A MEMORANDUM ON THE INVARIANCE OF PLURIGENERA (PRIVATE NOTE)

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Proposition 1. *Let X be a complex manifold. Let p, q be smooth nonnegative functions on X such that $q \neq 0$ almost everywhere. Assume that*

$$\mathrm{Vol}(X, \omega) = \int_X 1 dV_\omega < \infty$$

and

$$\int_X \frac{p}{q} dV_\omega \leq C$$

holds for some constant $C \geq 1$, where dV_ω is a volume form on X. Let $U \subset X$ be any open set of X and dV_{ω_U} a volume form on U such that $dV_{\omega_U} \leq C_U dV_\omega$ for some constant $C_U \geq 1$. Then, we have

$$\int_U \log \left(\frac{p}{q} \right) dV_{\omega_U} \leq \frac{1}{e} + C_U \mathrm{Vol}(X, \omega)(\log C_U + \log C).$$

Moreover, let a_i be a smooth nonnegative function on X for $1 \leq i \leq k$ such that $a_i \neq 0$ almost everywhere for $1 \leq i \leq k-1$. We put $a_0 \equiv 1$ and $a = a_k$. Assume that

$$\int_X \frac{a_{i+1}}{a_i} dV_\omega \leq C$$

holds for $0 \leq i \leq k-1$. Then

$$\frac{1}{k} \int_U \log a \, dV_{\omega_U} \leq \frac{1}{e} + C_U \mathrm{Vol}(X, \omega)(\log C_U + \log C).$$

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The arguments in this note are indispensable when we obtain local uniform supremum norm estimates from the global L^2-estimates. I like intrinsic formulations.

Proof. By Jensen's inequality,

$$\begin{aligned}
\int_U \log\left(\frac{p}{q}\right)\frac{dV_{\omega_U}}{\mathrm{Vol}(U,\omega_U)} &\leq \log\int_U \frac{p}{q}\frac{dV_{\omega_U}}{\mathrm{Vol}(U,\omega_U)}\\
&= \log\left(\frac{1}{\mathrm{Vol}(U,\omega_U)}\right)+\log\int_U \frac{p}{q}dV_{\omega_U}\\
&\leq \log\left(\frac{1}{\mathrm{Vol}(U,\omega_U)}\right)+\log\int_X \frac{p}{q}C_U dV_{\omega}\\
&\leq \log\left(\frac{1}{\mathrm{Vol}(U,\omega_U)}\right)+\log C_U+\log C,
\end{aligned}$$

where $\mathrm{Vol}(U,\omega_U)=\int_U 1dV_{\omega_U}<\infty$. Therefore, we have

$$\begin{aligned}
\int_U \log\left(\frac{p}{q}\right)dV_{\omega_U} \leq &\mathrm{Vol}(U,\omega_U)\log\left(\frac{1}{\mathrm{Vol}(U,\omega_U)}\right)\\
&+\mathrm{Vol}(U,\omega_U)(\log C_U+\log C).
\end{aligned}$$

Lemma 2. *We consider $g(x)=x\log\left(\frac{1}{x}\right)$ for $x>0$. Then $g(x)\leq g(\frac{1}{e})=\frac{1}{e}$ for any $x>0$. Moreover, $g(x)>0$ for $x<1$, $g(1)=0$, and $g(x)<0$ for $x>1$. We note that $g(x)\to 0$ as $x\to 0$.*

Proof. We have $g(x)=-x\log x$ and $g'(x)=-\log x-1$. Thus, we obtain the desired properties. □

Therefore, we obtain

$$\int_U \log\left(\frac{p}{q}\right)dV_{\omega_U}\leq \frac{1}{e}+C_U\mathrm{Vol}(X,\omega)(\log C_U+\log C).$$

The latter statement is obvious. □

Theorem 3. *In* Proposition 1*, we further assume that* $\log a$ *is a quasi-psh function. Let Y be a relatively compact open set of X. Then there exists a positive constant C' such that*

$$\sup_{x\in Y}\frac{1}{k}\log a(x)\leq C'<\infty.$$

Proof. Let $W_\alpha \Subset U_\alpha \Subset X$ be relatively compact open sets of X for $1\leq\alpha\leq N$. Assume that $Y\subset\bigcup_{\alpha=1}^N W_\alpha$ and U_α can be seen as a domain in $\mathbb{C}^n$ for any α, where $n=\dim X$, and $\mathcal{L}$ can be trivialized on each U_α. Let dV_{ω_α} be the Euclidean volume form on $U_\alpha\subset\mathbb{C}^n$. We note that we can assume that $dV_{\omega_\alpha}\leq C_\alpha dV_\omega$ on U_α for some constant $C_\alpha\geq 1$ (if we need, we can shrink U_α). Take a point $x\in Y$. Then there is an α such that $x\in W_\alpha$. On each U_α, we can further assume that $\log a=u_\alpha+v_\alpha$, where u_α is a psh function and v_α is a smooth function. For every $x \in W_\alpha$, there exists an open polydisk U_x whose center is x such that $U_x \Subset U_\alpha$ and

$$\mathrm{Vol}(U_x, \omega_\alpha) = \int_{U_x} 1 dV_{\omega_\alpha} = C_0.$$

Note that the positive constant C_0 is independent of $x \in W_\alpha$. By the sub-mean-value property of u_α, we have

$$u_\alpha(x) \leq \frac{1}{\mathrm{Vol}(U_x, \omega_\alpha)} \int_{U_x} u_\alpha dV_{\omega_\alpha} = \frac{1}{C_0} \int_{U_x} u_\alpha dV_{\omega_\alpha}.$$

Note that

$$\frac{1}{k} v_\alpha(x) \leq C_1$$

for any $x \in W_\alpha$ and

$$\frac{1}{kC_0} \int_{U_x} |v_\alpha| dV_{\omega_\alpha} \leq C_2$$

for any U_x, where C_1 and C_2 are positive constants independent of x and U_x. Thus, we obtain

$$\begin{aligned}
\frac{1}{k} \log a(x) &= \frac{1}{k} u_\alpha(x) + \frac{1}{k} v_\alpha(x) \\
&\leq \frac{1}{kC_0} \int_{U_x} u_\alpha dV_{\omega_\alpha} + C_1 \\
&\leq \frac{1}{kC_0} \int_{U_x} \log a \, dV_{\omega_\alpha} + C_1 + C_2 \\
&\leq \frac{1}{C_0} \left(\frac{1}{e} + C_\alpha \mathrm{Vol}(X, \omega)(\log C_\alpha + \log C) \right) + C_1 + C_2 \\
&=: C^\alpha.
\end{aligned}$$

Thus we have

$$\sup_{x \in W_\alpha} \frac{1}{k} \log a(x) \leq C^\alpha.$$

Then we obtain the upper bound $C' = \max_\alpha C^\alpha < \infty$. □

Remark 4. Let $\mathcal{L}$ be a holomorphic line bundle on X and h a smooth hermitian metric on $\mathcal{L}$. Let s_j be a holomorphic section of $\mathcal{L}$ on X for $1 \leq j \leq l$. Then $\log \sum_{j=1}^{l} |s_j|_h^2$ is a quasi-psh function on X, where $|s_j|_h$ is the pointwise norm of s_j with respect to h. For geometric applications, we use Theorem 3 for $a = \sum_{j=1}^{l} |s_j|_h^2$.

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