Iitaka dimensions, et cetra 2014/6/25

Osamu Fujino

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: fujino@math.kyoto-u.ac.jp

This note will be contained in the book: Foundation of the minimal model program.

Contents

CHAPTER 1

Supplements

1.1. Iitaka dimensions for $\mathbb{R}$-divisors

[1]

In this book, we adopt the following definition of the Iitaka dimension for $\mathbb{R}$-divisors. Although the Iitaka dimension for $\mathbb{R}$-divisors will not play important roles in this book, it seems to be useful when we discuss the abundance conjecture for higher-dimensional algebraic varieties. For the details of Iitaka's theory of D-dimension for $\mathbb{R}$-divisors, see [**Nak2**, Chapter II. §3].

Definition 1.1.1 (Iitaka dimension for $\mathbb{R}$-divisors). Let X be a smooth projective variety and let D be an $\mathbb{R}$-divisor on X. We put

$$\kappa(X, D) = \limsup_{m\to\infty} \frac{\log \dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{\log m}.$$

When X is a normal complete variety and D is an $\mathbb{R}$-Cartier divisor on X, we put

$$\kappa(X, D) = \kappa(Y, f^*D)$$

where $f : Y \to X$ is a resolution of singularities. We call $\kappa(X, D)$ the *Iitaka dimension* of D.

It is not difficult to see that $\kappa(X, D)$ is well-defined. We can check the following geometric characterization of the Iitaka dimension $\kappa(X, D)$.

Proposition 1.1.2. *Let X be a smooth projective variety and let D be an $\mathbb{R}$-divisor on X. We put*

$$\mathbb{N}(D) = \{m \in \mathbb{Z}_{>0} \mid H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) \neq 0\}.$$

Then we have

$$\kappa(X, D) = \begin{cases} \max\limits_{m\in\mathbb{N}(D)} \dim \Phi_{|mD|}(X) & \text{if } \mathbb{N}(D) \neq \emptyset, \\ -\infty & \text{if } \mathbb{N}(D) = \emptyset. \end{cases}$$

[1] I will add this section after Section 2.4 of the book.

Note that $\Phi_{|mD|}$ is a rational map defined by the linear system associated to $H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$. In particular, we see that

$$\kappa(X, D) \in \{-\infty, 0, 1, \cdots, \dim X\}.$$

Proof. For the proof and various related results, see [Nak2, Chapter II. §3.b]. Precisely speaking, Nakayama adopted the property described in this proposition as the definition of $\kappa(X, D)$ (see [Nak2, Chapter II. 3.2. Definition]). □

By Proposition 1.1.2, we can easily see that Definition 1.1.1 is compatible with Definition **??**.

Lemma 1.1.3. *Let X be a smooth projective variety and let D_1 and D_2 be $\mathbb{R}$-divisors on X such that $D_1 \sim_{\mathbb{Q}} D_2$. Then we have the equality*

$$\kappa(X, D_1) = \kappa(X, D_2).$$

Proof. This lemma is obvious by Definition 1.1.1 and Proposition 1.1.2. □

The following definition is due to Sung Rak Choi (see [Choi]). It seems to be natural from the minimal model theoretic viewpoint. We will need it for the generalized abundance conjecture (see Conjecture 1.2.1).

Definition 1.1.4 (Invariant Iitaka dimension). Let X be a smooth projective variety and let D be an $\mathbb{R}$-divisor on X. If there exists an effective $\mathbb{R}$-divisor D' on X such that $D \sim_{\mathbb{R}} D'$, then we put

$$\kappa_\iota(X, D) = \kappa(X, D').$$

Otherwise, we put

$$\kappa_\iota(X, D) = -\infty.$$

We call $\kappa_\iota(X, D)$ the *invariant Iitaka dimension* of D.

When X is a normal complete variety and D is an $\mathbb{R}$-Cartier divisor on X, we put

$$\kappa_\iota(X, D) = \kappa_\iota(Y, f^*D)$$

where $f : Y \to X$ is a resolution of singularities.

By Lemma 1.1.5 and Corollary 1.1.6, we see that $\kappa_\iota(X, D)$ is well-defined.

Lemma 1.1.5 (see [Choi, Proposition 2.1.2]). *Let D_1 and D_2 be $\mathbb{R}$-divisors on a smooth projective variety X such that $D_1 \sim_{\mathbb{R}} D_2$. Assume that D_2 is effective. Then we have the inequality*

$$\kappa(X, D_1) \le \kappa(X, D_2).$$

Proof. We may assume that

$$D_2 = D_1 + \sum_{i=1}^{k} r_i(f_i)$$

where f_i is a nonzero rational function on X and $r_i \in \mathbb{R} \setminus \mathbb{Q}$ for every i by replacing D_1 (see Lemma 1.1.3). We may further assume that $\kappa(X, D_1) \geq 0$. Therefore, there is a positive integer m such that $\dim \Phi_{|mD_1|}(X) = \kappa(X, D_1)$. In this case, $\dim \Phi_{|nD_1|}(X) = \kappa(X, D_1)$ when n is a positive integer divisible by m. Hence it is sufficient to find a positive integer n divisible by m such that there is an injection

$$H^0(X, \mathcal{O}_X(\lfloor nD_1 \rfloor)) \hookrightarrow H^0(X, \mathcal{O}_X(\lfloor (n+1)D_2 \rfloor))$$

given by $f \mapsto f/g$, where $f \in H^0(X, \mathcal{O}_X(\lfloor nD_1 \rfloor))$, for some appropriately chosen rational function g. From now on, let us find n and g with the desired properties. We put

$$\delta = \min\{\mathrm{mult}_P D_2 \,|\, P \text{ is an irreducible component of } \mathrm{Supp} D_2\}.$$

Since $r_i \in \mathbb{R} \setminus \mathbb{Q}$ for every i, we can find a positive integer n divisible by m such that

$$nr_i = m_i + \delta_i,$$

$m_i \in \mathbb{Z}$ for every i, and that

$$\alpha = \left\{ |\mathrm{mult}_P \sum_{i=1}^{k} \delta_i(f_i)| \,\middle|\, P \text{ is a prime component of } \mathrm{Supp} \sum_{i=1}^{k} \delta_i(f_i) \right\} < \min\{1, \delta\}$$

by Dirichlet's box principle. We put $g = \prod_{i=1}^{k} f_i^{m_i}$.

Claim. *Let f be a nonzero rational function on X such that $(f) + nD_1 \geq 0$. Then $(f/g) + (n+1)D_2 \geq 0$.*

Proof of Claim. We note that

$$\begin{aligned}(f/g) + (n+1)D_2 &= (f) - \sum_{i=1}^{k} m_i(f_i) + (n+1)D_2 \\ &= (f) - \sum m_i(f_i) + nD_1 + \sum nr_i(f_i) + D_2 \\ &= (f) + nD_1 + \sum \delta_i(f_i) + D_2.\end{aligned}$$

It is sufficient to prove $\mathrm{mult}_P((f/g) + (n+1)D_2) \geq 0$ for every prime divisor P on X.

STEP 1. If P is an irreducible component of $\mathrm{Supp}D_2$, then we have

$$\mathrm{mult}_P D_2 \geq \delta > \alpha \geq |\mathrm{mult}_P \sum \delta_i(f_i)|.$$

Therefore, we obtain

$$\mathrm{mult}_P((f/g) + (n+1)D_2) \geq 0.$$

Thus, from now on, we assume that $P \not\subset \mathrm{Supp}D_2$.

STEP 2. We further assume that $\mathrm{mult}_P \sum \delta_i(f_i) \geq 0$. In this case, we have

$$\mathrm{mult}_P((f) + nD_1 + \sum \delta_i(f_i) + D_2) \geq \mathrm{mult}_P((f) + nD_1) \geq 0.$$

Therefore, $\mathrm{mult}_P((f/g) + (n+1)D_2) \geq 0$.

STEP 3. Finally, we consider the case when $\mathrm{mult}_P \sum \delta_i(f_i) < 0$ and $P \not\subset \mathrm{Supp}D_2$. Note that

$$\begin{aligned} 0 = \mathrm{mult}_P nD_2 &= \mathrm{mult}_P(nD_1 + n\sum r_i(f_i)) \\ &= \mathrm{mult}_P(nD_1 + \sum m_i(f_i) + \sum \delta_i(f_i)). \end{aligned}$$

Therefore, we have

$$\{\mathrm{mult}_P nD_1\} + \mathrm{mult}_P \sum \delta_i(f_i) = 0.$$

This implies that

$$\begin{aligned} &\mathrm{mult}_P((f) + nD_1 + \sum \delta_i(f_i) + D_2) \\ &= \mathrm{mult}_P((f) + \lfloor nD_1 \rfloor + D_2) \\ &\geq 0 \end{aligned}$$

Thus, we have $\mathrm{mult}_P((f/g) + (n+1)D_2) \geq 0$.

Anyway, we always have $(f/g) + (n+1)D_2 \geq 0$. □

Thus we obtain the desired inequality $\kappa(X, D_1) \leq \kappa(X, D_2)$. □

COROLLARY 1.1.6. *Let D_1 and D_2 be effective $\mathbb{R}$-divisors on a smooth projective variety X such that $D_1 \sim_{\mathbb{R}} D_2$. Then we have $\kappa(X, D_1) = \kappa(X, D_2)$.*

PROOF. By Lemma 1.1.5, it is obvious that $\kappa(X, D_1) = \kappa(X, D_2)$. □

The following corollary is sometimes very useful.

COROLLARY 1.1.7. *Let D be an $\mathbb{R}$-divisor on a smooth projective variety X. Assume that there exists an effective $\mathbb{R}$-divisor D' on X such that $D \sim_{\mathbb{Q}} D'$. Then we have*

$$\kappa(X, D) = \kappa_\iota(X, D).$$

PROOF. By Lemma 1.1.3, we have $\kappa(X, D) = \kappa(X, D')$. On the other hand, by definition, we have $\kappa_\iota(X, D) = \kappa(X, D')$. Therefore, we have the desired equality $\kappa(X, D) = \kappa_\iota(X, D)$. □

Proposition 1.1.8 shows that we do not need $\kappa_\iota(X, D)$ when D is a $\mathbb{Q}$-Cartier divisor.

PROPOSITION 1.1.8. *Let X be a smooth projective variety and let D be a $\mathbb{Q}$-divisor on X. Then we have $\kappa_\iota(X, D) = \kappa(X, D)$.*

PROOF. We assume that there exists an effective $\mathbb{R}$-divisor D' on X such that $D \sim_{\mathbb{R}} D'$. Then we can write

$$D' = D + \sum_{i=1}^{k} r_i (f_i),$$

where $r_i \in \mathbb{R}$ and f_i is a rational function on X for every i. We define

$$\mathcal{S} = \left\{ (s_1, \cdots, s_k) \,\middle|\, D + \sum_{i=1}^{k} s_i (f_i) \geq 0 \right\} \subset \mathbb{R}^k.$$

Since D is a $\mathbb{Q}$-divisor, $\mathcal{S}$ is defined over $\mathbb{Q}$. Note that $\mathcal{S}$ is not empty because $(r_1, \cdots, r_k) \in \mathcal{S}$. Therefore, we can take $(r'_1, \cdots, r'_k) \in \mathcal{S} \cap \mathbb{Q}^k$ and put

$$D'' = D + \sum_{i=1}^{k} r'_i (f_i).$$

Then D'' is an effective $\mathbb{Q}$-divisor on X such that $D \sim_{\mathbb{Q}} D''$. Therefore, we have $\kappa(X, D) = \kappa_\iota(X, D)$. □

We close this section with the following easy but important example.

EXAMPLE 1.1.9. Let B be a principal Cartier divisor on a smooth projective variety X. We put $D = rB$ with $r \in \mathbb{R} \setminus \mathbb{Q}$. Then it is obvious that $\kappa_\iota(X, D) = 0$ since $D \sim_{\mathbb{R}} 0$. On the other hand, $\kappa(X, D) = -\infty$. This is because $H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) = 0$ for every positive integer m.

1.2. Generalized abundance conjecture

[2]

In this section, we discuss a generalized version of the abundance conjecture. We note that we need the invariant Iitaka dimension (see Definition 1.1.4) in order to formulate the generalized abundance conjecture.

Conjecture 1.2.1 (Generalized abundance conjecture). *Let (X, Δ) be a projective log canonical pair. Then the equality*

$$\kappa_\iota(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$$

holds.

Conjecture 1.2.1 is one of the most important conjectures in the minimal model program. Conjecture 1.2.1 can be seen as a generalization of the following well-known conjecture.

Conjecture 1.2.2 (Non-vanishing conjecture for dlt pairs). *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair. Assume that $K_X + \Delta$ is pseudo-effective. Then there is an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{\mathbb{R}} D$.*

This is because we have:

Proposition 1.2.3. *Conjecture 1.2.2 is a special case of Conjecture 1.2.1.*

Proof. Let (X, Δ) be a projective dlt pair such that $K_X + \Delta$ is pseudo-effective. Then $\kappa_\sigma(X, K_X + \Delta) \geq 0$ by definition. Conjecture 1.2.1 implies $\kappa_\iota(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta) \geq 0$. Therefore, we can find an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{\mathbb{R}} D$ by the definition of κ_ι. This means that Conjecture 1.2.2 follows from Conjecture 1.2.1. □

Let us discuss the relationship between Conjecture 1.2.1 and the following good minimal model conjecture, which is nothing but Conjecture **??** above.

Conjecture 1.2.4 (Good minimal model conjecture). *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair and let Δ be an $\mathbb{R}$-divisor. If $K_X + \Delta$ is pseudo-effective, then (X, Δ) has a good minimal model.*

The following result is the main result of this section. Although we have never seen it in the literature, Theorem 1.2.5 seems to be a folklore statement.

[2] I will add this section after Section 4.10 of the book.

THEOREM 1.2.5. *Conjecture 1.2.1 holds in dimension* $\leq n$ *if and only if Conjecture 1.2.4 holds in dimension* $\leq n$.

PROOF. First, we assume that Conjecture 1.2.1 holds in dimension $\leq n$. By Proposition 1.2.3, the non-vanishing conjecture for dlt pairs (see Conjecture 1.2.2) holds in dimension $\leq n$. Moreover, by induction, we may assume that Conjecture 1.2.4 holds in dimension $\leq n-1$. Therefore, (X, Δ) always has a minimal model by [**Bir3**, Theorem 1.4 and Corollary 1.7] when (X, Δ) is a $\mathbb{Q}$-factorial projective dlt pair such that $K_X + \Delta$ is pseudo-effective with $\dim X \leq n$. From now on, we will see that $K_X + \Delta$ is semi-ample when (X, Δ) is an n-dimensional projective log canonical pair such that $K_X + \Delta$ is nef. By taking a dlt blow-up (see Theorem **??**), we may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. By using Shokurov's polytope (see Theorem **??** (3) and Proposition **??**), we may further assume that Δ is a $\mathbb{Q}$-divisor. By Conjecture 1.2.1, $K_X + \Delta$ is nef and abundant, that is, $K_X + \Delta$ is nef and $\kappa(X, K_X + \Delta) = \nu(X, K_X + \Delta)$. Therefore, $K_X + \Delta$ is semi-ample by the result of Fujino–Gongyo (see [**FG1**, Theorem 1.7]). Anyway, we obtain that Conjecture 1.2.4 holds in dimension $\leq n$.

Next, we assume that Conjecture 1.2.4 holds in dimension $\leq n$. By dlt blow-ups (see Theorem **??**), it is sufficient to prove that

$$\kappa_\iota(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$$

holds for every $\mathbb{Q}$-factorial projective dlt pair (X, Δ) with $\dim X \leq n$. We note that we can freely run the minimal model program with ample scaling by Conjecture 1.2.4 (see, for example, [**Bir3**, Theorem 1.5]). If $K_X + \Delta$ is pseudo-effective, then we get a good minimal model. If $K_X + \Delta$ is not pseudo-effective, then we obtain a Mori fiber space structure. In each step of the minimal model program, we can easily see that κ_ι and κ_σ are preserved. Therefore, we obtain $\kappa_\iota(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$. This means that Conjecture 1.2.1 holds in dimension $\leq n$. □

Anyway, by Theorem 1.2.5, we see that the generalized abundance conjecture (see Conjecture 1.2.1) is equivalent to the good minimal model conjecture (see Conjecture 1.2.4). This means that we can translate the good minimal model conjecture (see Conjecture 1.2.4), which is geometric, into a numerical condition, that is, the generalized abundance conjecture (see Conjecture 1.2.1).

We close this section with an easy example.

EXAMPLE 1.2.6. We put $X = \mathbb{P}^1$ and $\Delta = \sum_{i=1}^{3} r_i P_i$, where $r_i \in \mathbb{R} \setminus \mathbb{Q}$ with $0 < r_i < 1$ for every i, $P_i \neq P_j$ for $i \neq j$, and $\sum_{i=1}^{3} r_i = 2$.

Then (X, Δ) is a projective klt curve and $K_X + \Delta \sim_{\mathbb{R}} 0$. Thus we have

$$\kappa_\iota(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta) = 0.$$

On the other hand, we have $\kappa(X, K_X + \Delta) = -\infty$.

1.3. On Iitaka conjectures

In this section, we quickly recall Iitaka's conjectures. The following theorem is the main theorem of [**F-sub**].

THEOREM 1.3.1. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let D_X (resp. D_Y) be a simple normal crossing divisor on X (resp. Y). Assume that* $\mathrm{Supp} f^* D_Y \subset \mathrm{Supp} D_X$. *Then we have*

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y)$$

where F is a sufficiently general fiber of $f : X \to Y$.

Theorem 1.3.1 is a variant of [**Nak2**, Chapter V. 4.1. Theorem]. The proof of Theorem 1.3.1 heavily depends on Nakayama's theory of ω-sheaves and $\widehat{\omega}$-sheaves (see [**Nak2**, Chapter V. §3]). As an obvious corollary of Theorem 1.3.1, we have:

COROLLARY 1.3.2. *If Conjecture 1.2.1 holds for (X, D_X) in Theorem 1.3.1, then we obtain*

$$\begin{aligned}\kappa(X, K_X + D_X) &= \kappa_\sigma(X, K_X + D_X) \\ &\geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y) \\ &\geq \kappa(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y).\end{aligned}$$

Let us recall Iitaka's famous conjecture (see [**Ii1**]).

CONJECTURE 1.3.3 (Iitaka conjecture for κ). *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then we have*

$$\kappa(X, K_X) \geq \kappa(F, K_F) + \kappa(Y, K_Y)$$

where F is a sufficiently general fiber of f.

Conjecture 1.3.3 is usually called Conjecture $C_{n,m}$ when $\dim X = n$ and $\dim Y = m$.

Let us introduce the notion of the logarithmic Kodaira dimension, which was also introduced by Shigeru Iitaka in [**Ii2**].

Definition 1.3.4 (Logarithmic Kodaira dimension). Let V be an irreducible algebraic variety. By Nagata, we have a complete algebraic variety $\overline{V}$ which contains V as a dense Zariski open subset. By Hironaka, we have a smooth projective variety $\overline{W}$ and a projective birational morphism $\mu : \overline{W} \to \overline{V}$ such that if $W = \mu^{-1}(V)$, then $\overline{D} = \overline{W} - W = \mu^{-1}(\overline{V} - V)$ is a simple normal crossing divisor on $\overline{W}$. The *logarithmic Kodaira dimension* $\overline{\kappa}(V)$ of V is defined as

$$\overline{\kappa}(V) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D})$$

where κ denotes Iitaka's D-dimension.

It is easy to see that $\overline{\kappa}(V)$ is well-defined, that is, it is independent of the choice of the pair $(\overline{W}, \overline{D})$. For $\overline{\kappa}$, we have:

Conjecture 1.3.5 (Iitaka conjecture for $\overline{\kappa}$). *Let $g : V \to W$ be a dominant morphism between varieties. Then we have*

$$\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$$

where F' is an irreducible component of a sufficiently general fiber of $g : V \to W$.

Conjecture 1.3.5 is called Conjecture $\overline{C}_{n,m}$ when $\dim V = n$ and $\dim W = m$. Note that Conjecture 1.3.3 is a special case of Conjecture 1.3.5. We also note that Conjecture 1.3.5 holds true when V is an affine variety by [**F-sub**].

Theorem 1.3.6 ([**F-sub**, Corollary 1.3]). *Let $g : V \to W$ be a dominant morphism from an affine variety V. Then we have*

$$\overline{\kappa}(V) \geq \overline{\kappa}(F') + \overline{\kappa}(W)$$

where F' is an irreducible component of a sufficiently general fiber of $g : V \to W$.

The proof of Theorem 1.3.6 in [**F-sub**] uses Theorem 1.3.1 and the minimal model program for projective klt pairs with big boundary divisor. For the details, see [**F-sub**].

Anyway, by Theorem 1.3.1, Conjecture 1.3.3 and Conjecture 1.3.5 follow from Conjecture 1.2.1. Therefore, Iitaka's conjectures (see Conjecture 1.3.3 and Conjecture 1.3.5) are now consequences of the minimal model conjecture and the abundance conjecture.

The following conjecture seems to be natural from the minimal model theoretic viewpoint. We call it Conjecture $C^{\log}_{n,m}$ when $\dim X = n$ and $\dim Y = m$.

CONJECTURE 1.3.7 (Log Iitaka conjecture). *Let (X, Δ) be a projective log canonical pair and let $f : X \to Y$ be a surjective morphism onto a normal projective variety Y with connected fibers. Then*

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta|_F) + \kappa(Y)$$

where F is a sufficiently general fiber of $f : X \to Y$. Note that $\kappa(Y)$ denotes the Kodaira dimension of Y, that is, $\kappa(Y) = \kappa(\widetilde{Y}, K_{\widetilde{Y}})$, where $\widetilde{Y} \to Y$ is a resolution of singularities.

It is also known that Conjecture 1.3.7 follows from Conjecture 1.2.1 when Δ is a $\mathbb{Q}$-divisor by [**Nak2**, Chapter V. 4.1. Theorem]. Since we can not find the proof of Conjecture $C^{\log}_{2,1}$ in the literature, we prove it here for the reader's convenience.

THEOREM 1.3.8. *Conjecture $C^{\log}_{2,1}$ holds true.*

PROOF. We use the same notation as in Conjecture 1.3.7. It is well known that Conjecture $C_{2,1}$ holds true. So we use it in this proof. More precisely, it is well known that Conjecture 1.2.1 holds true in dimension ≤ 3 by Theorem 1.2.5. Therefore, we know that Conjectures $C_{n,m}$ and Conjecture $\overline{C}_{n,m}$ hold for $n \leq 3$. By replacing X with its minimal resolution, we may assume that X is smooth. By [**Nak2**, Chapter V. 4.1. Theorem (1)], we have

$$\begin{aligned}\kappa_\sigma(X, K_X + \Delta) &\geq \kappa_\sigma(F, K_F + \Delta|_F) + \kappa_\sigma(Y, K_Y)\\ &\geq \kappa(F, K_F + \Delta|_F) + \kappa(Y, K_Y).\end{aligned}$$

For the proof of Conjecture $C^{\log}_{2,1}$, we may assume that $\kappa(Y, K_Y) \geq 0$ and $\kappa(F, K_F + \Delta|_F) \geq 0$. If we can take an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{\mathbb{Q}} D$, then we have

$$\kappa_\sigma(X, K_X + \Delta) = \kappa_\iota(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$$

by Corollary 1.1.7 and Conjecture 1.2.1. This implies the desired inequality. If $\kappa(F, K_F) \geq 0$, then we have $\kappa(X, K_X) \geq 0$ by Conjecture $C_{2,1}$. Therefore, we can find an effective $\mathbb{R}$-divisor D on X with $K_X + \Delta \sim_{\mathbb{Q}} D$. So, we may assume that $F = \mathbb{P}^1$. Since we assumed $\kappa(F, K_F + \Delta|_F) \geq 0$, we can find an effective $\mathbb{Q}$-divisor Δ' on X such that $\Delta' \leq \Delta$ and $F \cdot \Delta' = 2$. By the above argument, it is sufficient to prove $\kappa(X, K_X + \Delta') \geq 0$. We run the minimal model program with respect to $K_X + \Delta'$. In each step, we contract a rational curve. On the other hand, $\kappa(Y, K_Y) \geq 0$. Therefore, this minimal model program is a minimal model program over Y. Since $\kappa(Y, K_Y) \geq 0$ and $(K_X + \Delta') \cdot F = 0$, we can easily see that we obtain a good minimal model of (X, Δ') by this minimal model program (see, for example, Theorem **??** and Theorem **??**). Therefore, we see that $\kappa(X, K_X + \Delta') \geq 0$. Anyway, we obtain the desired inequality. $\square$

Finally, let us quickly review Kawamata's result on Iitaka's conjecture in [**Ka2.5**]. Before we state the main result of [**Ka2.5**], we recall Viehweg's definition of $\mathrm{Var}(f)$.

DEFINITION 1.3.9 (Viehweg's $\mathrm{Var}(f)$). Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. In this setting, $\mathrm{Var}(f)$ is defined to be the minimal number k such that there exists a subfield K of $\overline{\mathbb{C}(Y)}$ of transcendental degree k over $\mathbb{C}$ and a variety V over K with $V \times_{\mathrm{Spec} K} \mathrm{Spec}\,\overline{\mathbb{C}(Y)} \sim_{\mathrm{bir}} X_{\overline{\eta}}$, that is, $V \times_{\mathrm{Spec} K} \mathrm{Spec}\,\overline{\mathbb{C}(Y)}$ is birationally equivalent to the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$.

Theorem 1.3.10 is the main result of [**Ka2.5**]. The proof of Theorem 1.3.10 is out of scope of this book.

THEOREM 1.3.10 ([**Ka2.5**, Theorem 1.1]). *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers and let $\mathcal{L}$ be an invertible sheaf on Y. Assume that the geometric generic fiber $X_{\overline{\eta}}$ has a good minimal model defined over $\overline{\mathbb{C}(Y)}$. Then the following assertions hold:*

(i) *There exists a positive integer n such that*

$$\kappa(Y, \widehat{\det}(f_*\omega_{X/Y}^{\otimes n})) \geq \mathrm{Var}(f),$$

where $\widehat{\det}(f_*\omega_{X/Y}^{\otimes n}) = (\wedge^r f_*\omega_{X/Y}^{\otimes n})^{**}$ with $r = \mathrm{rank} f_*\omega_{X/Y}^{\otimes n}$.

(ii) *If $\kappa(Y, \mathcal{L}) \geq 0$, then*

$$\kappa(X, \omega_{X/Y} \otimes f^*\mathcal{L}) \geq \kappa(F, K_F) + \max\{\kappa(Y, \mathcal{L}), \mathrm{Var}(f)\},$$

where F is a sufficiently general fiber of $f : X \to Y$.

As an obvious corollary, we have:

COROLLARY 1.3.11 ([**Ka2.5**, Corollary 1.2]). *Under the same assumptions and notation as in Theorem 1.3.10, we have*

(i) $\kappa(X, \omega_{X/Y}) \geq \kappa(F, K_F) + \mathrm{Var}(f)$, *and*

(ii) *if $\kappa(Y, K_Y) \geq 0$, then*

$$\kappa(X, K_X) \geq \kappa(F, K_F) + \max\{\kappa(Y, K_Y), \mathrm{Var}(f)\}.$$

For the details of Theorem 1.3.10, see Kawamata's paper [**Ka2.5**].

1.4. Examples

[3]

EXAMPLE 1.4.1 (see [**F9**, Section 3]). We fix a lattice $N = \mathbb{Z}^3$. We take lattice points

$$\begin{aligned} &v_1 = (1,0,1), && v_2 = (0,1,1), && v_3 = (-1,-1,1), \\ &v_4 = (1,0,-1), && v_5 = (0,1,-1), && v_6 = (-1,-1,-1). \end{aligned}$$

We consider the following fan

$$\Delta = \left\{ \begin{array}{ccc} \langle v_1, v_2, v_4 \rangle, & \langle v_2, v_4, v_5 \rangle, & \langle v_2, v_3, v_5, v_6 \rangle, \\ \langle v_1, v_3, v_4, v_6 \rangle, & \langle v_1, v_2, v_3 \rangle, & \langle v_4, v_5, v_6 \rangle, \\ \text{and their faces} & & \end{array} \right\}.$$

Then the associated toric variety $X = X(\Delta)$ has the following properties.

(i) X is a *non-projective* complete toric variety with $\rho(X) = 1$.
(ii) There exists a Cartier divisor D on X such that D is positive on $\overline{NE}(X) \setminus \{0\}$. In particular, $\overline{NE}(X)$ is a half line.

Therefore, Kleiman's criterion for ampleness (see Theorem **??**) does not hold for this X. We note that X is not $\mathbb{Q}$-factorial and that there is a torus invariant curve $C \simeq \mathbb{P}^1$ on X such that C is numerically equivalent to zero. Precisely speaking, C is a torus invariant curve corresponding to the wall $\langle v_2, v_4 \rangle$.

By the similar construction, we have the following example.

EXAMPLE 1.4.2 (see [**F9**, Section 4]). We fix a lattice $N = \mathbb{Z}^3$. We take lattice points

$$\begin{aligned} &v_1 = (1,0,1), && v_2 = (0,1,1), && v_3 = (-1,-2,1), \\ &v_4 = (1,0,-1), && v_5 = (0,1,-1), && v_6 = (-1,-1,-1). \end{aligned}$$

We consider the following fan

$$\Delta = \left\{ \begin{array}{ccc} \langle v_1, v_2, v_4, v_5 \rangle, & \langle v_2, v_3, v_5, v_6 \rangle, & \langle v_1, v_3, v_4, v_6 \rangle, \\ \langle v_1, v_2, v_3 \rangle, & \langle v_4, v_5, v_6 \rangle, & \text{and their faces} \end{array} \right\}.$$

Then the associated toric variety $X = X(\Delta)$ is a complete toric threefold with $\mathrm{Pic}(X) = \{0\}$. Therefore, there are no effective Cartier divisors on Y. Thus X can not be embedded into a smooth variety. Of course, X is not projective. In this case, $\mathrm{Pic}(X)$ has no informations.

[3] I will include this section in Section 2.2 of the book.

1.5. A remark on dlt blow-ups

[4]

The following example helps us understand dlt blow-ups for non-lc singularities.

EXAMPLE 1.5.1. There exists a 2-dimensional normal Gorenstein singularity $P \in X$ with the following properties.

(i) Let $f : Y \to X$ be the unique minimal resolution. Then $\mathrm{Exc}(f) = C_1 \cup C_2 \cup C_3$ where $C_i \simeq \mathbb{P}^1$ for every i, and there is a point Q such that $Q \in C_i$ for every i with $C_i \cap C_j = Q$ for $i \neq j$.

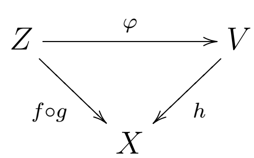
(ii) $K_Y = f^*K_X - \Delta_Y$ such that $\Delta_Y = C_1 + C_2 + C_3$.

(iii) (Y, Δ_Y) is not log canonical.

Of course, $f : (Y, \Delta_Y) \to X$ is not a dlt blow-up in the sense of Theorem **??**. Let $g : Z \to Y$ be the blow-up at Q. Let C'_i be the strict transform of C_i for every i. Let E be the exceptional curve of g. Then

$$K_Z + \Delta_Z = g^*f^*K_X - E,$$

where $\Delta_Z = C'_1 + C'_2 + C'_3 + E$, is a dlt blow-up of $P \in X$ in the sense of Theorem **??**. We note that $K_Z + \Delta_Z$ is not nef over X. We can contract $C'_1 + C'_2 + C'_3$ over X and obtain $h : V \to X$.



More precisely, if we run the minimal model program with respect to $K_Z + \Delta_Z$ over X, then we obtain $h : V \to X$ as a minimal model of (Z, Δ_Z) over X. It is easy to see that $h : V \to X$ is also a dlt blow-up of $P \in X$ in the sense of Theorem **??**. More precisely, $h : (V, E') \to X$ is a dlt blow-up in the sense of Theorem **??**, where $\mathrm{Exc}(h) = \varphi_*E =: E'$. Note that

$$K_V + E' = h^*K_X - E'.$$

In this case, $K_V + E'$ is ample over X.

From now on, let us construct such a singularity $P \in X$. By [**Ar2**, Corollary (1.6)] and [**Ar3**, Theorem (3.8)], it is sufficient to construct it in the category of analytic spaces.

Let L_i be a line on $\mathbb{P}^2$ for $1 \leq i \leq 3$ such that $L_i \cap L_j = Q$ for $i \neq j$. We take four distinct points $P_{i1}, P_{i2}, P_{i3}, P_{i4}$ on each L_i such that $P_{ij} \neq Q$ for $1 \leq j \leq 4$. We take blow-ups of $\mathbb{P}^2$ at P_{ij} for $1 \leq i \leq 3$

[4] I will add this section after Remark 4.4.22 of the book.

and $1 \leq j \leq 4$. Then we obtain a birational morphism $\varphi : Y \to \mathbb{P}^2$. Let C_i be the strict transform of L_i on Y. We can easily see that the intersection matrix $(C_i \cdot C_j)$ is

$$\begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix},$$

which is negative definite. Note that $C_i \cap C_j = Q$ for $i \neq j$. By the theorem of Grauert–Artin (see, for example, [Bă, Theorem 14.20]), we have a contraction morphism $f : Y \to X$ in the category of algebraic spaces which contracts $C_1 + C_2 + C_3$ to a point $P \in X$. Note that

$$K_{\mathbb{P}^2} + L_1 + L_2 + L_3 \sim 0$$

implies

$$K_Y + C_1 + C_2 + C_3 \sim 0.$$

Thus we obtain that $P \in X$ is an isolated normal Gorenstein singularity. By the above construction, $P \in X$ has the desired properties.

We make a remark on the author's paper [F26], which contains a small mistake.

Remark 1.5.2. We use the notation in [F26, Theorem 4.1]. In the proof of [F26, Theorem 4.1], some $E_i \in \mathcal{E}$ may be contracted in the minimal model program. Therefore, h is not necessarily a local isomorphism at the generic point of $E_i \in \mathcal{E}$. This means that the property (a) in [F26, Theorem 4.1] is not always true for the model $g : Z \to X$ constructed in the proof of [F26, Theorem 4.1]. For an explicit example, see the minimal model program obtaining $h : (V, E') \to X$ from $f \circ g : (Z, \Delta_Z) \to X$ in Example 1.5.1.

Acknowledgments. The author thanks Professors Kazuhiro Fujiwara, Yoshinori Gongyo, Shihoko Ishii, and Noboru Nakayama.

Bibliography

[Ar2] M. Artin, On the solutions of analytic equations, Invent. Math. **5** (1968), 277–291.

[Ar3] M. Artin, Algebraic approximation of structures over complete local rings, Inst. Hautes Études Sci. Publ. Math. No. **36** (1969), 23–58.

[Bă] L. Bădescu, *Algebraic surfaces*, Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author, Universitext. Springer-Verlag, New York, 2001.

[Bir3] C. Birkar, On existence of log minimal models II, J. Reine Angew. Math. **658** (2011), 99–113.

[Choi] S. Choi, The geography of log models and its applications, Thesis (Ph.D.)—The Johns Hopkins University. 2008.

[F9] O. Fujino, On the Kleiman–Mori cone, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 5, 80–84.

[F26] O. Fujino, Semi-stable minimal model program for varieties with trivial canonical divisor, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), no. 3, 25–30.

[F-book] O. Fujino, Foundation of the minimal model program, preprint (2014).

[F-sub] O. Fujino, Subadditivity of the logarithmic Kodaira dimension for affine varieties, preprint (2014).

[FG1] O. Fujino, Y. Gongyo, Log pluricanonical representations and the abundance conjecture, Compos. Math. **150** (2014), no. 4, 593–620.

[Ft4] T. Fujita, Zariski decomposition and canonical rings of elliptic threefolds, J. Math. Soc. Japan **38** (1986), no. 1, 19–37.

[Ii1] S. Iitaka, Genus and classification of algebraic varieties. I. (Japanese), Sûgaku **24** (1972), no. 1, 14–27.

[Ii2] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, *Complex analysis and algebraic geometry*, pp. 175–189. Iwanami Shoten, Tokyo, 1977.

[Ka2.5] Y. Kawamata, Minimal models and the Kodaira dimension of algebraic fiber spaces, J. Reine Angew. Math. **363** (1985), 1–46.

[Nak2] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, 2004.

Index