $R^i f_*(\mathcal{L} \otimes \mathcal{O}_X(D)) = 0$ for every $i > 2$ by the theorem on formal functions for proper morphisms between analytic spaces (see, for example, [BS, Chapter VI, Corollary 4.7]).

Anyway, we have $R^i f_*(\mathcal{L} \otimes \mathcal{O}_X(D)) = 0$ for every $i > 0$. □

We close this section with an obvious remark.

Remark 11.4. Theorems 11.1 and 11.3 can be formulated and proved easily for proper birational morphisms between algebraic surfaces defined over any algebraically closed field. Therefore, we can formulate and prove Theorem 3.11 for projective birational morphisms between (not necessarily complete) algebraic surfaces defined over any algebraically closed field. This is because the proof of Theorem 3.11 only needs the vanishing theorem: Theorem 11.3.

12. COMPLETE NON-PROJECTIVE ALGEBRAIC SURFACES

In this section, we construct some examples of complete non-projective log canonical algebraic surfaces. As was mentioned before, this section is based on a joint work with Kento Fujita. From Example 12.1 to Example 12.5, we will work over \mathbb{C} , the complex number field.

Let us start with Kollár's example. Although it is not stated explicitly in [K1], it does not satisfy Kleiman's ampleness criterion. We note that the arguments in Example 12.3 below help the reader understand Example 12.1. Therefore, we do not explain the details of Kollár's example.

Example 12.1 ([K1, Aside 3.46]). In this example, we will freely use Kollár's notation in [K1, Aside 3.46]. In [K1, Aside 3.46], we assume that C is an elliptic curve. Then the surface S constructed in [K1, Aside 3.46] is a complete non-projective algebraic surface with two simple elliptic singularities. In particular, S is Gorenstein and log canonical. Let C' be the strict transform of $\{1\} \times C$ on S . Then we have $\text{Pic}(S) = \mathbb{Z}\mathcal{O}_S(C')$. We can directly check that $\pi_1(S) = \{1\}$, that is, S is simply connected, $K_S \sim 0$, $H^1(S, \mathcal{O}_S) = 0$, and $H^2(S, \mathcal{O}_S) = \mathbb{C}$. Therefore, S is a log canonical Calabi–Yau algebraic surface. Of course, we have $\kappa(S, K_S) = 0$. Let F be a general fiber of the second projection $\mathbb{P}^1 \times C \rightarrow C$ and let F' be the strict transform of F on S . Then $\overline{NE}(S) = \mathbb{R}_{\geq 0}[F']$ holds. We note that $\mathcal{O}_S(C')$ is positive on $\overline{NE}(S) \setminus \{0\}$. However, C' is nef but is not ample. This means that Kleiman's ampleness criterion does not hold for S .

Let us prepare an easy lemma.

Lemma 12.2. *Let C be a smooth projective curve and let \mathcal{A} be an ample line bundle on C . We consider $\pi : X := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{A}) \rightarrow C$. Then $\mathcal{O}_X(1) \simeq \mathcal{O}_X(C_+)$ is semi-ample, where $C_+ := \mathbb{P}_C(\mathcal{A})$ is the positive section of π , and the complete linear system $|\mathcal{O}_X(m)|$ only contracts the negative section $C_- := \mathbb{P}_C(\mathcal{O}_C)$ of π to a point for some sufficiently large positive integer m .*

Proof. We can easily check that $\mathcal{O}_X(1)$ is semi-ample (see, for example, [La, Lemma 2.3.2]). We note that $C_- \cdot \mathcal{O}_X(1) = 0$ and $D \cdot \mathcal{O}_X(1) > 0$ for every irreducible curve D on X with $D \neq C_-$. Therefore, the complete linear system $|\mathcal{O}_X(m)|$ contracts C_- only. □

Let us construct complete non-projective normal algebraic surfaces S with $\text{Pic}(S) = \{0\}$.

Example 12.3. Let C be a smooth projective curve of genus ≥ 1 and let $\mathcal{L} = \mathcal{O}_C(L)$ be a non-torsion element of $\text{Pic}^0(C)$. We consider $\pi : V := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \rightarrow C$. Let C_1 (resp. C_2)

be the section of π corresponding to $\mathcal{O}_C \oplus \mathcal{L} \rightarrow \mathcal{O}_C \rightarrow 0$ (resp. $\mathcal{O}_C \oplus \mathcal{L} \rightarrow \mathcal{L} \rightarrow 0$). We note that $C_2 \sim \pi^*L + C_1$ holds. We take an arbitrary point $P \in C$ and blow up P_1 and P_2 , where $P_i := \pi^{-1}(P) \cap C_i$ for $i = 1, 2$, to get $p : W \rightarrow V$. Let C'_i be the strict transform of C_i on W for $i = 1, 2$ and let ℓ be the strict transform of $\pi^{-1}(P)$ on W . Let E_i denote the (-1) -curve on W with $p(E_i) = P_i$ for $i = 1, 2$. We put

$$U_1 := W \setminus (C'_2 \cup E_2 \cup \ell), \quad U_2 := W \setminus (C'_1 \cup E_1 \cup \ell), \quad \text{and} \quad U_0 := W \setminus (C'_1 \cup C'_2).$$

We note that $C'_i \subset U_i$ for $i = 1, 2$ and $\ell \subset U_0$ by construction. Then we can realize U_i as a Zariski open subset of $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{A}_i)$ with $\deg \mathcal{A}_i = 1$ such that C'_i corresponds to the negative section C_- on $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{A}_i)$. By Lemma 12.2, we can construct a projective birational morphism $U_i \rightarrow S_i$ onto a normal quasi-projective surface S_i such that the exceptional locus is C'_i for $i = 1, 2$. Since ℓ is a (-2) -curve on a smooth projective surface W , we can construct a projective birational morphism $U_0 \rightarrow S_0$ onto a normal quasi-projective surface S_0 such that the exceptional locus is ℓ and that ℓ is contracted to an A_1 singularity. Now S_0, S_1 , and S_2 can be glued together to get a birational contraction morphism $q : W \rightarrow S$ onto a complete normal algebraic surface S which only contracts C'_1, C'_2 , and ℓ .

Claim 4. $\text{Pic}(S) = \{0\}$ holds.

Proof of Claim 4. We take an arbitrary Cartier divisor D on S . We put $\bar{D} := q^*D$ and $D^\dagger := p_*\bar{D}$. Then we can write $D^\dagger \sim \pi^*M + kC_1$ for some integer k and some Cartier divisor M on C because V is $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$. We note that we can write

$$p^*D^\dagger = \bar{D} + m_1E_1 + m_2E_2$$

for some integers m_1 and m_2 . By construction, $\mathcal{O}_W(\bar{D})|_{C'_i}$ is trivial for $i = 1, 2$. Therefore, we have

$$\mathcal{O}_C(M) \simeq \mathcal{O}_V(D^\dagger)|_{C_2} \simeq \mathcal{O}_W(p^*D^\dagger)|_{C'_2} \simeq \mathcal{O}_W(\bar{D} + m_1E_1 + m_2E_2)|_{C'_2} \simeq \mathcal{O}_C(m_2P)$$

and

$$\mathcal{O}_C(M - kL) \simeq \mathcal{O}_V(D^\dagger)|_{C_1} \simeq \mathcal{O}_W(p^*D^\dagger)|_{C'_1} \simeq \mathcal{O}_W(\bar{D} + m_1E_1 + m_2E_2)|_{C'_1} \simeq \mathcal{O}_C(m_1P).$$

This implies that $\mathcal{O}_C(kL) \simeq \mathcal{O}_C((m_2 - m_1)P)$. Since $\deg L = 0$, we have $m_1 = m_2$. By assumption, $\mathcal{L} = \mathcal{O}_C(L)$ is a non-torsion element of $\text{Pic}^0(C)$. Thus, we get $k = 0$. By construction again, we have $\bar{D} \cdot \ell = 0$. Therefore,

$$0 = k = p^*D^\dagger \cdot \ell = (\bar{D} + m_1E_1 + m_2E_2) \cdot \ell = m_1 + m_2$$

holds. Since we have already known that $m_1 = m_2$ holds, we obtain that $m_1 = m_2 = 0$. Anyway, we have $\bar{D} = p^*D^\dagger \sim p^*\pi^*M$. Since $\mathcal{O}_W(\bar{D})|_{C'_i}$ is trivial for $i = 1, 2$, we see that $\mathcal{O}_C(M)$ is trivial. Therefore, we obtain $\bar{D} \sim 0$. So we get $D \sim 0$. This means that $\text{Pic}(S) = \{0\}$. \square

By Claim 4, we see that S is obviously non-projective. By construction, we can see that $\pi_1(S) = \{1\}$, that is, S is simply connected.

From now on, we assume that C is an elliptic curve. Then S has two simple elliptic singularities and one A_1 singularity. Moreover, S is Gorenstein and $K_S \sim 0$. We note that $K_V + C_1 + C_2 \sim 0$ and $K_W + C'_1 + C'_2 \sim 0$ by construction. We can easily check that $H^1(S, \mathcal{O}_S) = 0$ and $H^2(S, \mathcal{O}_S) = \mathbb{C}$ by using the Leray spectral sequence. Anyway, S is a complete non-projective log canonical Calabi–Yau algebraic surface with $\text{Pic}(S) = \{0\}$.

From now on, by taking blow-ups of S in Example 12.3, we construct complete non-projective algebraic surfaces with large Picard number.

Example 12.4. Let S be the surface constructed in Example 12.3, which is birationally equivalent to $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$. As we saw above, $\text{Pic}(S) = \{0\}$ holds. We take the minimal resolution $\mu : \tilde{S} \rightarrow S$ of the unique A_1 singularity of S . Since S is simply connected, so is \tilde{S} . Let E denote the exceptional curve of μ . Of course, E is the strict transform of ℓ in Example 12.3 and is a (-2) -curve on \tilde{S} .

Claim 5. $\text{Pic}(\tilde{S}) = \mathbb{Z}\mathcal{O}_{\tilde{S}}(E)$ holds.

Proof of Claim 5. By construction, we see that $E \simeq \mathbb{P}^1$ and that $-E$ and $-(K_{\tilde{S}} + E)$ are both μ -ample. Thus, by Theorem 3.11 (see also Remark 11.4), we have $\text{Pic}(\tilde{S}) \otimes \mathbb{Q} = \mathbb{Q}\mathcal{O}_{\tilde{S}}(E)$ since $\text{Pic}(S) = \{0\}$. Let \mathcal{N} be a torsion element of $\text{Pic}(\tilde{S})$. Then we have $\mathcal{N} \cdot E = 0$. Therefore, by Theorem 3.11 again, \mathcal{N} is trivial. This means that $\text{Pic}(\tilde{S})$ is torsion-free. Thus, we can write $\text{Pic}(\tilde{S}) = \mathbb{Z}\mathcal{M}$ for some line bundle \mathcal{M} on \tilde{S} . Therefore, there exists some integer a such that $\mathcal{O}_{\tilde{S}}(E) \simeq \mathcal{M}^{\otimes a}$. Since $E^2 = -2$, we have $aE \cdot \mathcal{M} = -2$. Note that $E \cdot \mathcal{M}$ is an integer. If $a = \pm 2$, then $-2 = E^2 = a^2\mathcal{M}^2 = 4\mathcal{M}^2$. This is a contradiction because \mathcal{M}^2 is an integer. Thus, we get $a = \pm 1$. This means that $\text{Pic}(\tilde{S}) = \mathbb{Z}\mathcal{O}_{\tilde{S}}(E)$. \square

By Claim 5, we have $N^1(\tilde{S}) = \mathbb{R}$ and $\overline{NE}(\tilde{S}) = N_1(\tilde{S}) = \mathbb{R}$. We note that $E^2 = -2$ and that there exists a curve G on \tilde{S} such that $G \cdot E > 0$.

We further assume that C is an elliptic curve. Then $K_{\tilde{S}} \sim 0$ and \tilde{S} has only two simple elliptic singularities. Therefore, \tilde{S} is a complete non-projective log canonical Calabi–Yau algebraic surface with $\overline{NE}(\tilde{S}) = \mathbb{R}$.

Example 12.5. Let \tilde{S} be the surface constructed in Example 12.4. We take finitely many smooth points $Q_1, Q_2, \dots, Q_{\rho-1}$ of \tilde{S} with $\rho \geq 2$ such that $Q_i \neq Q_j$ for $i \neq j$ and $Q_i \notin E$ for every i . We blow up $Q_1, Q_2, \dots, Q_{\rho-1}$ to get $\sigma : \bar{S} \rightarrow \tilde{S}$. Let B_i denote the (-1) -curve on \bar{S} with $\sigma(B_i) = Q_i$ for every i . Then, by Theorem 3.11 (see also Remark 11.4), we can easily check that

$$\text{Pic}(\bar{S}) = \mathbb{Z}\mathcal{O}_{\bar{S}}(\sigma^*E) \oplus \mathbb{Z}\mathcal{O}_{\bar{S}}(B_1) \oplus \cdots \oplus \mathbb{Z}\mathcal{O}_{\bar{S}}(B_{\rho-1}).$$

Claim 6. $\overline{NE}(\bar{S}) = N_1(\bar{S}) = \mathbb{R}^\rho$ holds.

Proof of Claim 6. Let D be a nef \mathbb{R} -Cartier \mathbb{R} -divisor on \bar{S} . It is sufficient to prove that D is numerically trivial. By the above description of $\text{Pic}(\bar{S})$, we can assume that $D = b_0\sigma^*E + \sum_{i=1}^{\rho-1} b_i B_i$, where $b_i \in \mathbb{R}$ for every i . Since $D \cdot B_i \geq 0$ for every i and $D \cdot \sigma^*E \geq 0$, $b_i \leq 0$ holds for every i . We assume that $D \neq 0$. Then we can take an irreducible curve G on \bar{S} such that $G \not\subset \text{Supp } D$ and that $G \cap \text{Supp } D \neq \emptyset$. This is because the smooth locus of \bar{S} is a quasi-projective open subset of \bar{S} and contains σ^*E and B_i for every i . Thus we get $D \cdot G < 0$. This is a contradiction. This means that $D = 0$. Hence we have $\overline{NE}(\bar{S}) = N_1(\bar{S}) = \mathbb{R}^\rho$. \square

If C is an elliptic curve, then \bar{S} is a complete non-projective log canonical algebraic surface with $\overline{NE}(\bar{S}) = N_1(\bar{S}) = \mathbb{R}^\rho$. By construction, we see that $K_{\bar{S}} = \sum_{i=1}^{\rho-1} B_i$. We can apply the minimal model program established in Theorem 1.5 to a complete non-projective log canonical algebraic surface \bar{S} . Then every B_i is contracted to a smooth point and we finally get a good minimal model \tilde{S} , which is a surface with $K_{\tilde{S}} \sim 0$ constructed in Example 12.4.

The following commutative diagram may help the reader understand the constructions in Examples 12.3, 12.4, and 12.5.

$$\begin{array}{ccccc}
& & W & \xrightarrow{\quad} & \tilde{S} & \xleftarrow{\sigma} & \bar{S} \\
& & \swarrow p & & \searrow \mu & & \\
& & V & & S & & \\
& & \downarrow \pi & & & & \\
& & C & & & &
\end{array}$$

The reader can find various examples of complete non-projective toric threefolds X with $\text{Pic}(X) = \{0\}$, $\overline{NE}(X) = \mathbb{R}_{\geq 0}$, or $\overline{NE}(X) = N_1(X)$ in [Fn1] and [FP].

Finally, we make a remark on complete non-projective algebraic surfaces defined over an algebraically closed field k with $k \neq \mathbb{C}$.

Remark 12.6. We note that every complete algebraic surface defined over $\overline{\mathbb{F}}_p$ is always \mathbb{Q} -factorial (see, for example, [T1, Theorem 4.5]). So it automatically becomes projective (see, for example, [Fn4, Lemma 2.2]). By the following lemma (see Lemma 12.7 below), the constructions in this section and Kollár's construction in [K1, Aside 3.46] can work for algebraic surfaces defined over an algebraically closed field k such that $k \neq \overline{\mathbb{F}}_p$ for every prime number p with some suitable modifications. We note that Theorem 3.11 holds true for algebraic surfaces defined over any algebraically closed field (see Remark 11.4). Anyway, we can construct complete non-projective algebraic surfaces over an algebraically closed field k such that $k \neq \overline{\mathbb{F}}_p$ for every prime number p .

Lemma 12.7. *Let C be a smooth projective curve defined over an algebraically closed field k whose genus $g(C)$ is positive. Let P be an arbitrary closed point of C . Assume that $k \neq \overline{\mathbb{F}}_p$ for every prime number p . Then we can take $Q \in C$ such that $\mathcal{O}_C(Q - P)$ is a non-torsion element of $\text{Pic}^0(C)$.*

Proof. By $g(C) \geq 1$ and $k \neq \overline{\mathbb{F}}_p$, we can take a non-torsion element \mathcal{L} of $\text{Pic}^0(C)$ (see, for example, [T1, Fact 2.3]). We take a large positive integer m . Then $\mathcal{L} \otimes \mathcal{O}_C(mP)$ is very ample. We consider a general member $Q_1 + \cdots + Q_m$ of $|\mathcal{L} \otimes \mathcal{O}_C(mP)|$. Then we have $\mathcal{L} \simeq \mathcal{O}_C((Q_1 - P) + \cdots + (Q_m - P))$. Therefore, there exists some i_0 such that $\mathcal{O}_C(Q_{i_0} - P)$ is a non-torsion element of $\text{Pic}^0(C)$. \square

13. REMARKS ON THE MINIMAL MODEL THEORY FOR ALGEBRAIC LOG SURFACES

In this section, we give some remarks on the minimal model theory for algebraic log surfaces established in [Fn4] and [T1]. We note that a log surface in this section means a pair (X, Δ) consisting of a normal algebraic surface defined over an algebraically closed field and an effective \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier and that the coefficients of Δ are in $[0, 1]$. One of the main purposes of this section is to completely generalize the minimal model theory for MR log canonical surfaces discussed in [A, Section 10]. More precisely, we introduce the notion of generalized MR log canonical surfaces and show that the minimal model theory established in [Fn4] and [T1] works for generalized MR log canonical surfaces.

We will treat algebraic surfaces defined over any algebraically closed field k throughout this section. This section is independent of the other sections. Of course, we will freely use the notation and the results in [Fn3], [Fn4], and [T1].

Let us start with the following well-known definition of log canonical surfaces, klt surfaces, and multiplier ideal sheaves.

Definition 13.1. Let X be a normal surface and let Δ be an \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a proper birational morphism from a normal surface Y . Then we can write $K_Y + \Delta_Y = f^*(K_X + \Delta)$ with $f_*(K_Y + \Delta_Y) = K_X + \Delta$ as usual.

If Δ is effective and the coefficients of Δ_Y are ≤ 1 (resp. < 1) for every $f : Y \rightarrow X$, then (X, Δ) is called a *log canonical surface* (resp. *klt surface*).

We assume that Y is smooth and $\text{Supp } \Delta_Y$ is a simple normal crossing divisor on Y . Then we put

$$\mathcal{J}(X, \Delta) := f_* \mathcal{O}_Y(-[\Delta_Y]) = f_* \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta) \rceil)$$

and call it the *multiplier ideal sheaf* of the pair (X, Δ) . It is not difficult to see that $\mathcal{J}(X, \Delta)$ is independent of the resolution $f : Y \rightarrow X$. We note that $\mathcal{J}(X, \Delta) \subset \mathcal{O}_X$ holds when Δ is effective.

Assume that Δ is an effective \mathbb{R} -divisor. Then we can check that (X, Δ) is klt if and only if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$.

Let us recall the definition of MR log canonical surfaces in [A].

Definition 13.2 (MR log canonical surfaces). Let (X, Δ) be a log surface, that is, X is a normal surface and Δ is a boundary \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be the minimal resolution of X . Then we can write $K_Y + \Delta_Y = f^*(K_X + \Delta)$ as usual. If all the coefficients of Δ_Y are less than or equal to one, then we say that (X, Δ) is *MR log canonical* (MRLC, for short). We note that Δ_Y is always effective by the negativity lemma.

By definition, it is obvious that log canonical surfaces are MR log canonical.

Let us introduce the notion of generalized MR log canonical surfaces.

Definition 13.3 (Generalized MR log canonical surfaces). Let (X, Δ) be a log surface. If there exists a proper birational morphism $f : Y \rightarrow X$ from a \mathbb{Q} -factorial surface Y such that Δ_Y is a boundary \mathbb{R} -divisor on Y , where $K_Y + \Delta_Y = f^*(K_X + \Delta)$ as usual, then we say that (X, Δ) is a *generalized MR log canonical* (GMRLC, for short) surface.

Of course, if (X, Δ) is MRLC, then it is GMRLC by definition. Moreover, if (X, Δ) is a \mathbb{Q} -factorial log surface, then (X, Δ) is obviously GMRLC.

The abundance theorem for GMRLC surfaces easily follows from [Fn4, Theorem 8.1] and [T1, Theorem 6.7].

Theorem 13.4 (Abundance theorem for GMRLC surfaces). *Let (X, Δ) be a GMRLC surface and let $\pi : X \rightarrow S$ be a proper surjective morphism onto a variety S . Assume that $K_X + \Delta$ is π -nef. Then $K_X + \Delta$ is π -semi-ample.*

Proof. By definition, there exist a proper birational morphism $f : Y \rightarrow X$ from a \mathbb{Q} -factorial surface Y and a boundary \mathbb{R} -divisor Δ_Y on Y such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$. It is obvious that $K_Y + \Delta_Y$ is $\pi \circ f$ -nef. Therefore, $K_Y + \Delta_Y$ is $\pi \circ f$ -semi-ample by the abundance theorem for \mathbb{Q} -factorial log surfaces (see [Fn4, Theorem 8.1] and [T1, Theorem 6.7]). Thus, we obtain that $K_X + \Delta$ is f -semi-ample. \square

As an easy consequence of Theorem 13.4, we have a weak version of the basepoint-free theorem for GMRLC surfaces. This weak version seems to be almost sufficient for many applications.

Corollary 13.5. *Let (X, Δ) be a GMRLC surface and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Let D be a π -nef Cartier divisor on X such that $D - (K_X + \Delta)$ is π -ample. Then D is π -semi-ample.*

Proof. Without loss of generality, we may assume that S is affine and $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$. By Bertini's theorem, we can write $D \sim_{\mathbb{R},\pi} K_X + \Delta + A$ such that $(X, \Delta + A)$ is GMRLC. By Theorem 13.4, D is π -semi-ample. \square

The following lemma is a key result for GMRLC surfaces.

Lemma 13.6. *Let (X, Δ) be a GMRLC (resp. an MRLC) surface and let $\varphi : X \rightarrow Z$ be a proper birational morphism onto a normal surface Z such that $-(K_X + \Delta)$ is φ -nef. Then (Z, Δ_Z) is GMRLC (resp. MRLC), where $\Delta_Z = \varphi_*\Delta$.*

Proof. First, we assume that (X, Δ) is GMRLC. We take a proper birational morphism $f : Y \rightarrow X$ from a \mathbb{Q} -factorial surface Y with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ as in Definition 13.3. Since $-(K_X + \Delta)$ is φ -nef, $-(K_Y + \Delta_Y)$ is nef over Z . Then, by the negativity lemma, we can uniquely take an \mathbb{R} -divisor Θ on Y with $\Theta \leq \Delta_Y$ such that $-(K_Y + \Theta)$ is numerically trivial over Z and $(\varphi \circ f)_*\Theta = \Delta_Z$. We write $\Theta = \Theta_+ - \Theta_-$, where Θ_+ and Θ_- are both effective \mathbb{R} -divisors with no common components. By construction, $K_Y + \Theta_+$ is numerically equivalent to Θ_- over Z and Θ_- is exceptional over Z . We note that Y is projective over Z since it is \mathbb{Q} -factorial. Thus we can run the minimal model program over Z with respect to $K_Y + \Theta_+$ (see [Fn4, Theorem 3.3] and [T1, Theorem 6.5]). Then we finally get a \mathbb{Q} -factorial log surface (Y^*, Θ_+^*) such that $K_{Y^*} + \Theta_+^*$ is numerically trivial over Z . By the abundance theorem (see [Fn4, Theorem 8.1] and [T1, Theorem 6.7]), $K_{Y^*} + \Theta_+^*$ is \mathbb{R} -linearly equivalent to zero over Z . By construction, $p_*(K_{Y^*} + \Theta_+^*) = K_Z + \Delta_Z$ holds, where $p : Y^* \rightarrow Z$, and $K_Z + \Delta_Z$ is \mathbb{R} -Cartier. Thus, we have $K_{Y^*} + \Theta_+^* = p^*(K_Z + \Delta_Z)$. This means that (Z, Δ_Z) is GMRLC because (Y^*, Θ_+^*) is a \mathbb{Q} -factorial log surface.

Next, we assume that (X, Δ) is MRLC. Then we may assume that $f : Y \rightarrow X$ is the minimal resolution of X in the above argument. Then we can easily see that Y^* is a smooth surface. More precisely, $Y \rightarrow Y^*$ is a composition of contractions of (-1) -curves. If there exists a (-1) -curve in the fiber of $p : Y^* \rightarrow Z$, then we contract it to a smooth point. By repeating this process finitely many times, we may further assume that $p : Y^* \rightarrow Z$ is the minimal resolution of Z . Therefore, (Z, Δ_Z) is MRLC. \square

As an easy application of Lemma 13.6, we prove the following useful lemma although we do not use it explicitly in this paper.

Lemma 13.7. *Let (X, Δ) be a GMRLC (resp. an MRLC) surface and let Δ' be an effective \mathbb{R} -divisor on X such that $\Delta' \leq \Delta$. Then (X, Δ') is GMRLC (resp. MRLC).*

Proof. We take a proper birational morphism $f : Y \rightarrow X$ from a \mathbb{Q} -factorial surface Y with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ as in Definition 13.3. We note that $f_*^{-1}(\Delta - \Delta')$ is f -nef. Therefore, $-(K_Y + \Delta_Y - f_*^{-1}(\Delta - \Delta'))$ is f -nef. We also note that $f_*(\Delta_Y - f_*^{-1}(\Delta - \Delta')) = \Delta'$ by construction. Of course, $(Y, \Delta_Y - f_*^{-1}(\Delta - \Delta'))$ is GMRLC since Y is \mathbb{Q} -factorial. Therefore, by Lemma 13.6, (X, Δ') is GMRLC. When (X, Δ) is MRLC, we can assume that Y is smooth in the above argument. In this case, $(Y, \Delta_Y - f_*^{-1}(\Delta - \Delta'))$ is MRLC. Therefore, (X, Δ) is MRLC by Lemma 13.6. \square

Let us recall a useful vanishing theorem for log surfaces.

Theorem 13.8 (Kawamata–Viehweg–Nadel vanishing theorem for log surfaces). *Let $\pi : X \rightarrow S$ be a proper surjective morphism from a normal surface X onto a variety S . When the characteristic of k is positive, we further assume that $\dim S \geq 1$. Let Δ be an \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let L be a Cartier divisor on X such that $L - (K_X + \Delta)$ is π -nef and π -big. Then $R^i\pi_*(\mathcal{O}_X(L) \otimes \mathcal{J}(X, \Delta)) = 0$ for every $i > 0$, where $\mathcal{J}(X, \Delta)$ is the multiplier ideal sheaf of (X, Δ) .*

Sketch of Proof. If the characteristic of k is zero, then we can assume that $k = \mathbb{C}$ by the Lefschetz principle. In this case, the statement is well-known (see, for example, [Fn7,

Theorem 3.4.2]). If the characteristic of k is positive, then the desired vanishing theorem follows from [T2, Theorem 2.7] (see also [T3, Section 3]). \square

Now we state the basepoint-free theorem for log surfaces (see also [T3, Theorem 4.2]).

Theorem 13.9 (Basepoint-freeness for log surfaces I). *Let X be a normal surface and let Δ be an effective \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier and $[\Delta] = 0$. Let $\pi : X \rightarrow S$ be a projective surjective morphism onto a variety S . Let D be a π -nef Cartier divisor on X such that $aD - (K_X + \Delta)$ is π -nef and π -big for some positive integer a . Then there exists a positive integer m_0 such that $\mathcal{O}_X(mD)$ is π -generated for every integer $m \geq m_0$.*

Before we explain the proof of Theorem 13.9, we prepare two easy non-vanishing theorems.

Lemma 13.10 (Non-vanishing theorem I). *Let C be a smooth projective curve C and let G be a \mathbb{Q} -divisor on C such that $[G] \leq 0$. Let D be a nef Cartier divisor on C such that $aD - (K_C + G)$ is ample for some positive integer a . Then $H^0(C, \mathcal{O}_C(mD + \lceil -G \rceil)) \neq 0$ for every integer $m \geq a$.*

Proof. If $C \simeq \mathbb{P}^1$, then $H^0(C, \mathcal{O}_C(mD + \lceil -G \rceil)) \neq 0$ for every non-negative integer m since $\deg D \geq 0$ and $\lceil -G \rceil \geq 0$. If $C \not\simeq \mathbb{P}^1$, then we can easily check that $H^0(C, \mathcal{O}_C(mD + \lceil -G \rceil)) \neq 0$ for every $m \geq a$ by the Riemann–Roch formula. More precisely, we have

$$\begin{aligned} & \dim H^0(C, \mathcal{O}_C(mD + \lceil -G \rceil)) - \dim H^1(C, \mathcal{O}_C(mD + \lceil -G \rceil)) \\ &= \deg(mD + \lceil -G \rceil) - g + 1 > \deg(K_C + \{G\}) - g + 1 \\ &\geq 2g - 2 - g + 1 = g - 1 \geq 0 \end{aligned}$$

for every $m \geq a$, where g denotes the genus of C . \square

Lemma 13.11 (Non-vanishing theorem II). *Let (X, Δ) be a klt surface and let $\pi : X \rightarrow S$ be a proper surjective morphism with $\dim S \geq 1$. Let D be a π -nef Cartier divisor on X such that $aD - (K_X + \Delta)$ is π -nef and π -big for some positive integer a . Then there exists a positive integer m' such that $\pi_* \mathcal{O}_X(mD) \neq 0$ for every integer $m \geq m'$.*

Proof. If $\dim S = 2$, then $\pi_* \mathcal{O}_X(mD) \neq 0$ obviously holds for every integer m . Therefore, we may assume that S is a smooth curve with $\pi_* \mathcal{O}_X \simeq \mathcal{O}_S$. We may further assume that every fiber of $\pi : X \rightarrow S$ is irreducible by shrinking S suitably. In this case, D is π -ample or π -numerically trivial since π is flat. When D is π -ample, the statement is obvious. When D is π -numerically trivial, $mD - (K_X + \Delta)$ is π -nef and π -big for every integer m . Therefore, $R^i \pi_* \mathcal{O}_X(mD) = 0$ for every integer m and every $i > 0$ by Theorem 13.8. This implies that $\pi_* \mathcal{O}_X(mD) \neq 0$ for every integer m since π is flat and $\pi_* \mathcal{O}_X \neq 0$. \square

Let us explain the proof of Theorem 13.9.

Sketch of Proof of Theorem 13.9. Here, we will explain how to prove Theorem 13.9. Without loss of generality, we may assume that S is affine and $\pi_* \mathcal{O}_X \simeq \mathcal{O}_S$. By Kodaira's lemma (see, for example, [Fn7, Lemma 2.1.29]) and perturbing the coefficients of Δ slightly, we further assume that $aD - (K_X + \Delta)$ is π -ample and that Δ is a \mathbb{Q} -divisor.

Case 1. We assume that the characteristic of k is positive and $\dim S = 0$.

If D is not numerically trivial, then the statement follows from [T2, Theorem 3.2]. If D is numerically trivial, then $\text{Pic}(X)$ is a free abelian group of finite rank by [OO, Theorem 1.5] since $-(K_X + \Delta)$ is ample. Therefore, we have $D \sim 0$. Anyway, we obtain the desired statement when $\dim S = 0$ and the characteristic of k is positive.

Case 2. We assume that (X, Δ) is klt and the characteristic of k is zero.

This case is a special case of the well-known basepoint-free theorem for klt pairs. It can be proved by the traditional X-method. For the details of the X-method, see, for example, [Fn3, Section 4.2].

Case 3. We assume that (X, Δ) is not klt. We further assume that $\dim S \geq 1$ holds when the characteristic of k is positive.

In this case, a slightly modified version of the X-method works by the vanishing theorem (see Theorem 13.8) and the non-vanishing theorem (see Lemma 13.10). There are no difficulties to adapt the arguments in the proof of [T2, Theorem 3.2] for our setting.

Case 4. We assume that the characteristic of k is positive, $\dim S \geq 1$, and (X, Δ) is klt.

In this case, we first use Lemma 13.11 and apply the traditional X-method. Then we can obtain the desired basepoint-freeness by the vanishing theorem (see Theorem 13.8) and the non-vanishing theorem (see Lemma 13.10).

Anyway, we can prove the desired basepoint-freeness for log surfaces. \square

As a Corollary of Theorem 13.9, we have the following statement.

Theorem 13.12 (Basepoint-freeness for log surfaces II). *Let (X, Δ) be a log surface and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Let D be a π -nef Cartier divisor on X such that $aD - (K_X + \Delta)$ is π -ample for some positive integer a . Then there exists a positive integer m_0 such that $\mathcal{O}_X(mD)$ is π -generated for every integer $m \geq m_0$.*

Proof. Without loss of generality, we may assume that S is affine and $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$. We take a π -ample Cartier divisor H on X . Then we can take a large positive integer b such that $[\Delta] + bH \sim G \geq 0$ and that $\text{Supp } G$ contains no irreducible components of $[\Delta]$. By replacing Δ with $\Delta - \varepsilon[\Delta] + \varepsilon G$, which is \mathbb{Q} -linearly equivalent to $\Delta + \varepsilon bH$, for $0 < \varepsilon \ll 1$, we may further assume that $[\Delta] = 0$. Then this theorem follows from Theorem 13.9. \square

For the reader's convenience, we explicitly state the cone and contraction theorem for log surfaces (see also [T3, Theorem 4.4]).

Theorem 13.13 (Cone and contraction theorem for log surfaces). *Let (X, Δ) be a log surface and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Then*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X + \Delta \geq 0} + \sum_j R_j$$

with the following properties:

- (i) R_j is a $(K_X + \Delta)$ -negative extremal ray of $\overline{NE}(X/S)$ for every j .
- (ii) Let H be a π -ample \mathbb{R} -divisor on X . Then there are only finitely many R_j 's included in $(K_X + \Delta + H)_{<0}$. In particular, the R_j 's are discrete in the half-space $(K_X + \Delta)_{<0}$.
- (iii) Let R be a $(K_X + \Delta)$ -negative extremal ray of $\overline{NE}(X/S)$. Then there exists a contraction morphism $\varphi_R : X \rightarrow Y$ over S with the following properties:
 - (a) Let C be an irreducible curve on X such that $\pi(C)$ is a point. Then $\varphi_R(C)$ is a point if and only if $[C] \in R$.
 - (b) $\mathcal{O}_Y \simeq (\varphi_R)_*\mathcal{O}_X$.
 - (c) Let \mathcal{L} be a line bundle on X such that $\mathcal{L} \cdot C = 0$ for every curve C with $[C] \in R$. Then there exists a line bundle \mathcal{M} on Y such that $\mathcal{L} \simeq \varphi_R^*\mathcal{M}$.

Sketch of Proof of Theorem 13.13. If the characteristic of the base field k is zero, then Theorem 13.13 follows from [Fn4, Theorem 3.2], which is a special case of [Fn3, Theorem 1.1]. Although [Fn4, Theorem 3.2] (see also [Fn3, Theorem 1.1]) is stated only for the case where $k = \mathbb{C}$, it holds true under the assumption that the characteristic of k is zero. This is because the Kodaira-type vanishing theorems in [Fn3] hold true when the characteristic

of k is zero by the Lefschetz principle. Anyway, for the details of the cone and contraction theorem in characteristic zero, we strongly recommend the reader to see [Fn3].

From now on, we assume that the characteristic of k is positive. Then the cone theorem, that is, (i) and (ii), holds true by [T1, Theorem 6.2]. Therefore, it is sufficient to prove (iii). We can take a π -nef Cartier divisor D on X such that $R = \overline{NE}(X/S) \cap D^\perp$. Then $aD - (K_X + \Delta)$ is π -ample for some positive integer a . By Theorem 13.12, there exists a positive integer m_0 such that $\mathcal{O}_X(mD)$ is π -generated for every integer $m \geq m_0$. We take the Stein factorization of the associated morphism. Then we have a contraction morphism $\varphi_R : X \rightarrow Y$ over S satisfying (a) and (b). By construction, $-(K_X + \Delta)$ is φ_R -ample. Therefore, $\mathcal{L} - (K_X + \Delta)$ is also φ_R -ample. By Theorem 13.12 again, we see that there exists a positive integer n_0 such that $\mathcal{L}^{\otimes n}$ is φ_R -generated for every integer $n \geq n_0$. This implies (c). \square

Theorem 13.14 is the main result of this section.

Theorem 13.14 (Minimal model program for GMRLC surfaces). *Let (X, Δ) be a GMRLC surface and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Then we can run the minimal model program over S with respect to $K_X + \Delta$. So, there exists a sequence of at most $\rho(X/S) - 1$ contractions starting from (X, Δ)*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} (X_1, \Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{l-1}} (X_l, \Delta_l) =: (X^*, \Delta^*)$$

over S such that (X_i, Δ_i) , where $\Delta_i := \varphi_{i-1*}\Delta_{i-1}$, is GMRLC for every i and that one of the followings holds.

- (i) (Good minimal model). *If $K_X + \Delta$ is pseudo-effective over S , then $K_{X^*} + \Delta^*$ is semi-ample over S .*
- (ii) (Mori fiber space). *If $K_X + \Delta$ is not pseudo-effective over S , then there is a contraction morphism $g : X^* \rightarrow W$ over S onto a normal variety W , which is projective over S , with connected fibers such that $-(K_{X^*} + \Delta^*)$ is g -ample, $\dim W < 2$, and the relative Picard number $\rho(X^*/W)$ is one.*

Moreover, we have the following properties.

- (1) *If (X, Δ) is log canonical, then so is (X_i, Δ_i) for every i .*
- (2) *If X is \mathbb{Q} -factorial then so is X_i for every i .*
- (3) *If (X, Δ) is MRLC, then so is (X_i, Δ_i) for every i .*

Sketch of Proof. Since (X, Δ) is a log surface that is projective over S , we can apply the cone and contraction theorem to (X, Δ) (see Theorem 13.13). By Lemma 13.6, we can run the minimal model program under the assumption that (X, Δ) is GMRLC. By the abundance theorem (see Theorem 13.4), $K_{X^*} + \Delta^*$ is semi-ample over S when $K_{X^*} + \Delta^*$ is nef over S .

If (X, Δ) is log canonical, then we can easily see that (X_i, Δ_i) is also log canonical for every i by the negativity lemma. Anyway, (1) is the classical minimal model theory for log canonical surfaces. For the details, see [Fn4] and [T1].

The case where (X, Δ) is a \mathbb{Q} -factorial log surface (see (2)) was established in characteristic 0 and $p > 0$ in [Fn4] and [T1], respectively.

If (X, Δ) is MRLC, then (X_i, Δ_i) is also MRLC by Lemma 13.6. Therefore, (3) holds. Note that (3) is the minimal model program sketched in [A, Section 10]. \square

Anyway, Theorem 13.14 covers the minimal model theory for log canonical surfaces, \mathbb{Q} -factorial log surfaces, and MR log canonical surfaces.

We note that some results in this section were already discussed in a more general setting by Hiromu Tanaka. We recommend the interested reader to see [T3].

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