INTRODUCTION TO THE THEORY OF QUASI-LOG VARIETIES

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Abstract. This paper is a gentle introduction to the theory of quasi-log varieties by Ambro. We explain the fundamental theorems for the log minimal model program for log canonical pairs. More precisely, we give a proof of the base point free theorem for log canonical pairs in the framework of the theory of quasi-log varieties.

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1. Introduction

The aim of this article is to explain the fundamental theorems for the log minimal model program for log canonical pairs. More explicitly, we describe the base point free theorem for log canonical pairs in the framework of the theory of quasi-log varieties (see Corollary 4.2). We also treat the cone theorem for log canonical pairs (see Theorem 5.3). This paper is a gentle introduction to Ambro’s theory of quasi-log varieties (cf. [A]). It contains no new statements. However, it must be valuable
because there are no introductory articles for the theory of quasi-log varieties. The original article [A] seems to be inaccessible even for experts. We basically follow Ambro’s arguments (see [A, Section 5]) but we change them slightly to clarify the basic ideas and to remove some ambiguities and mistakes. The book [F6] is a comprehensive survey of the fundamental theorems of the log minimal model program from the viewpoint of the theory of quasi-log varieties. A new approach to the log minimal model program for log canonical pairs without using quasi-log varieties was found in [F7]. It seems to be more natural and much easier than the theory of quasi-log varieties.

Note that we only use $\mathbb{Q}$-divisors for simplicity. Some of the results can be generalized for $\mathbb{R}$-divisors with a little care. We do not treat the relative versions of the fundamental theorems in order to make our arguments transparent. There are no difficulties for the reader to obtain the relative versions once he understands this paper. We hope that this article will make the theory of quasi-log varieties more accessible. Note that the reader does not have to refer [A] in order to read this article. Our formulation is slightly different from the one in [A]. So, if the reader wants to taste the original flavor of the theory of quasi-log varieties, then he has to see [A].

We summarize the contents of this paper. In Section 2, we quickly review the torsion-freeness and the vanishing theorem in [F6, Chapter 2]. In Section 3, we introduce the notion of qlc pairs, which is a special case of Ambro’s quasi-log varieties, and prove some important and useful lemmas. Theorem 3.6 is a key result in the theory of quasi-log varieties. Section 4 is devoted to the proof of the base point free theorem for qlc pairs. This section is the heart of this paper. In Section 5, we treat the rationality theorem and the cone theorem for lc pairs. We note that the rationality theorem directly implies the cone theorem and that we do not need the theory of quasi-log varieties for the proof of the rationality theorem. In the final section: Section 6, we explain some related topics.

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1.1. Notation and Conventions. We will work over the complex number field $\mathbb{C}$ throughout this paper. But we note that by using the Lefschetz principle, we can extend everything to the case where the base field is an algebraically closed field of characteristic zero. We will use the following notation and the notation in [KM] freely.
Notation. (i) For a $\mathbb{Q}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that $D_j$ is a prime divisor for every $j$ and $D_i \neq D_j$ for $i \neq j$, we define the round-up $\lceil D \rceil = \sum_{j=1}^{r} \lceil d_j \rceil D_j$ (resp. the round-down $\lfloor D \rfloor = \sum_{j=1}^{r} \lfloor d_j \rfloor D_j$), where for every rational number $x$, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The fractional part $\{ D \}$ of $D$ denotes $D - \lfloor D \rfloor$. We define

$$D^{=1} = \sum_{d_j=1} D_j, \quad D^{<1} = \sum_{d_j<1} d_j D_j.$$ 

We call $D$ a boundary (resp. subboundary) $\mathbb{Q}$-divisor if $0 \leq d_j \leq 1$ (resp. $d_j \leq 1$) for all $j$. Note that $\mathbb{Q}$-linear equivalence of two $\mathbb{Q}$-divisors $B_1$ and $B_2$ is denoted by $B_1 \sim_{\mathbb{Q}} B_2$.

(ii) For a proper birational morphism $f : X \to Y$, the exceptional locus $\text{Exc}(f) \subset X$ is the locus where $f$ is not an isomorphism.

(iii) Let $X$ be a normal variety and $B$ an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution such that $\text{Exc}(f) \cup f^{-1}_* B$ has a simple normal crossing support, where $f^{-1}_* B$ is the strict transform of $B$ on $Y$. We write $K_Y = f^*(K_X + B) + \sum a_i E_i$ and $a(E_i, X, B) = a_i$. We say that $(X, B)$ is lc if and only if $a_i \geq -1$ for all $i$. Here, lc is an abbreviation of log canonical. Note that the discrepancy $a(E, X, B) \in \mathbb{Q}$ can be defined for every prime divisor $E$ over $X$. Let $(X, B)$ be an lc pair. If $E$ is a prime divisor over $X$ such that $a(E, X, B) = -1$, then the center $c_X(E)$ is called an lc center of $(X, B)$.

2. Vanishing and torsion-free theorems

In this section, we quickly review Ambro’s formulation of torsion-free and vanishing theorems in a simplified form (see [F6, Chapter 2]). First, we fix the notation and the conventions to state theorems.

2.1 (Global embedded simple normal crossing pairs). Let $Y$ be a simple normal crossing divisor on a smooth variety $M$ and $D$ a $\mathbb{Q}$-divisor on $M$. Assume that $\text{Supp}(D + Y)$ is simple normal crossing and that $D$ and $Y$ have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair $(Y, B_Y)$. We call $(Y, B_Y)$ a global embedded simple normal crossing pair. Let $\nu : Y^\nu \to Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y)$. A stratum of $(Y, B_Y)$ is an irreducible component of $Y$ or the image of some lc center of $(Y^\nu, \Theta^{=1})$. When $Y$ is smooth and $B_Y$ is a $\mathbb{Q}$-divisor on $Y$ such that $\text{Supp} B_Y$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$ and $D = B_Y \times \mathbb{A}^1$. Then $(Y, B_Y) \simeq (Y \times \{0\}, B_Y \times \{0\})$ satisfies the above conditions, that
is, we can consider \((Y, B_Y)\) to be a global embedded simple normal crossing pair.

Theorem 2.2 is a special case of the main result in [F6, Chapter 2]. It will play crucial roles in the following sections.

**Theorem 2.2** (Torsion-freeness and vanishing theorem). Let \((Y, B_Y)\) be as above. Assume that \(B_Y\) is a boundary \(\mathbb{Q}\)-divisor. Let \(f : Y \to X\) be a proper morphism and \(L\) a Cartier divisor on \(Y\).

1. Assume that \(H \sim_{\mathbb{Q}} L - (K_Y + B_Y)\) is \(f\)-semi-ample. Then, for every integer \(q\), every non-zero local section of \(R^q f_* \mathcal{O}_Y(L)\) contains in its support the \(f\)-image of some strata of \((Y, B_Y)\).

2. Assume that \(X\) is projective and \(H \sim_{\mathbb{Q}} f^* H'\) for some ample \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(H'\) on \(X\). Then \(H^p(X, R^q f_* \mathcal{O}_Y(L)) = 0\) for every \(p > 0\).

The above theorem follows from the next theorem.

**Theorem 2.3** (Injectivity theorem). Let \((Y, B_Y)\) be as above. Assume that \(Y\) is proper and \(B_Y\) is a boundary \(\mathbb{Q}\)-divisor. Let \(D\) be an effective Cartier divisor whose support is contained in \(\text{Supp}\{B_Y\}\). Assume that \(L \sim_{\mathbb{Q}} K_Y + B_Y\). Then the homomorphism

\[H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + D)),\]

which is induced by the natural inclusion \(\mathcal{O}_Y \to \mathcal{O}_Y(D)\), is injective for every \(q\).

For the proof, which depends on the mixed Hodge theory, we recommend the reader to see [F6, Chapter 2]. It is because [A, Section 3] seems to be inaccessible.

2.1. **Idea of the proof.** We prove a very special case of Theorem 2.3. This subsection is independent of the other sections. So, the reader can skip it. We adopt Kollár’s principle (cf. [KM, Principle 2.46]) here instead of using the arguments by Esnault–Viehweg. We closely follow [KM, 2.4 The Kodaira Vanishing Theorem]. We note that [F5] may help the reader to understand Theorem 2.2. In [F5], we give a short and almost self-contained proof of Theorem 2.2 for the case when \(Y\) is smooth.

First, we recall the following result on the Hodge theory. Note that we compute the cohomology groups in the complex analytic setting throughout this subsection.

**Theorem 2.4.** Let \(V\) be a smooth projective variety and \(\Sigma\) a simple normal crossing divisor on \(V\). Let \(i : V \setminus \Sigma \to V\) be the natural open
immersion. Then the inclusion \( \iota_! \mathcal{C}_{V \setminus \Sigma} \subset \mathcal{O}_V(-\Sigma) \) induces surjections

\[
H^i_c(V \setminus \Sigma, \mathbb{C}) = H^i(V, \iota_! \mathcal{C}_{V \setminus \Sigma}) \to H^i(V, \mathcal{O}_V(-\Sigma))
\]

for all \( i \).

We note that \( \iota_! \mathcal{C}_{V \setminus \Sigma} \) is quasi-isomorphic to the complex \( \Omega^\bullet_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma) \) and the Hodge to de Rham spectral sequence

\[
E_1^{p,q} = H^q(V, \Omega^p_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)) \Rightarrow H^{p+q}_c(V \setminus \Sigma, \mathbb{C})
\]
degenerates at the \( E_1 \)-term. See, for example, [E, I.3.], [F6, Section 2.4], or Remark 2.5 below. Theorem 2.4 is a direct consequence of this \( E_1 \)-degeneration.

**Remark 2.5.** We put \( n = \dim V \). By the Poincaré duality, we have

\[
H^{2n-\langle p+q \rangle}(V \setminus \Sigma, \mathbb{C}) \simeq H^{p+q}_c(V \setminus \Sigma, \mathbb{C})^*.
\]

On the other hand, by the Serre duality, we see that

\[
H^{n-q}(V, \Omega^n_V(\log \Sigma)) \simeq H^q(V, \Omega^p_V(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma))^*.
\]

Therefore, the above \( E_1 \)-degeneration easily follows from the well-known \( E_1 \)-degeneration of

\[
E_1^{n-p,n-q} = H^{n-q}(V, \Omega^n_V(\log \Sigma)) \Rightarrow H^{2n-\langle p+q \rangle}(V \setminus \Sigma, \mathbb{C}).
\]

The next theorem is a special case of Theorem 2.3.

**Theorem 2.6.** Let \( X \) be a smooth projective variety and \( S \) a simple normal crossing divisor on \( X \). Let \( M \) be a Cartier divisor on \( X \). Assume that there exists a smooth divisor \( D \) on \( X \) such that \( dD \sim mM \) for some relatively prime positive integers \( d \) and \( m \) with \( d < m \), \( D \) and \( S \) have no common irreducible components, and \( D + S \) is a simple normal crossing divisor on \( X \). Then the homomorphism

\[
H^i(X, \mathcal{O}_X(K_X + S + M)) \to H^i(X, \mathcal{O}_X(K_X + S + M + bD))
\]

induced by the natural inclusion \( \mathcal{O}_X \to \mathcal{O}_X(bD) \) is injective for every positive integer \( b \) and every \( i \geq 0 \).

**Proof.** We take a usual \( m \)-fold cyclic cover \( \pi : Y \to X \) ramifying along \( D \) by \( dD \sim mM \). We put \( T = \pi^*S \). Then \( Y \) is smooth and \( T \) is simple normal crossing on \( Y \). Let \( \iota : Y \setminus T \to Y \) be the natural open immersion. Then the inclusion \( \iota_! \mathcal{C}_{Y \setminus T} \subset \mathcal{O}_Y(-T) \) induces the following surjections

\[
H^i(Y, \iota_! \mathcal{C}_{Y \setminus T}) \to H^i(Y, \mathcal{O}_Y(-T))
\]

for all \( i \) by Theorem 2.4. Since the fibers of \( \pi \) are zero-dimensional, there are no higher direct image sheaves, and

\[
H^i(X, \pi_* \iota_! \mathcal{C}_{Y \setminus T}) \to H^i(X, \pi_* \mathcal{O}_Y(-T))
\]
is surjective for every \( i \geq 0 \). The \( \mathbb{Z}/m\mathbb{Z} \)-action gives eigensheaf decompositions

\[
\pi_*\iota\mathcal{C}_{Y/T} = \bigoplus_{k=0}^{m-1} G_k
\]

and

\[
\pi_*\mathcal{O}_Y(-T) = \bigoplus_{k=0}^{m-1} \mathcal{O}_X(-S - kM + \frac{k}{m}D)
\]

such that

\[
G_k \subset \mathcal{O}_X(-S - kM + \frac{k}{m}D)
\]

for \( 0 \leq k \leq m-1 \). By taking a direct summand, we have the surjections

\[
H^i(X, G_1) \rightarrow H^i(X, \mathcal{O}_X(-S - M))
\]

for all \( i \). It is easy to see that \( G_1 \) is a subsheaf of \( \mathcal{O}_X(-S - M - bD) \) for every \( b \geq 0 \). See, for example, [KM, Corollary 2.54, Lemma 2.55]. Therefore,

\[
H^i(X, \mathcal{O}_X(-S - M - bD)) \rightarrow H^i(X, \mathcal{O}_X(-S - M))
\]

is surjective for every \( i \) (cf. [KM, Corollary 2.56]). By the Serre duality, we have the desired injections. \( \square \)

By Theorem 2.6, we can easily obtain a very special case of Theorem 2.2 (2). We omit the proof because it is a routine work.

**Theorem 2.7.** Let \( f : X \rightarrow Y \) be a morphism from a smooth projective variety \( X \) onto a projective variety \( Y \). Let \( S \) be a simple normal crossing divisor on \( X \) and \( L \) an ample Cartier divisor on \( Y \). Then

\[
H^i(Y, R^jf_*\mathcal{O}_X(K_X + S) \otimes \mathcal{O}_Y(L)) = 0
\]

for \( i > 0 \) and \( j \geq 0 \).

As a corollary, we obtain a generalization of the Kodaira vanishing theorem (cf. [F5, Theorem 4.4]).

**Corollary 2.8** (Kodaira vanishing theorem for log canonical varieties). Let \( Y \) be a projective variety with only log canonical singularities and \( L \) an ample Cartier divisor on \( Y \). Then

\[
H^i(Y, \mathcal{O}_Y(K_Y + L)) = 0
\]

for \( i > 0 \).

**Proof.** Let \( f : X \rightarrow Y \) be a resolution such that \( S = \text{Exc}(f) \) is a simple normal crossing divisor. Then \( f_*\mathcal{O}_X(K_X + S) \simeq \mathcal{O}_Y(K_Y) \). Therefore, we have the desired vanishing theorem by Theorem 2.7. \( \square \)
We close this subsection with Sommese’s example. For the details and other examples, see [F6, Section 2.8].

Example 2.9. We consider \( \pi : Y = \mathbb{P} \mathbb{P}^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 3}) \to \mathbb{P}^1 \). Let \( \mathcal{M} \) denote the tautological line bundle of \( \pi : Y \to \mathbb{P}^1 \). We take a general member \( X \) of \( |(\mathcal{M} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}| \). Then \( X \) is a normal Gorenstein projective threefold. Note that \( X \) is not lc. We put \( \mathcal{O}_Y(L) = \mathcal{M} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \). Then \( L \) is an ample Cartier divisor on \( Y \). We can check that \( H^1(X, \mathcal{O}_X(K_X + L)) = C \). Thus, the Kodaira vanishing theorem does not necessarily hold for non-lc varieties.

3. Adjunction for qlc varieties

To prove the base point free theorem for log canonical pairs following Ambro’s idea, it is better to introduce the notion of qlc varieties. For the details, see [F6, Section 3.2].

Definition 3.1 (Qlc varieties). A qlc variety is a variety \( X \) with a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( \omega \), and a finite collection \( \{C\} \) of reduced and irreducible subvarieties of \( X \) such that there is a proper morphism \( f : (Y, B_Y) \to X \) from a global embedded simple normal crossing pair as in 2.1 satisfying the following properties:

1. \( f^* \omega \sim \mathbb{Q} K_Y + B_Y \) such that \( B_Y \) is a subboundary \( \mathbb{Q} \)-divisor.
2. There is an isomorphism
   \[ \mathcal{O}_X \simeq f_* \mathcal{O}_Y(-B_Y^{\leq 1})^\tau. \]
3. The collection of subvarieties \( \{C\} \) coincides with the image of \( (Y, B_Y) \)-strata.

We use the following terminology. The subvarieties \( C \) are the qlc centers of \( X \), and \( f : (Y, B_Y) \to X \) is a quasi-log resolution of \( X \). We sometimes simply say that \( [X, \omega] \) is a qlc pair, or the pair \([X, \omega]\) is qlc.

Remark 3.2. By the condition (2), we have an isomorphism \( \mathcal{O}_X \simeq f_* \mathcal{O}_Y \). In particular, \( f \) is a surjective morphism with connected fibers and \( X \) is semi-normal.

Proposition 3.3. Let \( (X, B) \) be an lc pair. Then \([X, K_X + B]\) is a qlc pair.

Proof. Let \( f : Y \to X \) be a resolution such that \( K_Y + B_Y = f^*(K_X + B) \) and \( \text{Supp}B_Y \) is a simple normal crossing divisor. Then \( \mathcal{O}_X \simeq f_* \mathcal{O}_Y(-B_Y^{\leq 1})^\tau \) because \( -B_Y^{\leq 1} \) is effective and \( f \)-exceptional. We note that a qlc center \( C \) is \( X \) itself or an lc center of \( (X, B) \). \( \square \)

We start an easy lemma.
Lemma 3.4. Let $f : Z \to Y$ be a proper birational morphism between smooth varieties and $B_Y$ a subboundary $\mathbb{Q}$-divisor on $Y$ such that $\text{Supp} B_Y$ is simple normal crossing. Assume that $K_Z + B_Z = f^*(K_Y + B_Y)$ and that $\text{Supp} B_Z$ is simple normal crossing. Then we have

$$f_*\mathcal{O}_Z(\gamma - (B_Z^\leq)^\gamma) \cong \mathcal{O}_Y(\gamma - (B_Y^\leq)^\gamma).$$

Proof. By $K_Z + B_Z = f^*(K_Y + B_Y)$, we obtain

$$K_Z = f^*(K_Y + B_Y^\leq + \{B_Y\}) + f^*(\cup B_Y^\leq - \cup B_Z^\leq - B_Z - \{B\}).$$

If $a(\nu, Y, B_Y^\leq + \{B_Y\}) = -1$ for a prime divisor $\nu$ over $Y$, then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $f^*(\cup B_Y^\leq - \cup B_Z^\leq )$ is Cartier, we can easily see that $f^*(\cup B_Y^\leq) = \cup B_Z^\leq + E$, where $E$ is an effective $f$-exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_Z(\gamma - (B_Z^\leq)^\gamma) \cong \mathcal{O}_Y(\gamma - (B_Y^\leq)^\gamma).$$

This completes the proof. \qed

The following lemma is very important in the study of qlc pairs.

Lemma 3.5. We use the same notation and assumption as in Lemma 3.4. Let $S$ be a simple normal crossing divisor on $Y$ such that $S \subset \text{Supp} B_Y^\leq$. Let $T$ be the union of the irreducible components of $B_Z^\leq$ that are mapped into $S$ by $f$. Assume that $\text{Supp} f_*^{-1} B_Y \cup \text{Exc}(f)$ is simple normal crossing on $Z$. Then we have

$$f_*\mathcal{O}_T(\gamma - (B_T^\leq)^\gamma) \cong \mathcal{O}_S(\gamma - (B_S^\leq)^\gamma),$$

where $(K_Z + B_Z)|_T = K_T + B_T$ and $(K_Y + B_Y)|_S = K_S + B_S$.

Proof. We use the same notation as in the proof of Lemma 3.4. We consider the short exact sequence

$$0 \to \mathcal{O}_Z(\gamma - (B_T^\leq)^\gamma - T) \to \mathcal{O}_Z(\gamma - (B_Z^\leq)^\gamma) \to \mathcal{O}_T(\gamma - (B_T^\leq)^\gamma) \to 0.$$ 

Since $T = f^*S - F$, where $F$ is an effective $f$-exceptional divisor, we can easily see that

$$f_*\mathcal{O}_Z(\gamma - (B_T^\leq)^\gamma - T) \cong \mathcal{O}_Y(\gamma - (B_Y^\leq)^\gamma - S).$$

We note that

$$\gamma - (B_Z^\leq)^\gamma - T - (K_Z + \{B_Z\} + (B_Z^\leq - T)) = -f^*(K_Y + B_Y).$$

Therefore, every local section of $R^1 f_*\mathcal{O}_Z(\gamma - (B_Z^\leq)^\gamma - T)$ contains in its support the $f$-image of some strata of $(Z, \{B_Z\} + B_Z^\leq - T)$ by Theorem 2.2 (1).
Claim. No strata of \((Z, \{B_Z\} + B_Z^{-1} - T)\) are mapped into \(S\) by \(f\).

Proof of Claim. Assume that there is a stratum \(C \in (Z, \{B_Z\} + B_Z^{-1} - T)\) such that \(f(C) \subset S\). Note that \(\text{Supp} f^*\mathcal{S} \subset \text{Supp} f^*_{-1}\mathcal{B_I} \cup \text{Exc}(f)\) and \(\text{Supp}B_Z^{-1} \subset \text{Supp} f^*_{-1}\mathcal{B_I} \cup \text{Exc}(f)\). Since \(C\) is also a stratum of \((Z, B_Z^{-1})\) and \(C \subset \text{Supp} f^*\mathcal{S}\), there exists an irreducible component \(G\) of \(B_Z^{-1}\) such that \(C \subset G \subset \text{Supp} f^*\mathcal{S}\). Therefore, by the definition of \(T\), \(G\) is an irreducible component of \(T\) because \(f(G) \subset S\) and \(G\) is an irreducible component of \(B_Z^{-1}\). So, \(C\) is not a stratum of \((Z, \{B_Z\} + B_Z^{-1} - T)\). It is a contradiction.

On the other hand, \(f(T) \subset S\). Therefore,

\[ f_*\mathcal{O}_T(\gamma -(B_T^{<1})^\gamma ) \rightarrow R^1f_*\mathcal{O}_Z(\gamma -(B_Z^{<1})^\gamma - T) \]

is a zero-map by the above claim. Thus, we obtain

\[ f_*\mathcal{O}_T(\gamma -(B_T^{<1})^\gamma ) \simeq \mathcal{O}_S(\gamma -(B_S^{<1})^\gamma ). \]

This completes the proof.

The following theorem (cf. [A, Theorem 4.4]) is one of the key results for the theory of qlc varieties. It is a consequence of Theorem 2.2.

**Theorem 3.6** (Adjunction and vanishing theorem). Let \([X, \omega]\) be a qlc pair and \(X'\) a union of some qlc centers of \([X, \omega]\).

(i) Then \([X', \omega']\) is a qlc pair, where \(\omega' = \omega|_{X'}\). Moreover, the qlc centers of \([X', \omega']\) are exactly the qlc centers of \([X, \omega]\) that are included in \(X'\).

(ii) Assume that \(X\) is projective. Let \(L\) be a Cartier divisor on \(X\) such that \(L - \omega\) is ample. Then \(H^q(X, \mathcal{O}_X(L)) = 0\) and \(H^q(X, \mathcal{I}_{X'}, \mathcal{O}_X(L)) = 0\) for \(q > 0\), where \(\mathcal{I}_{X'}\) is the defining ideal sheaf of \(X'\) on \(X\). Note that \(H^q(X', \mathcal{O}_{X'}(L)) = 0\) for any \(q > 0\) because \([X', \omega']\) is a qlc pair by (i) and \(L|_{X'} - \omega'\) is ample.

Proof. (i) Let \(f : (Y, B_Y) \rightarrow X\) be a quasi-log resolution. Let \(M\) be the ambient space of \(Y\) and \(D\) a subboundary \(\mathbb{Q}\)-divisor on \(M\) such that \(B_Y = D|_Y\). By taking blow-ups of \(M\), we can assume that the union of all strata of \((Y, B_Y)\) mapped into \(X'\), which is denoted by \(Y'\), is a union of irreducible components of \(Y\) (cf. Lemma 3.5). We put \(Y'' = Y - Y'\). We define \((K_Y + B_Y)|_{Y''} = K_{Y''} + B_{Y''}\) and consider \(f : (Y', B_{Y'}) \rightarrow X'\). We claim that \([X', \omega']\) is a qlc pair, where \(\omega' = \omega|_{X'}\), and \(f : (Y', B_{Y'}) \rightarrow X'\) is a quasi-log resolution. By the definition, \(B_{Y'}\) is a subboundary and \(f^*\omega' \sim \mathbb{Q} K_{Y''} + B_{Y''}\) on \(Y'\). We consider the following short exact sequence

\[ 0 \rightarrow \mathcal{O}_{Y''}(-Y') \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{Y'} \rightarrow 0. \]
We put \( A = \gamma - (B_Y^{\leq 1})^\gamma \). Then we have
\[
0 \to \mathcal{O}_{X''}(A - Y') \to \mathcal{O}_Y(A) \to \mathcal{O}_{Y'}(A) \to 0.
\]
Applying \( f_* \), we obtain
\[
0 \to f_*\mathcal{O}_{Y''}(A - Y') \to \mathcal{O}_X \to f_*\mathcal{O}_{Y'}(A) \to R^1 f_*\mathcal{O}_{Y''}(A - Y') \to \cdots.
\]
The support of every non-zero local section of \( R^1 f_*\mathcal{O}_{Y''}(A - Y') \) can not be contained in \( f \) since \( \gamma > 0 \).

We use the next definition in Section 4.

\[ \text{Corollary 3.7.} \text{ Let } [X, \omega] \text{ be a qlc pair and } X' \text{ an irreducible component of } X. \text{ Then } [X', \omega'], \text{ where } \omega' = \omega|_{X'}, \text{ is a qlc pair.} \]

\[ \text{Proof.} \text{ It is because } X' \text{ is a qlc center of } [X, \omega] \text{ by Remark 3.2.} \]

We use the next definition in Section 4.
Definition 3.8. Let $[X, \omega]$ be a qlc pair. Let $X'$ be the union of qlc centers of $X$ that are not any irreducible components of $X$. Then $X'$ with $\omega' = \omega|_{X'}$ is a qlc variety by Theorem 3.6 (i). We denote it by $N_{q\text{qlc}}(X, \omega)$.

We close this section with the following very useful lemma, which seems to be indispensable for the proof of the base point free theorem in Section 4.

Lemma 3.9. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of a qlc pair $[X, \omega]$. Let $E$ be a Cartier divisor on $X$ such that $\text{Supp}(E)$ contains no qlc centers of $[X, \omega]$. By blowing up $M$, the ambient space of $Y$, inside $\text{Supp}(f^*E)$, we can assume that $(Y, B_Y + f^*E)$ is a global embedded simple normal crossing pair.

Proof. First, we take a blow-up of $M$ along $f^*E$ and apply Hironaka’s resolution theorem to $M$. Then we can assume that there exists a Cartier divisor $F$ on $M$ such that $\text{Supp}(F \cap Y) = \text{Supp}(f^*E)$. Next, we apply Szabó’s resolution lemma to $\text{Supp}(D + Y + F)$ on $M$. Thus, we obtain the desired properties by Lemma 3.5. \hfill \Box

4. BASE POINT FREE THEOREM

The next theorem is the main theorem of this section. It is a special case of [A, Theorem 5.1]. This formulation is indispensable for the inductive treatment of log canonical pairs in the framework of the theory of quasi-log varieties. For the details, see [F6, Section 3.2.2].

Theorem 4.1. Let $[X, \omega]$ be a projective qlc pair and $L$ a nef Cartier divisor on $X$. Assume that $qL - \omega$ is ample for some $q > 0$. Then $\mathcal{O}_X(mL)$ is generated by global sections for $m \gg 0$, that is, there exists a positive number $m_0$ such that $\mathcal{O}_X(mL)$ is generated by global sections for every $m \geq m_0$.

Proof. First, we note that the statement is obvious when $\dim X = 0$.

Claim 1. We can assume that $X$ is irreducible.

Let $X'$ be an irreducible component of $X$. Then $X'$ with $\omega' = \omega|_{X'}$ has a natural qlc structure induced by $[X, \omega]$ by adjunction (see Corollary 3.7). By the vanishing theorem (see Theorem 3.6 (ii)), we have $H^1(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for all $m \geq q$. We consider the
following commutative diagram.

\[
\begin{array}{ccc}
H^0(X, \mathcal{O}_X(mL)) \otimes \mathcal{O}_X & \xrightarrow{\alpha} & H^0(X', \mathcal{O}_{X'}(mL)) \otimes \mathcal{O}_{X'} \\
\downarrow & & \downarrow \\
\mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) \\
\end{array}
\]

Since \(\alpha\) is surjective for \(m \geq q\), we can assume that \(X\) is irreducible when we prove this theorem.

Claim 2. \(\mathcal{O}_X(mL)\) is generated by global sections around \(N_{qklt}(X, \omega)\) for \(m \gg 0\).

We put \(X' = N_{qklt}(X, \omega)\). Then \([X', \omega]\), where \(\omega = \omega|_{X'}\), is a qlc pair by adjunction (see Definition 3.8 and Theorem 3.6 (i)). By the induction on the dimension, \(\mathcal{O}_{X'}(mL)\) is generated by global sections for \(m \gg 0\). By the following commutative diagram:

\[
\begin{array}{ccc}
H^0(X, \mathcal{O}_X(mL)) \otimes \mathcal{O}_X & \xrightarrow{\alpha} & H^0(X', \mathcal{O}_{X'}(mL)) \otimes \mathcal{O}_{X'} \\
\downarrow & & \downarrow \\
\mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) \\
\end{array}
\]

we know that \(\mathcal{O}_X(mL)\) is generated by global sections around \(X'\) for \(m \gg 0\).

Claim 3. \(\mathcal{O}_X(mL)\) is generated by global sections on a non-empty Zariski open set for \(m \gg 0\).

By Claim 2, we can assume that \(N_{qklt}(X, \omega)\) is empty. If \(L\) is numerically trivial, then \(H^0(X, \mathcal{O}_X(L)) = H^0(X, \mathcal{O}_X(-L)) = \mathbb{C}\). It is because \(h^0(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X) = 1\) by Theorem 3.6 (ii) and [Kl, Chapter II §2 Theorem 1]. Therefore, \(\mathcal{O}_X(L)\) is trivial. So, we can assume that \(L\) is not numerically trivial. Let \(f : (Y, B_Y) \to X\) be a quasi-log resolution. Let \(x \in X\) be a general smooth point.

Then we can take a \(\mathbb{Q}\)-divisor \(D\) such that \(\text{mult}_x D > \dim X\) and that \(D \sim_Q (q + r)L - \omega\) for some \(r > 0\) (see [KM, 3.5 Step 2]). By blowing up \(M\), we can assume that \((Y, B_Y + f^*D)\) is a global embedded simple normal crossing pair by Lemma 3.9. We note that every stratum of \((Y, B_Y)\) is mapped onto \(X\) by the assumption. By the construction of \(D\), we can find a positive rational number \(c < 1\) such that \(B_Y + cf^*D\) is a subboundary and some stratum of \((Y, B_Y + cf^*D)\) does not dominate \(X\). Note that \(f_* \mathcal{O}_Y(\langle - (B_Y^{c+1}) \rangle) \simeq \mathcal{O}_X\). Then the pair \([X, \omega + cD]\) is qlc and \(f : (Y, B_Y + cf^*D) \to X\) is a quasi-log resolution. We note that \(q'L - (\omega + cD)\) is ample by \(c < 1\), where \(q' = q + cr\). By the
construction, $\text{Nqklt}(X, \omega + cD)$ is non-empty. Therefore, by applying Claim 2 to $[X, \omega + cD]$, $\mathcal{O}_X(mL)$ is generated by global sections around $\text{Nqklt}(X, \omega + cD)$ for $m \gg 0$. So, we obtain Claim 3.

Let $p$ be a prime number and $l$ a large integer. Then $|p^lL| \neq \emptyset$ by Claim 3 and $|p^lL|$ is free around $\text{Nqklt}(X, \omega)$ by Claim 2.

**Claim 4.** If the base locus $\text{Bs}|p^lL|$ (with reduced scheme structure) is not empty, then $\text{Bs}|p^lL|$ is not contained in $\text{Bs}|p'^lL|$ for $l' \gg l$.

Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. We take a general member $D \in |p^lL|$. We note that $|p^lL|$ is free around $\text{Nqklt}(X, \omega)$. Thus, $f^*D$ intersects all strata of $(Y, \text{Supp}B_Y)$ transversally over $X \setminus \text{Bs}|p^lL|$ by Bertini and $f^*D$ contains no strata of $(Y, B_Y)$. By taking blow-ups of $M$ suitably, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair (cf. Lemmas 3.9 and 3.5). We take the maximal positive rational number $c$ such that $B_Y + cf^*D$ is a subboundary. We note that $c \leq 1$. Here, we used $\mathcal{O}_X \cong f_*\mathcal{O}_Y(f^*(B_Y^{-1})^\sim)$. Then $f : (Y, B_Y + cf^*D) \to X$ is a quasi-log resolution of $[X, \omega' = \omega + cD]$. Note that $[X, \omega']$ has a qlc center $C$ that intersects $\text{Bs}|p^lL|$ by the construction. By the induction on the dimension, $\mathcal{O}_C(mL)$ is generated by global sections for $m \gg 0$. We can lift the sections of $\mathcal{O}_C(mL)$ to $X$ for $m \geq q + cp'$ by Theorem 3.6 (ii). Then we obtain that $\mathcal{O}_X(mL)$ is generated by global sections around $C$ for $m \gg 0$. Therefore, $\text{Bs}|p'^lL|$ is strictly smaller than $\text{Bs}|p^lL|$ for $l' \gg l$.

**Claim 5.** $\mathcal{O}_X(mL)$ is generated by global sections for $m \gg 0$.

By Claim 4 and the noetherian induction, $\mathcal{O}_X(p^lL)$ and $\mathcal{O}_X(p'^lL)$ are generated by global sections for large $l$ and $l'$, where $p$ and $p'$ are prime numbers and they are relatively prime. So, there exists a positive number $m_0$ such that $\mathcal{O}_X(mL)$ is generated by global sections for any $m \geq m_0$.

The next corollary is obvious by Theorem 4.1 and Proposition 3.3.

**Corollary 4.2** (Base point free theorem). Let $(X, B)$ be a projective lc pair and $L$ a nef Cartier divisor on $X$. Assume that $qL - (K_X + B)$ is ample for some $q > 0$. Then $\mathcal{O}_X(mL)$ is generated by global sections for $m \gg 0$.

5. **Cone theorem**

In this section, we treat the cone theorem for log canonical pairs.
5.1. **Rationality theorem.** Here, we explain the rationality theorem for log canonical pairs. It implies the cone theorem for log canonical pairs.

**Theorem 5.1** (Rationality theorem). Let $(X, B)$ be a projective lc pair such that $a(K_X + B)$ is Cartier for a positive integer $a$. Let $H$ be an ample Cartier divisor on $X$. Assume that $K_X + B$ is not nef. We put

$$r = \max\{t \in \mathbb{R} : H + t(K_X + B) \text{ is nef}\}.$$

Then $r$ is a rational number of the form $u/v$ ($u, v \in \mathbb{Z}$) where $0 < v \leq a(\text{dim } X + 1)$.

The following theorem is a very special case of Theorem 3.6 (i i).

**Theorem 5.2** (Vanishing theorem for lc centers). Let $X$ be a projective variety and $B$ a boundary $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is log canonical. Let $D$ be a Cartier divisor on $X$. Assume that $D - (K_X + B)$ is ample. Let $C$ be an lc center of the pair $(X, B)$ with a reduced scheme structure. Then we have

$$H^i(X, \mathcal{I}_C \otimes \mathcal{O}_X(D)) = 0, \quad H^i(C, \mathcal{O}_C(D)) = 0$$

for all $i > 0$, where $\mathcal{I}_C$ is the defining ideal sheaf of $C$ on $X$. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(C, \mathcal{O}_C(D))$$

is surjective.

We can prove Theorem 5.2 much more easily than Theorem 3.6 (ii) (see [F5, Theorem 4.1] or [F7, Theorem 2.2]). Theorem 5.1 can be proved by the standard traditional argument using Theorem 5.2. We omit the proof here. For the details, see [F7, Section 5]. Note that we do not need the theory of quasi-log varieties nor any vanishing theorems on reducible varieties to prove the rationality theorem for log canonical pairs.

5.2. **Cone theorem.** Let us state the main theorem of this section.

**Theorem 5.3** (Cone theorem). Let $(X, B)$ be a projective lc pair. Then we have

(i) There are (countably many) rational curves $C_j \subset X$ such that $0 < -(K_X + B) \cdot C_j \leq 2 \text{ dim } X$, and

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + B) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$
(ii) For any $\varepsilon > 0$ and ample $\mathbb{Q}$-divisor $H$,

\[
\overline{NE}(X) = \overline{NE}(X)_{(K_X+B+\varepsilon H)\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j].
\]

(iii) Let $F \subset \overline{NE}(X)$ be a $(K_X + B)$-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ such that $(\varphi_F)_* O_X \simeq O_Z$, $Z$ is projective, and an irreducible curve $C \subset X$ is mapped to a point by $\varphi_F$ if and only if $[C] \in F$. The map $\varphi_F$ is called the contraction of $F$.

(iv) Let $F$ and $\varphi_F$ be as in (iii). Let $L$ be a line bundle on $X$ such that $(L \cdot C) = 0$ for every curve $C$ with $[C] \in F$. Then there is a line bundle $L_Z$ on $Z$ such that $L \simeq \varphi_F^* L_Z$.

Proof. The estimate $\leq 2 \dim X$ and the fact that $C_j$ is a rational curve in (i) can be proved by Kawamata’s argument in [Ka] with the aid of [BCHM]. For the details, see [F6, Section 3.1.3]. The other statements in (i) and (ii) are formal consequences of the rationality theorem (cf. Theorem 5.1). For the proof, see [KM, Theorem 3.15]. The statements (iii) and (iv) are obvious by Corollary 4.2 and the statements (i) and (ii). See Steps 7 and 9 in [KM, 3.3 The Cone Theorem].

6. Related topics

In this paper, we did not prove Theorem 2.2, which is a key result for the theory of quasi-log varieties. For the proof, see [F6, Chapter 2]. The paper [F5] is a gentle introduction to the vanishing and torsion-free theorems. In [F6, Chapters 3, 4], we gave a proof of the existence of fourfold lc flips and proved the base point free theorem of Reid–Fukuda type for lc pairs. The base point free theorem for lc pairs was generalized in [F1], where we obtained Kollár’s effective base point free theorem for lc pairs. In [F2], we proved the effective base point free theorem of Angenhu–Siu type for lc pairs. Recently, we introduced the notion of non-lc ideal sheaves and proved the restriction theorem (see [F3]). It is a generalization of Kawakita’s inversion of adjunction on log canonicity for normal divisors. In [F4], we proved that the log canonical ring is finitely generated in dimension four. In [F7], we succeeded in proving the fundamental theorems of the log minimal model program for log canonical pairs without using the theory of quasi-log varieties. Our new approach in [F7] seems to be more natural and simpler than Ambro’s theory of quasi-log varieties.
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