

Fundamental theorems for the log minimal model program

Dedicated to Professor Shigefumi Mori on the occasion of his sixtieth birthday

by

Osamu FUJINO

Abstract

In this paper, we prove the cone theorem and the contraction theorem for pairs (X, B) , where X is a normal variety and B is an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier.

2010 Mathematics Subject Classification: Primary 14E30; Secondary 14C20, 14F17.

Keywords: log minimal model program, log canonical pairs, vanishing theorems, non-vanishing theorem, base point free theorem, rationality theorem, cone theorem, lengths of extremal rays

Contents

§1. Introduction	2
§2. Warm-ups	6
§3. Kawamata–Viehweg, Nadel, Kollár, . . .	8
§4. Preliminaries	10
§5. Hodge theoretic injectivity theorem	15
§6. Injectivity, torsion-free, and vanishing theorems	19
§7. Non-lc ideal sheaves	23
§8. Vanishing theorem	25
§9. Lc centers	28
§10. Dlt blow-ups	28
§11. Vanishing theorem for minimal lc centers	33
§12. Non-vanishing theorem	34
§13. Base point free theorem	38

Communicated by S. Mori. Received November 30, 2009. Revised XXXX.

O. Fujino: Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan;
e-mail: fujino@math.kyoto-u.ac.jp

§14. Shokurov's differentials	40
§15. Rationality theorem	42
§16. Cone theorem	47
§17. Base point free theorem revisited	53
§18. Lengths of extremal rays	53
§19. Ambro's theory of quasi-log varieties	60
§20. Related topics	62
References	63

§1. Introduction

The main purpose of this paper is to prove the following cone and contraction theorem. It is the culmination of the works of several authors: Ambro, Benveniste, Birkar, Kawamata, Kollár, Mori, Reid, Shokurov, and others. It is indispensable for the study of the log minimal model program for varieties with bad singularities (cf. [F17]).

Theorem 1.1 (cf. Theorems 16.4, 16.6, 18.9, and 18.10). *Let X be a normal variety defined over \mathbb{C} and let B be an effective \mathbb{R} -divisor such that $K_X + B$ is \mathbb{R} -Cartier, and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Then we have*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+B \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)} + \sum R_j$$

with the following properties.

- (1) $\text{Nlc}(X, B)$ is the non-lc locus of (X, B) and

$$\overline{NE}(X/S)_{\text{Nlc}(X,B)} = \text{Im}(\overline{NE}(\text{Nlc}(X, B)/S) \rightarrow \overline{NE}(X/S)).$$

- (2) R_j is a $(K_X + B)$ -negative extremal ray of $\overline{NE}(X/S)$ such that

$$R_j \cap \overline{NE}(X/S)_{\text{Nlc}(X,B)} = \{0\}$$

for every j .

- (3) Let A be a π -ample \mathbb{R} -divisor on X . Then there are only finitely many R_j 's included in $(K_X + B + A)_{<0}$. In particular, the R_j 's are discrete in the half-space $(K_X + B)_{<0}$.
- (4) Let F be a face of $\overline{NE}(X/S)$ such that

$$F \cap (\overline{NE}(X/S)_{K_X+B \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)}) = \{0\}.$$

Then there exists a contraction morphism $\varphi_F : X \rightarrow Y$ over S .

- (i) Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.
 - (ii) $\mathcal{O}_Y \simeq (\varphi_F)_* \mathcal{O}_X$.
 - (iii) Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Y on Y such that $L \simeq \varphi_F^* L_Y$.
- (5) Every $(K_X + B)$ -negative extremal ray R with

$$R \cap \overline{NE}(X/S)_{\text{Nlc}(X,B)} = \{0\}$$

is spanned by a rational curve C with $0 < -(K_X + B) \cdot C \leq 2 \dim X$.

From now on, we further assume that (X, B) is a log canonical pair, that is, $\text{Nlc}(X, B) = \emptyset$. Then we have the following properties.

- (6) Let H be an effective \mathbb{R} -Cartier \mathbb{R} -divisor on X such that $K_X + B + H$ is π -nef and $(X, B + H)$ is log canonical. Then, either $K_X + B$ is also π -nef or there is a $(K_X + B)$ -negative extremal ray R such that

$$(K_X + B + \lambda H) \cdot R = 0$$

where

$$\lambda := \inf\{t \geq 0 \mid K_X + B + tH \text{ is } \pi\text{-nef}\}.$$

Of course, $K_X + B + \lambda H$ is π -nef.

The first half of Theorem 1.1, that is, (1), (2), (3), and (4) in Theorem 1.1, is the main result of [A1]. His proof depends on the theory of *quasi-log varieties*. Unfortunately, the theory of quasi-log varieties is inaccessible even for experts because it requires very technical arguments on reducible varieties. In this paper, we give a proof of the above cone and contraction theorem without using the notion of quasi-log varieties. Our approach is much more direct than Ambro's theory of quasi-log varieties. We note that the reader does not have to refer to [A1] nor the book [F11] in order to read this paper. The latter half of Theorem 1.1, that is, (5) and (6) in Theorem 1.1, is a generalization of the results obtained by Kollár, Kawamata, Shokurov, and Birkar. We note that the formulation of (5) is new. It will play important roles in the log minimal model program with scaling. So, we include this part in our cone and contraction theorem.

Here we would like to compare our results with the theory of quasi-log varieties ([A1], [F11]). The first part of Theorem 1.1, the main theorem of our article, was first proved by the theory of quasi-log varieties; the proof reduced a problem on an irreducible normal variety to one on the union of certain proper closed subvarieties (called non-klt centers) of various dimensions. The notion of quasi-log varieties

was a framework to treat such reducible varieties. Thus various strong vanishing theorems on quasi-log varieties were needed and a significant part of [F11] was devoted to the proof of such theorems. Delicate arguments were also needed to overcome several technical difficulties including partial resolutions of reducible closed subvarieties.

The main idea of our paper first appeared in [F16], which treated a special kind of non-klt centers called minimal lc centers and proved their existence and normality. The point of our approach was a fully general treatment of minimal lc centers, whose existence and normality were settled earlier under special assumptions.

Vanishing theorems required for these proofs are stronger than the Kawamata–Viehweg–Nadel vanishing theorem but not as difficult as the one on quasi-log varieties. The next step is to reduce a problem on a normal variety to one on its minimal lc centers, where a vanishing theorem plays a central role. Thus it is enough to consider only normal varieties in our treatment. Though we need to prepare vanishing theorems stronger than the Kawamata–Viehweg–Nadel vanishing theorem, they are all proved in our paper (without quoting [A1] or [F11]). It is our view that the most important contribution of our paper and [F16] is the correct formulation of various vanishing theorems and non-vanishing theorem (Theorem 12.1), by which the cone and contraction theorems can be proved without any difficulties. It is not needed to treat reducible varieties or precise partial resolutions of singularities of reducible varieties. As already mentioned, the vanishing theorems needed and formulated in our paper are stronger than the Kawamata–Viehweg–Nadel vanishing theorem. It is our belief that this advancement of vanishing theorems distinguishes our treatment from those in [KMM], [KM] and [L].

Let us briefly recall the history of the cone and contraction theorem. In the epoch-making paper [Mo], Mori invented the cone theorem for smooth projective varieties and the contraction theorem for smooth projective threefolds by his ingenious ideas. See, for example, [KM, Theorems 1.24 and 1.32]. After Mori’s pioneering works, the cone and contraction theorem was proved and generalized for singular varieties by the completely different method, which is now called X-method (cf. [Ka1], [Ko1], [R], and [S1]). In [A1], Ambro introduced the notion of quasi-log varieties and generalized the cone and contraction theorem. See, for example, [F11, Chapter 3]. For the details of the history of the cone and contraction theorem up to [KMM], we recommend the reader to see the introductions of Chapters 2, 3, and 4 of [KMM].

We summarize the contents of this paper. Section 2 is a warm-up. Here, we discuss the base point free theorem for projective log canonical surfaces to moti-

vate the reader. This section clarifies the difference between our new approach and Ambro’s theory of quasi-log varieties. In Section 3, we explain our philosophy on various vanishing theorems. This section helps the reader to understand the subsequent sections on our new vanishing theorems. Section 4 collects the preliminary definitions and results. In Section 5, we explain the Hodge theoretic aspect of the injectivity theorem. It is an easy consequence of the theory of mixed Hodge structures on compact support cohomology groups of smooth quasi-projective varieties. Section 6 treats generalizations of Kollár’s injectivity, torsion-free, and vanishing theorems. These results play crucial roles in the following sections. They replace the Kawamata–Viehweg vanishing theorem. In Section 7, we introduce the notion of non-lc ideal sheaves. It is an analogue of the well-known multiplier ideal sheaves. Section 8 contains a very important vanishing theorem. It is a generalization of the Nadel vanishing theorem. It is very useful for the study of log canonical pairs. In Section 9, we recall the basic properties of lc centers. Section 10 treats the dlt blow-up following Hacon and its slight refinement, which will be useful for future studies (cf. [G]). Here, we need [BCHM]. In Section 11, we give a vanishing theorem for minimal lc centers. By the dlt blow-up obtained in Section 10, we can easily prove this very important vanishing theorem. Section 12 is devoted to the proof of the non-vanishing theorem. In Section 13, we prove the base point free theorem. It is a direct consequence of the non-vanishing theorem. In Section 14, we quickly recall Shokurov’s differentials. Section 15 is devoted to the rationality theorem. In Section 16, we obtain the cone theorem and contraction theorem by using the rationality theorem and base point free theorem. Section 17 is a supplement to the base point free theorem. In Section 18, we discuss estimates of lengths of extremal rays. It is very important for the study of the log minimal model program with scaling. Our formulation for non-lc pairs is new. In Section 19, we quickly explain some results which were obtained by the theory of quasi-log varieties but can not be covered by our new approach. In the final section: Section 20, we briefly explain some related topics obtained by the author for the reader’s convenience.

This paper grew out from the ideas in [F16]. The result in Section 10 heavily depends on [BCHM]. We use it to prove the vanishing theorem for minimal lc centers in Section 11. We note that we can prove the result in Section 11 without applying [BCHM] if we discuss the theory of mixed Hodge structures on compact support cohomology groups of *reducible* varieties. It was carried out in [F11, Chapter 2]. We note that [F11, Chapter 2] is independent of the log minimal model program for klt pairs. So, the non-vanishing theorem: Theorem 12.1, the base point free theorem: Theorem 13.1, the rationality theorem: Theorem 15.1, and the cone theorem: Theorem 16.6 do not depend on the corresponding results for klt pairs.

Therefore, our proofs are new even for klt pairs. In Section 18, we need Theorem 10.4, which is a consequence of [BCHM], to prove Theorems 18.2 and 18.10. At present there are no proofs of Theorems 18.2 and 18.10 without using [BCHM]. However, Theorem 18.2 can be directly proved if we have an appropriate vanishing theorem for projective morphisms between analytic spaces. For the details, see [F11, Remark 3.22].

We will work over \mathbb{C} , the complex number field, throughout this paper.

§2. Warm-ups

In this section, we explain the base point free theorem for projective log canonical surfaces to motivate the reader. This section clarifies the difference between our new approach and Ambro's theory of quasi-log varieties. We recommend the reader to see [F15, Section 4] for Ambro's approach. The following theorem is a very special case of Theorem 13.1.

Theorem 2.1 (Base point free theorem for lc surfaces). *Let (X, B) be a projective log canonical surface. Let L be a nef Cartier divisor on X such that $aL - (K_X + B)$ is ample for some $a > 0$. Then $|mL|$ is base point free for $m \gg 0$.*

It can not be proved by the traditional X-method. A key ingredient of this paper is the following generalization of Kollár's vanishing theorem. We will describe it in Section 8.

Theorem 2.2 (cf. Theorem 8.1). *Let (X, B) be a projective log canonical pair. Let D be a Cartier divisor on X such that $D - (K_X + B)$ is ample. Let C be an lc center of (X, B) with the reduced scheme structure. Then*

$$H^i(X, \mathcal{I}_C \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, where \mathcal{I}_C is the defining ideal sheaf of C . In particular, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(C, \mathcal{O}_C(D))$$

is surjective.

In Theorem 2.2, we do not assume that C is isolated in the non-klt locus of the pair (X, B) , neither do we assume that there exists another boundary \mathbb{R} -divisor B' on X such that (X, B') is klt. Therefore, it can not be proved by the traditional arguments depending on the Kawamata–Viehweg–Nadel vanishing theorem.

The next theorem is a special case of Theorem 12.1. This formulation was first introduced in [F16]. We will see that it is equivalent to Theorem 2.1.

Theorem 2.3 (Non-vanishing theorem for lc surfaces). *Let X be a projective log canonical surface. Let L be a nef Cartier divisor on X such that $aL - (K_X + B)$ is ample for some $a > 0$. Then the base locus $\text{Bs}|mL|$ of $|mL|$ contains no lc centers of (X, B) for $m \gg 0$.*

Proof. It is sufficient to check that $\text{Bs}|mL|$ contains no minimal lc centers of (X, B) for $m \gg 0$. Let C be a minimal lc center of (X, B) . If C is a point P , then $\text{Bs}|mL|$ does not contain C for every $m \geq a$. It is because the evaluation map

$$H^0(X, \mathcal{O}_X(mL)) \rightarrow \mathbb{C}(P) \simeq H^0(P, \mathcal{O}_P(mL))$$

is surjective for every $m \geq a$ by Theorem 2.2. If C is a curve, then $C \subset \lfloor B \rfloor$ and (X, B) is plt around C . Therefore,

$$K_C + B_C = (K_X + B)|_C$$

is klt by adjunction. Since $aL|_C - (K_C + B_C)$ is ample, there exists m_1 such that $|mL|_C$ is base point free for every $m \geq m_1$. By Theorem 2.2, the restriction map

$$H^0(X, \mathcal{O}_X(mL)) \rightarrow H^0(C, \mathcal{O}_C(mL))$$

is surjective for every $m \geq a$. Thus, $\text{Bs}|mL|$ does not contain C for $m \gg 0$. So, we finish the proof since there are only finitely many minimal lc centers. \square

In the above proof, C is a point or a divisor on X . So, there are no difficulties in investigating minimal lc centers. When $\dim X \geq 3$, we need a more powerful vanishing theorem (cf. Theorem 11.1) to study linear systems on minimal lc centers.

Let us explain the proof of Theorem 2.1.

Proof of Theorem 2.1. If (X, B) is klt, then the statement is well-known as the Kawamata–Shokurov base point free theorem (cf. [KM, Theorem 3.3]). So, we assume that (X, B) is lc but not klt for simplicity. By Theorem 2.3, we can take general members $D_1, D_2, D_3 \in |m_1L|$ for some $m_1 > 0$. If $\text{Bs}|m_1L| = \emptyset$, then L is semi-ample. Therefore, we assume that $\text{Bs}|m_1L| \neq \emptyset$. We note that $(X, B + D)$, where $D = D_1 + D_2 + D_3$, is log canonical outside $\text{Bs}|m_1L|$ and that $(X, B + D)$ is not log canonical at the generic point of every irreducible component of $\text{Bs}|m_1L|$. Let c be the log canonical threshold of (X, B) with respect to D . Then $0 < c$ by Theorem 2.3 and $c < 1$ because $(X, B + D)$ is not log canonical. By the above construction, $(X, B + cD)$ is log canonical and there is an lc center C of $(X, B + cD)$ such that C is contained in $\text{Bs}|m_1L|$. By applying Theorem 2.3 to

$$(3cm_1 + a)L - (K_X + B + cD) \sim_{\mathbb{Q}} aL - (K_X + B)$$

on $(X, B + cD)$, we see that $\text{Bs}|m_2m_1L|$ does not contain C for $m_2 \gg 0$. Therefore, $\text{Bs}|m_2m_1L| \subsetneq \text{Bs}|m_1L|$ holds. By the noetherian induction, we obtain that L is semi-ample. With a little care, we can check that $|mL|$ is base point free for $m \gg 0$. We omit some details here. For details, see the proof of Theorem 13.1. \square

In Ambro's framework of quasi-log varieties (cf. [A1], [F11], and [F15]), we have to discuss the base point free theorem for certain reducible curves (cf. [F1]) to prove Theorem 2.1. We note that the ultimate generalization of Theorem 2.1 for surfaces is proved in [F17].

One of the main purposes of this paper is to generalize Theorem 2.3 for pairs (X, B) , where X is an n -dimensional normal variety and B is an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier (see Theorems 12.1 and 13.1).

§3. Kawamata–Viehweg, Nadel, Kollár, ...

In this section, we explain our philosophy on vanishing theorems. There exists a big conceptual difference between our new approach described in this paper and the traditional arguments based on the Kawamata–Viehweg–Nadel vanishing theorem (cf. [KMM], [KM], and [L]).

In the traditional X-method, the following type of the Kawamata–Viehweg vanishing theorem plays crucial roles (cf. [KM, Theorem 3.1], [L, Theorem 9.1.18]).

3.1 (The Kawamata–Viehweg vanishing theorem). Let X be a smooth projective variety and let B be an effective \mathbb{Q} -divisor such that $\text{Supp} B$ is simple normal crossing and $\lfloor B \rfloor = 0$. Let L be a Cartier divisor on X such that $L - (K_X + B)$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for every $i > 0$.

Recently, the (algebraic version of) Nadel vanishing theorem, which is a generalization of the above Kawamata–Viehweg vanishing theorem, is very often used for the study of linear systems (cf. [L, Theorem 9.4.17]).

3.2 (The Nadel vanishing theorem). Let X be a normal projective variety and let B be an effective \mathbb{Q} -divisor on X such that $K_X + B$ is \mathbb{Q} -Cartier. Let L be a Cartier divisor on X such that $L - (K_X + B)$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(L) \otimes \mathcal{J}(X, B)) = 0$$

for every $i > 0$, where $\mathcal{J}(X, B)$ is the multiplier ideal sheaf of the pair (X, B) (see Remark 7.3 below).

The following relative version of the Kawamata–Viehweg vanishing theorem sometimes plays very important roles implicitly (cf. [L, Theorem 9.4.17], [KM, Corollary 2.68]).

3.3 (The relative Kawamata–Viehweg vanishing theorem). Let X be a normal projective variety and let B be an effective \mathbb{Q} -divisor on X such that $K_X + B$ is \mathbb{Q} -Cartier. Let $f : Y \rightarrow X$ be a projective resolution such that $K_Y + B_Y = f^*(K_X + B)$ and that $\text{Supp} B_Y$ is simple normal crossing. Then

$$R^i f_* \mathcal{O}_Y(-\lfloor B_Y \rfloor) = 0$$

for every $i > 0$.

It is obvious that 3.1 is a special case of 3.2. It is a routine exercise to prove 3.3 by 3.1. We note that 3.2 can be obtained as a consequence of 3.1 and 3.3 by Hironaka’s resolution theorem and Leray’s spectral sequence. In this paper, we see the Nadel vanishing theorem 3.2 (resp. the relative Kawamata–Viehweg vanishing theorem 3.3) as a special case of Kollár’s vanishing theorem 3.4 (ii) (resp. Kollár’s torsion-free theorem 3.4 (i)).

Let us recall Kollár’s theorems (cf. [Ko4, 10.15 Corollary]).

3.4 (Kollár’s torsion-free and vanishing theorems). Let Y be a smooth projective variety and let Δ be an effective \mathbb{Q} -divisor on Y such that $\text{Supp} \Delta$ is simple normal crossing and $\lfloor \Delta \rfloor = 0$. Let $f : Y \rightarrow X$ be a surjective morphism onto a projective variety X and let D be a Cartier divisor on Y .

- (i) If $D - (K_Y + \Delta) \sim_{\mathbb{Q}} f^* M$ for some \mathbb{Q} -Cartier \mathbb{Q} -divisor M on X , then $R^i f_* \mathcal{O}_Y(D)$ is torsion-free for every $i \geq 0$. In particular, $R^i f_* \mathcal{O}_Y(D) = 0$ for every $i > 0$ if f is birational.
- (ii) If $D - (K_Y + \Delta) \sim_{\mathbb{Q}} f^* M$, where M is an ample \mathbb{Q} -divisor on X , then

$$H^i(X, R^j f_* \mathcal{O}_Y(D)) = 0$$

for every $i > 0$ and $j \geq 0$.

We will completely generalize it in Theorem 6.3. As we stated above, in this paper, 3.2 is not seen as a combination of 3.1 and 3.3. It should be recognized as a special case of Kollár’s vanishing theorem 3.4 (ii). We do not see the vanishing theorem 3.3 as a relative vanishing theorem but as a special case of Kollár’s torsion-free theorem 3.4 (i). This change of viewpoint opens the door to the study of log canonical pairs.

3.5 (Philosophy). We note that 3.4 follows from the theory of pure Hodge structures. In our philosophy, we have the following correspondences.

$$\boxed{\text{Kawamata log terminal pairs}} \iff \boxed{\text{Pure Hodge structures}}$$

and

$$\boxed{\text{Log canonical pairs}} \iff \boxed{\text{Mixed Hodge structures}}$$

Therefore, it is very natural to prove a “mixed” version of 3.4 for the study of log canonical pairs. We will carry it out in Sections 5 and 6. There is a big difference between our framework discussed in this paper (cf. Sections 12, 13, and 15) and the traditional X-method from the Hodge theoretic viewpoint. We believe that all the results for klt pairs can be proved without using the theory of mixed Hodge structures (cf. [F14]).

3.6 (Further discussions). When we consider various extension theorems, which play crucial roles in the proof of the existence of pl flips (cf. [HM]), we think that the following correspondence is natural.

$$\boxed{\text{Kawamata log terminal pairs}} \iff \boxed{L^2\text{-method}}$$

The extension theorem in [HM] can be proved as a consequence of the usual vanishing theorems. However, we note that the origin of the extension theorem is the Ohsawa–Takegoshi L^2 extension theorem. The Nadel vanishing theorem also has its origin in the L^2 -method. It is very natural to try to generalize the above correspondence for log canonical pairs. However, we do not know what should be in the right box in the correspondence below.

$$\boxed{\text{Log canonical pairs}} \iff \boxed{?}$$

It is very desirable to fill the right box correctly. Here, we do not discuss this topic any more.

§4. Preliminaries

We work over the complex number field \mathbb{C} throughout this paper. But we note that by using the Lefschetz principle, we can extend almost everything to the case where the base field is an algebraically closed field of characteristic zero. In this paper, an *algebraic scheme* denotes a scheme which is separated and of finite type over \mathbb{C} . We collect the basic notation and definitions.

4.1 ($m \gg 0$). The expression ‘... for $m \gg 0$ ’ means that ‘there exists a positive number m_0 such that ... for every $m \geq m_0$.’

4.2 (Operations on \mathbb{R} -divisors). For an \mathbb{R} -Weil divisor $D = \sum_{j=1}^r d_j D_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\lceil D \rceil = \sum_{j=1}^r \lceil d_j \rceil D_j$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{j=1}^r \lfloor d_j \rfloor D_j$), where for every real number x , $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We define

$$D^{\leq 1} = \sum_{d_j=1} D_j, \quad D^{\leq 1} = \sum_{d_j \leq 1} d_j D_j,$$

$$D^{< 1} = \sum_{d_j < 1} d_j D_j, \quad \text{and} \quad D^{> 1} = \sum_{d_j > 1} d_j D_j.$$

We call D a *boundary* \mathbb{R} -divisor if $0 \leq d_j \leq 1$ for every j . We note that $\sim_{\mathbb{Q}}$ (resp. $\sim_{\mathbb{R}}$) denotes the \mathbb{Q} -linear (resp. \mathbb{R} -linear) equivalence of \mathbb{Q} -divisors (resp. \mathbb{R} -divisors). Let D_1 and D_2 be \mathbb{R} -Cartier \mathbb{R} -divisors on X and let $f : X \rightarrow Y$ be a morphism. We say that D_1 and D_2 are *\mathbb{R} -linearly f -equivalent*, denoted by $D_1 \sim_{\mathbb{R},f} D_2$, if and only if there is an \mathbb{R} -Cartier \mathbb{R} -divisor B on Y such that $D_1 \sim_{\mathbb{R}} D_2 + f^*B$. We can define $D_1 \sim_{\mathbb{Q},f} D_2$ for \mathbb{Q} -Cartier \mathbb{Q} -divisors D_1 and D_2 similarly.

Definition 4.3 (Exceptional locus). For a proper birational morphism $f : X \rightarrow Y$, the *exceptional locus* $\text{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

4.4 (Discrepancy, singularities of pairs, etc.). Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a resolution such that $\text{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y . We write

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i$$

and $a(E_i, X, B) = a_i$. We say that (X, B) is *lc* (resp. *klt*) if and only if $a_i \geq -1$ (resp. $a_i > -1$) for every i . Note that the *discrepancy* $a(E, X, B) \in \mathbb{R}$ can be defined for every prime divisor E over X . If $a(E, X, B) > -1$ for every exceptional divisor E over X , then the pair (X, B) is called *plt*. Here, *lc* (resp. *klt*, *plt*) is an abbreviation of *log canonical* (resp. *Kawamata log terminal*, *purely log terminal*). By the definition, there exists the largest Zariski open set U (resp. U') of X such that (X, B) is *lc* (resp. *klt*) on U (resp. U'). We put $\text{Nlc}(X, B) = X \setminus U$ (resp. $\text{Nklt}(X, B) = X \setminus U'$) and call it the *non-lc locus* (resp. *non-klt locus*) of the pair (X, B) . We sometimes simply denote $\text{Nlc}(X, B)$ by X_{NLC} .

Let (X, B) be a log canonical pair and let M be an effective \mathbb{R} -Cartier \mathbb{R} -divisor on X . The *log canonical threshold* of (X, B) with respect to M is defined by

$$c = \sup\{t \in \mathbb{R} \mid (X, B + tM) \text{ is log canonical}\}.$$

Definition 4.5 (Center). Let E be a prime divisor over X . The closure of the image of E on X is denoted by $c_X(E)$ and called the *center* of E on X .

Definition 4.6 (Lc center). Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. If $a(E, X, B) = -1$ and $c_X(E)$ is not contained in $\text{Nlc}(X, B)$, then $c_X(E)$ is called an *lc center* of (X, B) . It is obvious that there are at most finitely many lc centers.

We note that our definition of lc centers is slightly different from the usual one. For details, see [FST, Section 3].

Definition 4.7 (Stratum). Let (X, B) be a log canonical pair. A *stratum* of (X, B) denotes X itself or an lc center of (X, B) .

Let T be a simple normal crossing divisor on a smooth variety Y . A *stratum* of T denotes a stratum of the pair (Y, T) contained in T .

4.8 (Kleiman–Mori cone). Let X be an algebraic scheme over \mathbb{C} and let $\pi : X \rightarrow S$ be a proper morphism to an algebraic scheme S . Let $\text{Pic}(X)$ be the group of line bundles on X . Take a complete curve on X which is mapped to a point by π . For $\mathcal{L} \in \text{Pic}(X)$, we define the intersection number $\mathcal{L} \cdot C = \deg_{\overline{C}} f^* \mathcal{L}$, where $f : \overline{C} \rightarrow C$ is the normalization of C . Via this intersection pairing, we introduce a bilinear form

$$\cdot : \text{Pic}(X) \times Z_1(X/S) \rightarrow \mathbb{Z},$$

where $Z_1(X/S)$ is the free abelian group generated by integral curves which are mapped to points on S by π .

Now we have the notion of numerical equivalence both in $Z_1(X/S)$ and in $\text{Pic}(X)$, which is denoted by \equiv , and we obtain a perfect pairing

$$N^1(X/S) \times N_1(X/S) \rightarrow \mathbb{R},$$

where

$$N^1(X/S) = \{\text{Pic}(X)/\equiv\} \otimes \mathbb{R} \quad \text{and} \quad N_1(X/S) = \{Z_1(X/S)/\equiv\} \otimes \mathbb{R},$$

namely $N^1(X/S)$ and $N_1(X/S)$ are dual to each other through this intersection pairing. It is well known that

$$\dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S) < \infty.$$

We write $\rho(X/S) = \dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S)$. We define the Kleiman–Mori cone $\overline{NE}(X/S)$ as the closed convex cone in $N_1(X/S)$ generated by integral curves on X which are mapped to points on S by π . When $S = \text{Spec}\mathbb{C}$, we drop $/\text{Spec}\mathbb{C}$ from the notation, e.g., we simply write $N_1(X)$ instead of $N_1(X/\text{Spec}\mathbb{C})$.

Definition 4.9. An element $D \in N^1(X/S)$ is called π -nef (or *relatively nef* for π), if $D \geq 0$ on $\overline{NE}(X/S)$. When $S = \text{Spec}\mathbb{C}$, we simply say that D is *nef*.

Theorem 4.10 (Kleiman’s criterion for ampleness). *Let $\pi : X \rightarrow S$ be a projective morphism between algebraic schemes. Then $\mathcal{L} \in \text{Pic}(X)$ is π -ample if and only if the numerical class of \mathcal{L} in $N^1(X/S)$ gives a positive function on $\overline{NE}(X/S) \setminus \{0\}$.*

In Theorem 4.10, we note that the projectivity of π is indispensable (cf. [F4]).

Definition 4.11 (Semi-ample \mathbb{R} -divisors). An \mathbb{R} -Cartier \mathbb{R} -divisor D on X is π -semi-ample if $D \sim_{\mathbb{R}} \sum_i a_i D_i$, where D_i is a π -semi-ample Cartier divisor on X and a_i is a positive real number for every i .

Remark 4.12. In Definition 4.11, we can replace $D \sim_{\mathbb{R}} \sum_i a_i D_i$ with $D = \sum_i a_i D_i$ since every principal Cartier divisor on X is π -semi-ample.

The following two lemmas seem to be missing in the literature.

Lemma 4.13. *Let D be an \mathbb{R} -Cartier \mathbb{R} -divisor on X . Then the following conditions are equivalent.*

- (1) D is π -semi-ample.
- (2) *There exists a morphism $f : X \rightarrow Y$ over S such that $D \sim_{\mathbb{R}} f^*A$, where A is an \mathbb{R} -Cartier \mathbb{R} -divisor on Y which is ample over S .*

Proof. It is obvious that (1) follows from (2). If D is π -semi-ample, then we can write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition 4.11. By replacing D_i with its multiple, we can assume that $\pi^* \pi_* \mathcal{O}_X(D_i) \rightarrow \mathcal{O}_X(D_i)$ is surjective for every i . Let $f : X \rightarrow Y$ be a morphism over S obtained by the surjection $\pi^* \pi_* \mathcal{O}_X(\sum_i D_i) \rightarrow \mathcal{O}_X(\sum_i D_i)$. Then it is easy to see that $f : Y \rightarrow X$ has the desired property. \square

Lemma 4.14. *Let D be a Cartier divisor on X . If D is π -semi-ample in the sense of Definition 4.11, then D is π -semi-ample in the usual sense, that is, $\pi^* \pi_* \mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD)$ is surjective for some positive integer m . In particular, Definition 4.11 is well-defined.*

Proof. We write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition 4.11. Let $f : X \rightarrow Y$ be a morphism in Lemma 4.13 (2). By taking the Stein factorization, we can assume that

f has connected fibers. By the construction, $D_i \sim_{\mathbb{Q},f} 0$ for every i . By replacing D_i with its multiple, we can assume that $D_i \sim f^*D'_i$ for some Cartier divisor D'_i on Y for every i . Let U be any Zariski open set of Y on which $D'_i \sim 0$ for every i . On $f^{-1}(U)$, we have $D \sim_{\mathbb{R}} 0$. This implies $D \sim_{\mathbb{Q}} 0$ on $f^{-1}(U)$ since D is Cartier. Therefore, there exists a positive integer m such that $f^*f_*\mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD)$ is surjective. By this surjection, we have $mD \sim f^*A$ for a Cartier divisor A on Y which is ample over S . This means that D is π -semi-ample in the usual sense. \square

We will repeatedly use the following easy lemma. We give a detailed proof for the reader's convenience.

Lemma 4.15. *Let X be a normal variety and let B be an effective \mathbb{R} -Cartier \mathbb{R} -divisor on X such that $\lfloor B \rfloor = 0$. Let A be a Cartier divisor on X . Assume that $A \sim_{\mathbb{R}} B$. Then there exists a \mathbb{Q} -Cartier \mathbb{Q} -divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\lfloor C \rfloor = 0$, and $\text{Supp}C = \text{Supp}B$.*

Proof. We can write $B = A + \sum_{i=1}^k r_i(f_i)$, where $r_i \in \mathbb{R}$ and f_i is a rational function on X for every i . We put

$$E = \text{Supp}A \cup \text{Supp}B \cup \bigcup_{i=1}^k \text{Supp}(f_i).$$

Let $E = \sum_{j=1}^n E_j$ be the irreducible decomposition of E . We can write

$$A = \sum_j a_j E_j, \quad B = \sum_j b_j E_j,$$

and

$$(f_i) = \sum_j m_{ij} E_j \quad \text{for every } i.$$

We can assume that $b_j \in \mathbb{Q}$ for $1 \leq j \leq l$ and $b_j \notin \mathbb{Q}$ for $l+1 \leq j \leq n$. We note that $a_j \in \mathbb{Z}$ for every j and that $m_{ij} \in \mathbb{Z}$ for every i, j . We define

$$\mathcal{S} = \left\{ (v_1, \dots, v_k) \in \mathbb{R}^k ; b_j = a_j + \sum_{i=1}^k v_i m_{ij} \text{ for } 1 \leq j \leq l \right\}.$$

Then \mathcal{S} is an affine subspace of \mathbb{R}^k defined over \mathbb{Q} . We note that \mathcal{S} is not empty since $(r_1, \dots, r_k) \in \mathcal{S}$. If we take $(r'_1, \dots, r'_k) \in \mathcal{S} \cap \mathbb{Q}^k$ which is very close to (r_1, \dots, r_k) and put $C = A + \sum_i r'_i(f_i)$, then it is obvious that C satisfies the desired properties. \square

The next lemma is well known as the negativity lemma.

Lemma 4.16 (Negativity lemma). *Let $h : Z \rightarrow Y$ be a proper birational morphism between normal varieties. Let $-B$ be an h -nef \mathbb{R} -Cartier \mathbb{R} -divisor on Z . Then we have the following statements.*

- (1) B is effective if and only if h_*B is.
- (2) Assume that B is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset \text{Supp}B$ or $h^{-1}(y) \cap \text{Supp}B = \emptyset$.

Lemma 4.16 is essentially an application of the Hodge index theorem for smooth projective surfaces. For the proof, see, for example, [KM, Lemma 3.39].

We close this section with the following useful lemma. It is a consequence of Szabó's resolution lemma.

Lemma 4.17. *Let Z be a smooth variety and let B be an \mathbb{R} -divisor on Z such that $\text{Supp}B$ is simple normal crossing. Let $f : Z \rightarrow X$ be a projective morphism and let \bar{X} be a projective variety such that \bar{X} contains X as a Zariski open set. Then there exist a smooth projective variety \bar{Z} and an \mathbb{R} -divisor \bar{B} on \bar{Z} such that*

- (i) $f : Z \rightarrow X$ is extended to $\bar{f} : \bar{Z} \rightarrow \bar{X}$.
- (ii) $\text{Supp}\bar{B}$ is simple normal crossing.
- (iii) $\text{Supp}\bar{B} \cup \text{Supp}(\bar{Z} \setminus Z)$ is simple normal crossing.
- (iv) $\bar{B}|_Z = B$.

Proof. Let Z' be an arbitrary compactification of Z . By taking the graph of $f : Z' \dashrightarrow \bar{X}$ and using Hironaka's resolution, we can assume that Z' is smooth projective, $\text{Supp}(Z' \setminus Z)$ is simple normal crossing, and $f : Z \rightarrow X$ is extended to $f' : Z' \rightarrow \bar{X}$. Let B' be the closure of B on Z' . We apply Szabó's resolution lemma (see, for example, [F6]) to $\text{Supp}B' \cup \text{Supp}(Z' \setminus Z)$. Then we obtain the desired variety \bar{Z} and \bar{B} . By the above construction, f can be extended to $\bar{f} : \bar{Z} \rightarrow \bar{X}$. \square

§5. Hodge theoretic injectivity theorem

In this section, we will prove the following injectivity theorem, which is a generalization of [EV, 5.1. b)] for \mathbb{R} -divisors. We use the classical topology throughout this section.

Proposition 5.1 (Fundamental injectivity theorem). *Let X be a smooth projective variety and let $S + B$ be a boundary \mathbb{R} -divisor on X such that the support of $S + B$ is simple normal crossing and $\lrcorner S + B \lrcorner = S$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in $\text{Supp}B$.*

Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \rightarrow H^q(X, \mathcal{O}_X(L + D))$$

which are induced by the natural inclusion $\mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ are injective for all q .

Let us recall some results on the theory of mixed Hodge structures.

5.2 (Mixed Hodge structures). Let V be a smooth projective variety and Σ a simple normal crossing divisor on V . Let $\iota : V \setminus \Sigma \rightarrow V$ be the natural open immersion. Then $\iota_! \mathbb{C}_{V \setminus \Sigma}$ is quasi-isomorphic to the complex $\Omega_V^\bullet(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)$. By this quasi-isomorphism, we can construct the following spectral sequence

$$E_1^{p,q} = H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)) \Rightarrow H_c^{p+q}(V \setminus \Sigma, \mathbb{C}).$$

By the Serre duality, the right hand side

$$H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma))$$

is dual to

$$H^{n-q}(V, \Omega_V^{n-p}(\log \Sigma)),$$

where $n = \dim V$. By the Poincaré duality, $H_c^{p+q}(V \setminus \Sigma, \mathbb{C})$ is dual to $H^{2n-(p+q)}(V \setminus \Sigma, \mathbb{C})$. Therefore,

$$\dim H_c^k(V \setminus \Sigma, \mathbb{C}) = \sum_{p+q=k} \dim H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma))$$

by Deligne (cf. [D, Corollaire (3.2.13) (ii)]). Thus, the above spectral sequence degenerates at E_1 . We will use this E_1 -degeneration in the proof of Proposition 5.1. By the above E_1 -degeneration, we obtain

$$H_c^k(V \setminus \Sigma, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)).$$

In particular, the natural inclusion $\iota_! \mathbb{C}_{V \setminus \Sigma} \subset \mathcal{O}_V(-\Sigma)$ induces surjections

$$H_c^p(V \setminus \Sigma, \mathbb{C}) \simeq H^p(V, \iota_! \mathbb{C}_{V \setminus \Sigma}) \rightarrow H^p(V, \mathcal{O}_V(-\Sigma))$$

for all p .

Proof of Proposition 5.1. By Lemma 4.15, we can assume that B is a \mathbb{Q} -divisor and that $L \sim_{\mathbb{Q}} K_X + S + B$. We put $\mathcal{L} = \mathcal{O}_X(L - K_X - S)$. Let ν be the smallest positive integer such that $\nu L \sim \nu(K_X + S + B)$. In particular, νB is an integral Weil divisor. We take the ν -fold cyclic cover $\pi' : Y' = \text{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i} \rightarrow X$ associated to the section $\nu B \in |\mathcal{L}^\nu|$. More precisely, let $s \in H^0(X, \mathcal{L}^\nu)$ be a section

whose zero divisor is νB . Then the dual of $s : \mathcal{O}_X \rightarrow \mathcal{L}^\nu$ defines an \mathcal{O}_X -algebra structure on $\bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i}$. Let $Y \rightarrow Y'$ be the normalization and let $\pi : Y \rightarrow X$ be the composition morphism. For the details, see [EV, 3.5. Cyclic covers]. We can take a finite cover $\varphi : V \rightarrow Y$ such that V is smooth and that T is a simple normal crossing divisor on V , where $\psi = \pi \circ \varphi$ and $T = \psi^* S$, by Kawamata's covering trick (cf. [EV, 3.17. Lemma]). Let $\iota' : Y \setminus \pi^* S \rightarrow Y$ be the natural open immersion and let U be the smooth locus of Y . We denote the natural open immersion $U \rightarrow Y$ by j . We put $\tilde{\Omega}_Y^p(\log(\pi^* S)) = j_* \Omega_U^p(\log(\pi^* S))$ for every p . Then it can be checked easily that

$$\iota'_! \mathbb{C}_{Y \setminus \pi^* S} \xrightarrow{qis} \tilde{\Omega}_Y^\bullet(\log(\pi^* S)) \otimes \mathcal{O}_Y(-\pi^* S)$$

is a direct summand of

$$\varphi_*(\iota' \mathbb{C}_{V \setminus T}) \xrightarrow{qis} \varphi_*(\Omega_V^\bullet(\log T) \otimes \mathcal{O}_V(-T)),$$

where qis means a quasi-isomorphism. On the other hand, we can decompose $\pi_*(\tilde{\Omega}_Y^\bullet(\log(\pi^* S)) \otimes \mathcal{O}_Y(-\pi^* S))$ and $\pi_*(\iota'_! \mathbb{C}_{Y \setminus \pi^* S})$ into eigen components of the Galois action of $\pi : Y \rightarrow X$. We write these decompositions as follows,

$$\pi_*(\iota'_! \mathbb{C}_{Y \setminus \pi^* S}) = \bigoplus_{i=0}^{\nu-1} \mathcal{C}_i \subset \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i}(\lfloor i B \rfloor - S) = \pi_* \mathcal{O}_Y(-\pi^* S),$$

where $\mathcal{C}_i \subset \mathcal{L}^{-i}(\lfloor i B \rfloor - S)$ for every i . We put $\mathcal{C} = \mathcal{C}_1$. We have that

$$\mathcal{C} \xrightarrow{qis} \Omega_X^\bullet(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$$

is a direct summand of

$$\psi_*(\iota' \mathbb{C}_{V \setminus T}) \xrightarrow{qis} \psi_*(\Omega_V^\bullet(\log T) \otimes \mathcal{O}_V(-T)).$$

The E_1 -degeneration of the spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(V, \Omega_V^p(\log T) \otimes \mathcal{O}_V(-T)) \\ &\Rightarrow \mathbb{H}^{p+q}(V, \Omega_V^\bullet(\log T) \otimes \mathcal{O}_V(-T)) \simeq H^{p+q}(V, \iota' \mathbb{C}_{V \setminus T}) \end{aligned}$$

(cf. 5.2) implies the E_1 -degeneration of

$$\begin{aligned} E_1^{pq} &= H^q(X, \Omega_X^p(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \simeq H^{p+q}(X, \mathcal{C}) \end{aligned}$$

Therefore, the inclusion $\mathcal{C} \subset \mathcal{L}^{-1}(-S)$ induces surjections

$$H^p(X, \mathcal{C}) \rightarrow H^p(X, \mathcal{L}^{-1}(-S))$$

for all p . We can check the following simple property by seeing the monodromy action of the Galois group of $\pi : Y \rightarrow X$ on \mathcal{C} around $\text{Supp} B$.

Corollary 5.3 (cf. [KM, Corollary 2.54]). *Let $U \subset X$ be a connected open set such that $U \cap \text{Supp}B \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

This property is utilized via the following fact. The proof is obvious.

Lemma 5.4 (cf. [KM, Lemma 2.55]). *Let F be a sheaf of Abelian groups on a topological space X and let $F_1, F_2 \subset F$ be subsheaves. Let $Z \subset X$ be a closed subset. Assume that*

- (1) $F_2|_{X \setminus Z} = F|_{X \setminus Z}$, and
- (2) if U is connected, open and $U \cap Z \neq \emptyset$, then $H^0(U, F_1|_U) = 0$.

Then F_1 is a subsheaf of F_2 .

As a corollary, we obtain:

Corollary 5.5 (cf. [KM, Corollary 2.56]). *Let $M \subset \mathcal{L}^{-1}(-S)$ be a subsheaf such that $M|_{X \setminus \text{Supp}B} = \mathcal{L}^{-1}(-S)|_{X \setminus \text{Supp}B}$. Then the injection*

$$\mathcal{C} \rightarrow \mathcal{L}^{-1}(-S)$$

factors as

$$\mathcal{C} \rightarrow M \rightarrow \mathcal{L}^{-1}(-S).$$

Therefore,

$$H^i(X, M) \rightarrow H^i(X, \mathcal{L}^{-1}(-S))$$

is surjective for every i .

Proof. The first part is clear from Corollary 5.3 and Lemma 5.4. This implies that we have maps

$$H^i(X, \mathcal{C}) \rightarrow H^i(X, M) \rightarrow H^i(X, \mathcal{L}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right. \square

Therefore, we obtain that

$$H^q(X, \mathcal{L}^{-1}(-S - D)) \rightarrow H^q(X, \mathcal{L}^{-1}(-S))$$

is surjective for every q . By the Serre duality, we obtain

$$H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S)) \rightarrow H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S + D))$$

is injective for every q . This means that

$$H^q(X, \mathcal{O}_X(L)) \rightarrow H^q(X, \mathcal{O}_X(L + D))$$

is injective for every q . \square

§6. Injectivity, torsion-free, and vanishing theorems

In this section, we prove generalizations of Kollár's torsion-freeness and vanishing theorem (cf. Theorem 6.3). First, we prove a generalization of Kollár's injectivity theorem (cf. [A1, Theorem 3.1]). It is a straightforward consequence of Proposition 5.1 and will produce the desired torsion-free and vanishing theorems.

Theorem 6.1 (Injectivity theorem). *Let X be a smooth projective variety and let Δ be a boundary \mathbb{R} -divisor such that $\text{Supp}\Delta$ is simple normal crossing. Let L be a Cartier divisor on X and let D be an effective Cartier divisor that contains no lc centers of (X, Δ) . Assume the following conditions.*

- (i) $L \sim_{\mathbb{R}} K_X + \Delta + H$,
- (ii) H is a semi-ample \mathbb{R} -Cartier \mathbb{R} -divisor, and
- (iii) $tH \sim_{\mathbb{R}} D + D'$ for some positive real number t , where D' is an effective \mathbb{R} -Cartier \mathbb{R} -divisor whose support contains no lc centers of (X, Δ) .

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \rightarrow H^q(X, \mathcal{O}_X(L + D))$$

which are induced by the natural inclusion $\mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ are injective for all q .

Proof. We put $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. We can take a resolution $f : Y \rightarrow X$ such that f is an isomorphism outside $\text{Supp}(D + D' + B)$, and that the union of the support of $f^*(S + B + D + D')$ and the exceptional locus of f has a simple normal crossing support on Y . Let B' be the strict transform of B on Y . We write

$$K_Y + S' + B' = f^*(K_X + S + B) + E,$$

where S' is the strict transform of S and E is f -exceptional. It is easy to see that $E_+ = \lceil E \rceil \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is an effective \mathbb{R} -Cartier \mathbb{R} -divisor with $\lfloor E_- \rfloor = 0$. Since f^*H is semi-ample, we can write $f^*H \sim_{\mathbb{R}} \sum_i a_i H'_i$, where $0 < a_i < 1$ and H'_i is a general Cartier divisor on Y for every i . We put

$$B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D + D') + (1 - \varepsilon) \sum_i a_i H'_i$$

for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{R}} K_Y + S' + B''$. By the construction, $\lfloor B'' \rfloor = 0$, the support of $S' + B''$ is simple normal crossing on Y , and $\text{Supp}B'' \supset \text{Supp}f^*D$.

So, Proposition 5.1 implies that the homomorphisms

$$H^q(Y, \mathcal{O}_Y(L')) \rightarrow H^q(Y, \mathcal{O}_Y(L' + f^*D))$$

are injective for all q . It is easy to see that $f_*\mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By Lemma 4.15, we can write $L' \sim_{\mathbb{Q}} K_Y + S' + B'''$, where B''' is a \mathbb{Q} -divisor on Y such that $\lfloor B''' \rfloor = 0$ and $\text{Supp} B''' = \text{Supp} B''$. Thus, by Lemma 6.2 below, $R^q f_*\mathcal{O}_Y(L') = 0$ for all $q > 0$. By the Leray spectral sequence, the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \rightarrow H^q(X, \mathcal{O}_X(L + D))$$

are injective for all q . □

Let us recall the following well-known easy lemma.

Lemma 6.2 (Reid–Fukuda type). *Let V be a smooth projective variety and let B be a boundary \mathbb{Q} -divisor on V such that $\text{Supp} B$ is simple normal crossing. Let $f : V \rightarrow W$ be a projective birational morphism onto a variety W . Assume that f is an isomorphism at the generic point of every lc center of (V, B) and that D is a Cartier divisor on V such that $D - (K_V + B)$ is nef. Then $R^i f_*\mathcal{O}_V(D) = 0$ for every $i > 0$.*

Proof. We use the induction on the number of irreducible components of $\lfloor B \rfloor$ and on the dimension of V . If $\lfloor B \rfloor = 0$, then the lemma follows from the Kawamata–Viehweg vanishing theorem (cf. [KM, Corollary 2.68]). Therefore, we can assume that there is an irreducible divisor $S \subset \lfloor B \rfloor$. We consider the following short exact sequence

$$0 \rightarrow \mathcal{O}_V(D - S) \rightarrow \mathcal{O}_V(D) \rightarrow \mathcal{O}_S(D) \rightarrow 0.$$

By induction, we see that $R^i f_*\mathcal{O}_V(D - S) = 0$ and $R^i f_*\mathcal{O}_S(D) = 0$ for every $i > 0$. Thus, we have $R^i f_*\mathcal{O}_V(D) = 0$ for $i > 0$. □

The next theorem is the main theorem of this section (cf. [A1]). See also [F3].

Theorem 6.3 (Torsion-freeness and vanishing theorem). *Let Y be a smooth variety and let B be a boundary \mathbb{R} -divisor such that $\text{Supp} B$ is simple normal crossing. Let $f : Y \rightarrow X$ be a projective morphism and let L be a Cartier divisor on Y such that $L - (K_Y + B)$ is f -semi-ample.*

- (i) *Let q be an arbitrary non-negative integer. Every non-zero local section of $R^q f_*\mathcal{O}_Y(L)$ contains in its support the f -image of some stratum of (Y, B) .*
- (ii) *Let $\pi : X \rightarrow S$ be a projective morphism. Assume that $L - (K_X + B) \sim_{\mathbb{R}} f^*H$ for some π -ample \mathbb{R} -Cartier \mathbb{R} -divisor H on X . Then $R^p \pi_* R^q f_*\mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.*

Remark 6.4. It is obvious that the statement of Theorem 6.3 (i) is equivalent to the following one.

(i') Let q be an arbitrary non-negative integer. Then every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f -image of some stratum of (Y, B) .

Let us start the proof of Theorem 6.3.

Proof of Theorem 6.3. We take an f -semi-ample \mathbb{R} -Cartier \mathbb{R} -divisor M on Y such that $M \sim_{\mathbb{R}} L - (K_Y + B)$.

(i) We divide the proof into two steps.

Step 1. First, we assume that X is projective. We can assume that M is semi-ample by replacing L (resp. M) with $L + f^* A'$ (resp. $M + f^* A'$), where A' is a very ample Cartier divisor on X . Assume that $R^q f_* \mathcal{O}_Y(L)$ has a local section whose support does not contain the images of any (Y, B) -strata. More precisely, let U be a non-empty Zariski open set and let $s \in \Gamma(U, R^q f_* \mathcal{O}_Y(L))$ be a non-zero section of $R^q f_* \mathcal{O}_Y(L)$ on U whose support $V \subset U$ does not contain the f -images of any strata of (Y, B) . Let \bar{V} be the closure of V in X . We note that $\bar{V} \setminus V$ may contain the f -image of some stratum of (Y, B) . By replacing Y with its blow-up along an lc center which is mapped into $\bar{V} \setminus V$, we can assume that an irreducible component B_0 of $\perp B \perp$ is mapped into $\bar{V} \setminus V$ by f . We note that $M \sim_{\mathbb{R}} L - B_0 - (K_X + B - B_0)$. We replace L (resp. B) with $L - B_0$ (resp. $B - B_0$). By repeating this process finitely many times, we can assume that \bar{V} does not contain the f -images of any strata of (Y, B) . Then we can find a very ample Cartier divisor A with the following properties.

- (a) $f^* A$ contains no lc centers of (Y, B) , and
- (b) $R^q f_* \mathcal{O}_Y(L) \rightarrow R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We can assume that $M - f^* A$ is semi-ample by replacing L (resp. M) with $L + f^* A$ (resp. $M + f^* A$). If necessary, we replace L (resp. M) with $L + f^* A''$ (resp. $M + f^* A''$), where A'' is a very ample Cartier divisor on X . Then, we have

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \simeq H^q(Y, \mathcal{O}_Y(L))$$

and

$$H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y, \mathcal{O}_Y(L + f^* A)).$$

We see that

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \rightarrow H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A))$$

is not injective by (b) if A'' is sufficiently ample. So,

$$H^q(Y, \mathcal{O}_Y(L)) \rightarrow H^q(Y, \mathcal{O}_Y(L + f^* A))$$

is not injective. It contradicts Theorem 6.1. We finish the proof when X is projective.

Step 2. Next, we assume that X is not projective. Note that the problem is local. So, we can shrink X and assume that X is affine. By the argument similar to the one in Step 1 in the proof of (ii) below, we can assume that M is a semi-ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. We compactify X and apply Lemma 4.17. Then we obtain a compactification $\bar{f} : \bar{Y} \rightarrow \bar{X}$ of $f : Y \rightarrow X$. Let \bar{M} be the closure of M on \bar{Y} . If \bar{M} is not a semi-ample \mathbb{Q} -Cartier \mathbb{Q} -divisor, then we take blowing-ups of \bar{Y} inside $\bar{Y} \setminus Y$ and obtain a semi-ample \mathbb{Q} -Cartier \mathbb{Q} -divisor \bar{M} on \bar{Y} such that $\bar{M}|_Y = M$. Let \bar{L} (resp. \bar{B}) be the closure of L (resp. B) on \bar{Y} . We note that $\bar{M} \sim_{\mathbb{R}} \bar{L} - (K_{\bar{Y}} + \bar{B})$ does not necessarily hold. We can write $M + \sum_i a_i(f_i) = L - (K_Y + B)$, where a_i is a real number and f_i is a rational function on Y for every i . We put

$$E = \bar{M} + \sum_i a_i(f_i) - (\bar{L} - (K_{\bar{Y}} + \bar{B})).$$

We replace \bar{L} (resp. \bar{B}) with $\bar{L} + {}^\square E^\square$ (resp. $\bar{B} + \{-E\}$). Then we obtain the desired property of $R^q \bar{f}_* \mathcal{O}_{\bar{Y}}(\bar{L})$ since \bar{X} is projective. We note that $\text{Supp } E$ is in $\bar{Y} \setminus Y$. So, this completes the whole proof.

(ii) We divide the proof into three steps.

Step 1. We assume that $\dim S = 0$. The following arguments are well known and standard. We describe them for the reader's convenience. In this case, we can write $H \sim_{\mathbb{R}} H_1 + H_2$, where H_1 (resp. H_2) is a π -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor (resp. π -ample \mathbb{R} -Cartier \mathbb{R} -divisor) on X . So, we can write $H_2 \sim_{\mathbb{R}} \sum_i a_i H'_i$, where $0 < a_i < 1$ and H'_i is a general very ample Cartier divisor on X for every i . Replacing B (resp. H) with $B + \sum_i a_i f^* H'_i$ (resp. H_1), we can assume that H is a π -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. We take a general member $A \in |mH|$, where m is a sufficiently divisible positive integer, such that $A' = f^* A$ and $R^q f_* \mathcal{O}_Y(L + A')$ is π_* -acyclic for all q . By (i), we have the following short exact sequences,

$$0 \rightarrow R^q f_* \mathcal{O}_Y(L) \rightarrow R^q f_* \mathcal{O}_Y(L + A') \rightarrow R^q f_* \mathcal{O}_{A'}(L + A') \rightarrow 0.$$

for all q . Note that $R^q f_* \mathcal{O}_{A'}(L + A')$ is π_* -acyclic by induction on $\dim X$ and $R^q f_* \mathcal{O}_Y(L + A')$ is also π_* -acyclic by the above assumption. Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{array}{ccc} E_2^{pq} = R^p \pi_* R^q f_* \mathcal{O}_Y(L) & \Longrightarrow & R^{p+q}(\pi \circ f)_* \mathcal{O}_Y(L) \\ \varphi^{pq} \downarrow & & \varphi^{p+q} \downarrow \\ \bar{E}_2^{pq} = R^p \pi_* R^q f_* \mathcal{O}_Y(L + A') & \Longrightarrow & R^{p+q}(\pi \circ f)_* \mathcal{O}_Y(L + A') \end{array}$$

We note that φ^{1+q} is injective by Theorem 6.1. We have $E_2^{1q} \rightarrow R^{1+q}(\pi \circ f)_* \mathcal{O}_Y(L)$ is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$. We also have that $\overline{E}_2^{1q} = 0$ by the above assumption. Therefore, we obtain $E_2^{1q} = 0$ since the injection $E_2^{1q} \rightarrow R^{1+q}(\pi \circ f)_* \mathcal{O}_Y(L + A')$ factors through $\overline{E}_2^{1q} = 0$. This implies that $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.

Step 2. We assume that S is projective. By replacing H (resp. L) with $H + \pi^* G$ (resp. $L + (\pi \circ f)^* G$), where G is a very ample Cartier divisor on S , we can assume that H is an ample \mathbb{R} -Cartier \mathbb{R} -divisor. By the same argument as in Step 1, we can assume that H is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor and $M \sim_{\mathbb{Q}} f^* H$. If G is a sufficiently ample Cartier divisor on S , $H^k(S, R^p \pi_* R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_S(G)) = 0$ for every $k \geq 1$,

$$\begin{aligned} & H^0(S, R^p \pi_* R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_S(G)) \\ & \simeq H^p(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(\pi^* G)) \\ & \simeq H^p(X, R^q f_* \mathcal{O}_Y(L + f^* \pi^* G)), \end{aligned}$$

and $R^p \pi_* R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_S(G)$ is generated by its global sections. Since

$$\begin{aligned} M + f^* \pi^* G & \sim_{\mathbb{R}} L + f^* \pi^* G - (K_Y + B), \\ M + f^* \pi^* G & \sim_{\mathbb{Q}} f^*(H + \pi^* G), \end{aligned}$$

and $H + \pi^* G$ is ample, we can apply Step 1 and obtain $H^p(X, R^q f_* \mathcal{O}_Y(L + f^* \pi^* G)) = 0$ for every $p > 0$ and $q \geq 0$. Thus, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$ by the above arguments.

Step 3. When S is not projective, we shrink S and assume that S is affine. By the same argument as in Step 1 above, we can assume that H is \mathbb{Q} -Cartier. We compactify S and X , and can assume that S and X are projective. By Lemma 4.17, we can reduce it to the case when S is projective. This step is essentially the same as Step 2 in the proof of (i). So, we omit the details here.

We obtained the statement (ii). \square

§7. Non-lc ideal sheaves

We introduce the notion of *non-lc ideal sheaves*. It is an analogue of the usual *multiplier ideal sheaves* (see, for example, [L, Chapter 9]). For details, see [F10] and [FST].

Definition 7.1 (Non-lc ideal sheaf). Let X be a normal variety and let B be an \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a resolution

with $K_Y + B_Y = f^*(K_X + B)$ such that $\text{Supp}B_Y$ is simple normal crossing. Then we put

$$\begin{aligned}\mathcal{J}_{NLC}(X, B) &= f_*\mathcal{O}_Y(\Gamma-(B_Y^{<1})^\Gamma - \lfloor B_Y^{>1} \rfloor) \\ &= f_*\mathcal{O}_Y(-\lfloor B_Y \rfloor + B_Y^{\bar{1}})\end{aligned}$$

and call it the *non-lc ideal sheaf associated to (X, B)* . If B is effective, then $\mathcal{J}_{NLC}(X, B) \subset \mathcal{O}_X$.

The ideal sheaf $\mathcal{J}_{NLC}(X, B)$ is well-defined by the following easy lemma.

Lemma 7.2. *Let $g : Z \rightarrow Y$ be a proper birational morphism between smooth varieties and let B_Y be an \mathbb{R} -divisor on Y such that $\text{Supp}B_Y$ is simple normal crossing. Assume that $K_Z + B_Z = g^*(K_Y + B_Y)$ and that $\text{Supp}B_Z$ is simple normal crossing. Then we have*

$$g_*\mathcal{O}_Z(\Gamma-(B_Z^{<1})^\Gamma - \lfloor B_Z^{>1} \rfloor) \simeq \mathcal{O}_Y(\Gamma-(B_Y^{<1})^\Gamma - \lfloor B_Y^{>1} \rfloor).$$

Proof. By $K_Z + B_Z = g^*(K_Y + B_Y)$, we obtain

$$\begin{aligned}K_Z &= g^*(K_Y + B_Y^{\bar{1}} + \{B_Y\}) \\ &\quad + g^*(\lfloor B_Y^{<1} \rfloor + \lfloor B_Y^{>1} \rfloor) - (\lfloor B_Z^{<1} \rfloor + \lfloor B_Z^{>1} \rfloor) - B_Z^{\bar{1}} - \{B_Z\}.\end{aligned}$$

If $a(\nu, Y, B_Y^{\bar{1}} + \{B_Y\}) = -1$ for a prime divisor ν over Y , then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $g^*(\lfloor B_Y^{<1} \rfloor + \lfloor B_Y^{>1} \rfloor) - (\lfloor B_Z^{<1} \rfloor + \lfloor B_Z^{>1} \rfloor)$ is Cartier, we can easily see that

$$g^*(\lfloor B_Y^{<1} \rfloor + \lfloor B_Y^{>1} \rfloor) = \lfloor B_Z^{<1} \rfloor + \lfloor B_Z^{>1} \rfloor + E,$$

where E is an effective f -exceptional Cartier divisor. Thus, we obtain

$$g_*\mathcal{O}_Z(\Gamma-(B_Z^{<1})^\Gamma - \lfloor B_Z^{>1} \rfloor) \simeq \mathcal{O}_Y(\Gamma-(B_Y^{<1})^\Gamma - \lfloor B_Y^{>1} \rfloor).$$

This completes the proof. \square

Remark 7.3. We use the same notation as in Definition 7.1. We put

$$\mathcal{J}(X, B) = f_*\mathcal{O}_Y(-\lfloor B_Y \rfloor).$$

This sheaf $\mathcal{J}(X, B)$ is well known as the (algebraic version of) *multiplier ideal sheaf* of the pair (X, B) . See, for example, [L, Chapter 9].

By the definition, the following proposition is obvious.

Proposition 7.4. *Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Then (X, B) is log canonical if and only if $\mathcal{J}_{NLC}(X, B) = \mathcal{O}_X$.*

The next proposition is a kind of Bertini's theorem.

Proposition 7.5. *Let X be a smooth variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let Λ be a linear system on X and let $D \in \Lambda$ be a general member of Λ . Then*

$$\mathcal{J}_{NLC}(X, B + tD) = \mathcal{J}_{NLC}(X, B)$$

outside the base locus $\text{Bs}\Lambda$ of Λ for all $0 \leq t \leq 1$.

Proof. By replacing X with $X \setminus \text{Bs}\Lambda$, we can assume that $\text{Bs}\Lambda = \emptyset$. Let $f : Y \rightarrow X$ be a resolution as in Definition 7.1. Since D is a general member of Λ , $f^*D = f_*^{-1}D$ is a smooth divisor on Y such that $\text{Supp}f^*D \cup \text{Supp}B_Y$ is simple normal crossing. Therefore, we can check that $\mathcal{J}_{NLC}(X, B + tD) = \mathcal{J}_{NLC}(X, B)$ for all $0 \leq t \leq 1$. \square

We close this section with an important remark.

Remark 7.6. In the subsequent sections (Sections 8, 12, 13, and 15), we consider the scheme structure of $\text{Nlc}(X, B)$ defined by $\mathcal{J}_{NLC}(X, B)$. However, we can use $\mathcal{J}'(X, B)$ or $\mathcal{J}'_l(X, B)$ for any negative integer l in place of $\mathcal{J}_{NLC}(X, B)$. For the definitions and basic properties of $\mathcal{J}'(X, B)$ and $\mathcal{J}'_l(X, B)$, see [FST]. We adopt $\mathcal{J}_{NLC}(X, B)$ since we think $\mathcal{J}_{NLC}(X, B)$ is the most natural defining ideal sheaf of $\text{Nlc}(X, B)$.

§8. Vanishing theorem

The following vanishing theorem, which is a special case of [A1, Theorem 4.4], is one of the key results in this paper. We note that the importance of Theorem 8.1 is in its formulation best suited for new frameworks explained in subsequent sections. For the details of Ambro's original statement, see [A1, Theorem 4.4] or [F11, Theorem 3.39].

Theorem 8.1. *Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let D be a Cartier divisor on X . Assume that $D - (K_X + B)$ is π -ample, where $\pi : X \rightarrow S$ is a projective morphism onto a variety S . Let $\{C_i\}$ be any set of lc centers of the pair (X, B) . We put $W = \bigcup C_i$ with the reduced scheme structure. Assume that W is disjoint from $\text{Nlc}(X, B)$. Then we have*

$$R^i \pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, where $\mathcal{J} = \mathcal{I}_W \cdot \mathcal{J}_{NLC}(X, B) \subset \mathcal{O}_X$ and \mathcal{I}_W is the defining ideal sheaf of W on X . Therefore, the restriction map

$$\pi_* \mathcal{O}_X(D) \rightarrow \pi_* \mathcal{O}_W(D) \oplus \pi_* \mathcal{O}_{\text{Nlc}(X, B)}(D)$$

is surjective and

$$R^i \pi_* \mathcal{O}_W(D) = 0$$

for every $i > 0$. In particular, the restriction maps

$$\pi_* \mathcal{O}_X(D) \rightarrow \pi_* \mathcal{O}_W(D)$$

and

$$\pi_* \mathcal{O}_X(D) \rightarrow \pi_* \mathcal{O}_{\text{Nlc}(X, B)}(D)$$

are surjective.

Proof. Let $f : Y \rightarrow X$ be a resolution such that $\text{Supp} f_*^{-1}B \cup \text{Exc}(f)$ is a simple normal crossing divisor. We can further assume that $f^{-1}(W)$ is a simple normal crossing divisor on Y . We can write

$$K_Y + B_Y = f^*(K_X + B).$$

Let T be the union of the irreducible components of B_Y^{-1} that are mapped into W by f . We consider the following short exact sequence

$$0 \rightarrow \mathcal{O}_Y(A - N - T) \rightarrow \mathcal{O}_Y(A - N) \rightarrow \mathcal{O}_T(A - N) \rightarrow 0,$$

where $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$. Note that A is an effective f -exceptional divisor. We obtain the following long exact sequence

$$\begin{aligned} 0 \rightarrow f_* \mathcal{O}_Y(A - N - T) \rightarrow f_* \mathcal{O}_Y(A - N) \rightarrow f_* \mathcal{O}_T(A - N) \\ \xrightarrow{\delta} R^1 f_* \mathcal{O}_Y(A - N - T) \rightarrow \cdots \end{aligned}$$

Since

$$\begin{aligned} A - N - T - (K_Y + \{B_Y\} + B_Y^{-1} - T) &= -(K_Y + B_Y) \\ &\sim_{\mathbb{R}} -f^*(K_X + B), \end{aligned}$$

every non-zero local section of $R^1 f_* \mathcal{O}_Y(A - N - T)$ contains in its support the f -image of some stratum of $(Y, \{B_Y\} + B_Y^{-1} - T)$ by Theorem 6.3 (i). On the other hand, $W = f(T)$. Therefore, the connecting homomorphism δ is a zero map. Thus, we have a short exact sequence

$$(\diamond) \quad 0 \rightarrow f_* \mathcal{O}_Y(A - N - T) \rightarrow f_* \mathcal{O}_Y(A - N) \rightarrow f_* \mathcal{O}_T(A - N) \rightarrow 0.$$

We put $\mathcal{J} = f_*\mathcal{O}_Y(A - N - T) \subset \mathcal{O}_X$. Since W is disjoint from $\text{Nlc}(X, B)$, the ideal sheaf \mathcal{J} coincides with \mathcal{I}_W (resp. $\mathcal{J}_{NLC}(X, B)$) in a neighborhood of W (resp. $\text{Nlc}(X, B)$). Therefore, $\mathcal{J} = \mathcal{I}_W \cdot \mathcal{J}_{NLC}(X, B)$. We put $U = X \setminus \text{Nlc}(X, B)$ and $V = f^{-1}(U)$. By restricting (\diamond) to U , we obtain

$$0 \rightarrow f_*\mathcal{O}_V(A - T) \rightarrow f_*\mathcal{O}_V(A) \rightarrow f_*\mathcal{O}_T(A) \rightarrow 0.$$

Since $f_*\mathcal{O}_V(A) \simeq \mathcal{O}_U$, we have $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$. The isomorphism $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$ plays crucial roles in the next section. Thus we write it as a proposition.

Proposition 8.2. *We have $f_*\mathcal{O}_T(A) \simeq \mathcal{O}_W$. It obviously implies that $f_*\mathcal{O}_T \simeq \mathcal{O}_W$ since A is effective.*

Remark 8.3. We did not use D nor $\pi : X \rightarrow S$ to obtain Proposition 8.2.

Since

$$f^*D + A - N - T - (K_Y + \{B_Y\} + B_Y^{-1} - T) \sim_{\mathbb{R}} f^*(D - (K_X + B)),$$

we have

$$R^i\pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) \simeq R^i\pi_*(f_*\mathcal{O}_Y(A - N - T) \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$ by Theorem 6.3 (ii). By considering the short exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{J}_{NLC}(X, B) \rightarrow \mathcal{O}_W \rightarrow 0,$$

we obtain

$$\begin{aligned} \cdots &\rightarrow R^i\pi_*(\mathcal{J}_{NLC}(X, B) \otimes \mathcal{O}_X(D)) \\ &\rightarrow R^i\pi_*\mathcal{O}_W(D) \rightarrow R^{i+1}\pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) \rightarrow \cdots \end{aligned}$$

Since we have already checked

$$R^i\pi_*(\mathcal{J}_{NLC}(X, B) \otimes \mathcal{O}_X(D)) = R^i\pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, we have $R^i\pi_*\mathcal{O}_W(D) = 0$ for all $i > 0$. Finally, we consider the following short exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_W \oplus \mathcal{O}_{\text{Nlc}(X, B)} \rightarrow 0.$$

By taking $\otimes \mathcal{O}_X(D)$ and $R^i\pi_*$, we obtain

$$0 \rightarrow \pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) \rightarrow \pi_*\mathcal{O}_X(D) \rightarrow \pi_*\mathcal{O}_W(D) \oplus \pi_*\mathcal{O}_{\text{Nlc}(X, B)}(D) \rightarrow 0.$$

This completes the proof. \square

§9. Lc centers

We prove the basic properties of lc centers as an application of the result in the preceding section (cf. Proposition 8.2). Theorem 9.1 is very useful in the study of linear systems on log canonical pairs. It can not be proved by the traditional method based on the Kawamata–Viehweg–Nadel vanishing theorem in the sense that the coefficients of B cannot be perturbed in general. For related topics, see [FG].

Theorem 9.1 (cf. [A1, Propositions 4.7 and 4.8]). *Let X be a normal variety and let B be an effective \mathbb{R} -divisor such that (X, B) is log canonical. Then we have the following properties.*

- (1) (X, B) has at most finitely many lc centers.
- (2) An intersection of two lc centers is a union of lc centers.
- (3) Any union of lc centers of (X, B) is semi-normal.
- (4) Let $x \in X$ be a closed point such that (X, B) is lc but not klt at x . Then there is a unique minimal lc center W_x passing through x , and W_x is normal at x .

Proof. We use the notation in the proof of Theorem 8.1. (1) is obvious. (3) is also obvious by Proposition 8.2 since T is a simple normal crossing divisor. Let C_1 and C_2 be two lc centers of (X, B) . We fix a closed point $P \in C_1 \cap C_2$. For the proof of (2), it is enough to find an lc center C such that $P \in C \subset C_1 \cap C_2$. We put $W = C_1 \cup C_2$. By Proposition 8.2, we obtain $f_*\mathcal{O}_T \simeq \mathcal{O}_W$. This means that $f : T \rightarrow W$ has connected fibers. We note that T is a simple normal crossing divisor on Y . Thus, there exist irreducible components T_1 and T_2 of T such that $T_1 \cap T_2 \cap f^{-1}(P) \neq \emptyset$ and that $f(T_i) \subset C_i$ for $i = 1, 2$. Therefore, we can find an lc center C with $P \in C \subset C_1 \cap C_2$. We finish the proof of (2). Finally, we will prove (4). The existence and the uniqueness of the minimal lc center follow from (2). We take the unique minimal lc center $W = W_x$ passing through x . By Proposition 8.2, we have $f_*\mathcal{O}_T \simeq \mathcal{O}_W$. By shrinking W around x , we can assume that every stratum of T dominates W . Thus, $f : T \rightarrow W$ factors through the normalization W^ν of W . Since $f_*\mathcal{O}_T \simeq \mathcal{O}_W$, we obtain that $W^\nu \rightarrow W$ is an isomorphism. So, we obtain (4). \square

§10. Dlt blow-ups

In this section, we discuss *dlt blow-ups* by Hacon (cf. Theorem 10.4). In the subsequent sections, we will only use Lemma 10.2 (well known to experts) and Theorem 10.4. For details, see Sections 11 and 18. We also discuss a slight refinement of dlt blow-ups (cf. Theorem 10.5), which is useful for future studies of log canonical

pairs and has already played crucial roles in the study of log canonical weak Fano pairs (cf. [G]).

Let us recall the definition of *dlt pairs*. For another definition and the basic properties of dlt pairs, see [KM, Section 2.3] and [F6].

Definition 10.1 (Dlt pair). Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. If there exists a resolution $f : Y \rightarrow X$ such that

- (i) both $\text{Exc}(f)$ and $\text{Exc}(f) \cup \text{Supp} f_*^{-1} B$ are simple normal crossing divisors on Y , and
- (ii) $a(E, X, B) > -1$ for every exceptional divisor $E \subset Y$,

then (X, B) is called *divisorial log terminal* (*dlt*, for short).

We will use the following lemma in Section 11. For the details, see [F6, 3.9 Adjunction for dlt pairs].

Lemma 10.2. *Let (X, B) be a dlt pair and let V be an lc center of (X, B) . Then $K_V + B_V = (K_X + B)|_V$ is dlt by adjunction.*

We borrow the next theorem from [BCHM].

Theorem 10.3 (cf. [BCHM, Theorem 1.2]). *Let (X, B) be a klt pair, where $K_X + B$ is \mathbb{R} -Cartier. Let $\pi : X \rightarrow S$ be a projective birational morphism of quasi-projective varieties. Then (X, B) has a log terminal model over S . This means that there exists a projective birational morphism $f : X' \rightarrow X$ such that*

- (i) X' is \mathbb{Q} -factorial,
- (ii) ϕ^{-1} has no exceptional divisors, where $\phi = f^{-1} \circ \pi : X \dashrightarrow X'$,
- (iii) $K_{X'} + B'$ is f -nef, where $B' = \phi_* B$, and
- (iv) $a(E, X, B) < a(E, X', B')$ for every ϕ -exceptional divisor $E \subset X$.

The following theorem is very useful. It is a consequence of Theorem 10.3. The proof which is the same as [KK, Theorem 3.1] is nevertheless given below since Theorem 10.5 is based on the close examination of it and Section 6.

Theorem 10.4 (Dlt blow-ups by Hacon, cf. [KK, Theorem 3.1]). *Let X be a normal quasi-projective variety and let B be a boundary \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. In this case, we can construct a projective birational morphism $f : Y \rightarrow X$ from a normal quasi-projective variety Y with the following properties.*

- (i) Y is \mathbb{Q} -factorial.

(ii) $a(E, X, B) \leq -1$ for every f -exceptional divisor E on Y .

(iii) We put

$$B_Y = f_*^{-1}B + \sum_{E:f\text{-exceptional}} E.$$

Then (Y, B_Y) is dlt and

$$K_Y + B_Y = f^*(K_X + B) + \sum_{a(E, X, B) < -1} (a(E, X, B) + 1)E.$$

In particular, if (X, B) is lc, then $K_Y + B_Y = f^*(K_X + B)$. Moreover, if (X, B) is dlt, then we can make f small, that is, f is an isomorphism in codimension one.

Proof. Let $\pi : V \rightarrow X$ be a resolution such that $\pi_*^{-1}B \cup \text{Exc}(\pi)$ has a simple normal crossing support. We can assume that π is a composite of blow-ups of centers of codimension at least two. Then there exists an effective π -exceptional Cartier divisor C on V such that $-C$ is π -ample. We put

$$F = \sum_{\substack{a(E, X, B) > -1, \\ E:\pi\text{-exceptional}}} E$$

and

$$E^+ = - \sum_{a(E, X, B) \leq -1} a(E, X, B)E.$$

We note that E^+ is not necessarily π -exceptional. We put $E = \text{Supp}E^+$. We note that $E^+ - E$ is π -exceptional.

Let H be a sufficiently ample Cartier divisor on X . We choose $0 < \epsilon, \nu, \mu \ll 1$ and note that

$$\begin{aligned} (\spadesuit) \quad & E + (1 - \nu)F + \mu(-C + \pi^*H) \\ & = (1 - \epsilon\mu)E + (1 - \nu)F + \mu(\epsilon E - C + \pi^*H). \end{aligned}$$

Since $-C + \pi^*H$ and $\epsilon E - C + \pi^*H$ are ample, we can take effective \mathbb{Q} -divisors H_1 and H_2 with small coefficients such that $E + F + \pi^*B + H_1 + H_2$ has a simple normal crossing support and that $-C + \pi^*H \sim_{\mathbb{Q}} H_1$, $\epsilon E - C + \pi^*H \sim_{\mathbb{Q}} H_2$. Then $(V, (1 - \epsilon\mu)E + (1 - \nu)F + \pi_*^{-1}B^{<1} + \mu H_2)$ is klt. By Theorem 10.3, it has a log terminal model $f : Y \rightarrow X$. By the above equation (\spadesuit) , this is also a relative minimal model of the pair $(V, E + (1 - \nu)F + \pi_*^{-1}B^{<1} + \mu H_1)$, which is therefore dlt.

For any divisor G on V appearing above, let G' denote its transform on Y . By the above construction,

$$N = K_Y + (1 - \epsilon\mu)E' + (1 - \nu)F' + f_*^{-1}B^{<1} + \mu H'_2$$

is f -nef and $K_Y + \overline{B} = f^*(K_X + B)$ is \mathbb{R} -linearly f -trivial. We put

$$D = \overline{B} - E' - (1 - \nu)F' - f_*^{-1}B^{<1} + \mu C'.$$

Then

$$\begin{aligned} -D &\sim_{\mathbb{R},f} N - (K_Y + \overline{B}) \\ &= -\overline{B} + (1 - \epsilon\mu)E' + (1 - \nu)F' + f_*^{-1}B^{<1} + \mu H'_2, \end{aligned}$$

hence it is f -nef. Since $f_*D = 0$, we see that D is effective by the negativity lemma (cf. Lemma 4.16).

Every divisor in F has a negative coefficient in

$$\tilde{B} - E - (1 - \nu)F - \pi_*^{-1}B^{<1} + \mu C,$$

where $K_V + \tilde{B} = \pi^*(K_X + B)$. Therefore, F is contracted on Y . So, every f -exceptional divisor has discrepancy ≤ -1 . By the above construction, $(Y, E' + f_*^{-1}B^{<1} + \mu H'_1)$ is dlt since $F' = 0$. Therefore, $(Y, E' + f_*^{-1}B^{<1})$ is also dlt. This means that (Y, B_Y) is dlt because $B_Y = E' + \sum f_*^{-1}B^{<1}$.

When (X, B) is dlt, we can assume that $E^+ = \pi_*^{-1}B^{=1}$ by the definition of dlt pairs. Therefore, we can make f small. \square

The study of lc centers of a given lc pair often requires a refinement of Theorem 10.4, that is, Theorem 10.5 which relates lc centers on X to those on Y . It is actually needed in [G], which studies weak Fano varieties with log canonical singularities.

Theorem 10.5. *Let X be a normal quasi-projective variety and let B be an effective \mathbb{R} -divisor on X such that (X, B) is lc. In this case, we can construct a projective birational morphism $f : Y \rightarrow X$ from a normal quasi-projective variety Y with the following properties.*

- (i) Y is \mathbb{Q} -factorial.
- (ii) $a(E, X, B) = -1$ for every f -exceptional divisor E on Y .
- (iii) We put

$$B_Y = f_*^{-1}B + \sum_{E:f\text{-exceptional}} E.$$

Then (Y, B_Y) is dlt and $K_Y + B_Y = f^*(K_X + B)$.

(iv) Let $\{C_i\}$ be any set of lc centers of (X, B) . We put $W = \bigcup C_i$ with the reduced scheme structure. Let S be the union of the irreducible components of B_Y^{-1} which are mapped into W by f . Then $f_*\mathcal{O}_S \simeq \mathcal{O}_W$.

Proof. Let $\pi : V \rightarrow X$ be a resolution such that

- (1) $\pi^{-1}(C)$ is a simple normal crossing divisor on V for every lc center C of (X, B) , and
- (2) $\pi_*^{-1}B \cup \text{Exc}(\pi) \cup \pi^{-1}(\text{Nklt}(X, B))$ has a simple normal crossing support.

We apply the arguments in the proof of Theorem 10.4. From now on, we use the same notation as in the proof of Theorem 10.4. In this case, we have

$$E = \text{Supp}E^+ = E^+.$$

When we construct $f : Y \rightarrow X$, we can run the log minimal model program with scaling with respect to

$$\begin{aligned} & K_V + E + (1 - \nu)F + \pi_*^{-1}B^{<1} + \mu H_1 \\ & \sim_{\mathbb{R}} K_V + (1 - \varepsilon\mu)E + (1 - \nu)F + \pi_*^{-1}B^{<1} + \mu H_2 \end{aligned}$$

(cf. [BCHM]). So, we can assume that $\varphi : V \dashrightarrow Y$ is a composition of $(K_V + E + (1 - \nu)F + \pi_*^{-1}B^{<1} + \mu H_1)$ -negative divisorial contractions and log flips. Let Σ be an lc center of (Y, B_Y) . Then it is also an lc center of $(Y, B_Y + \mu H_1')$. By the negativity lemma (cf. Lemma 4.16), $\varphi : V \dashrightarrow Y$ is an isomorphism around the generic point of Σ . Therefore, if $f(\Sigma) \subset W$, then $\Sigma \subset S$ by the conditions (1) and (2) for $\pi : V \rightarrow X$. This means that no lc centers of $(Y, B_Y - S)$ are mapped into W by f . Let $g : Z \rightarrow Y$ be a resolution such that

- (a) $K_Z + B_Z = g^*(K_Y + B_Y)$,
- (b) $\text{Supp}B_Y$ is a simple normal crossing divisor, and
- (c) g is an isomorphism over the generic point of any lc center of (Y, B_Y) .

Let S_Z be the strict transform of S on Z . We consider the following short exact sequence

$$\begin{aligned} (\heartsuit) \quad 0 \rightarrow \mathcal{O}_Z(\Gamma - (B_Z^{<1})^\Gamma - S_Z) &\rightarrow \mathcal{O}_Z(\Gamma - (B_Z^{<1})^\Gamma) \\ &\rightarrow \mathcal{O}_{S_Z}(\Gamma - (B_Z^{<1})^\Gamma) \rightarrow 0. \end{aligned}$$

We note that

$$\Gamma - (B_Z^{<1})^\Gamma - S_Z - (K_Z + \{B_Z\} + B_Z^{-1} - S_Z) \sim_{\mathbb{R}} -h^*(K_X + B),$$

where $h = f \circ g$. Then we obtain

$$\begin{aligned} 0 \rightarrow h_* \mathcal{O}_Z(\Gamma - (B_Z^{\leq 1})^\vee - S_Z) \rightarrow h_* \mathcal{O}_Z(\Gamma - (B_Z^{\leq 1})^\vee) \rightarrow h_* \mathcal{O}_{S_Z}(\Gamma - (B_Z^{\leq 1})^\vee) \\ \xrightarrow{\delta} R^1 h_* \mathcal{O}_Z(\Gamma - (B_Z^{\leq 1})^\vee - S_Z) \rightarrow \dots \end{aligned}$$

Every associated prime of $R^1 h_* \mathcal{O}_Z(\Gamma - (B_Z^{\leq 1})^\vee - S_Z)$ is the generic point of the h -image of some stratum of $(Z, \{B_Z\} + B_Z^{\leq 1} - S_Z)$ by Theorem 6.3 (i) and no lc centers of $(Z, \{B_Z\} + B_Z^{\leq 1} - S_Z)$ are mapped into W by h . Therefore, δ is a zero map. Thus, we obtain

$$0 \rightarrow \mathcal{I}_W \rightarrow \mathcal{O}_X \rightarrow h_* \mathcal{O}_{S_Z}(\Gamma - (B_Z^{\leq 1})^\vee) \rightarrow 0$$

and $\mathcal{O}_W \simeq h_* \mathcal{O}_{S_Z} \simeq h_* \mathcal{O}_{S_Z}(\Gamma - (B_Z^{\leq 1})^\vee)$ (cf. Proposition 8.2), where \mathcal{I}_W is the defining ideal sheaf of W . Here, we used the fact that $\Gamma - (B_Z^{\leq 1})^\vee$ is effective and h -exceptional. By applying g_* to (\heartsuit) , we obtain

$$0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{O}_Y \rightarrow g_* \mathcal{O}_{S_Z}(\Gamma - (B_Z^{\leq 1})^\vee) \rightarrow 0$$

and $\mathcal{O}_S \simeq g_* \mathcal{O}_{S_Z} \simeq g_* \mathcal{O}_{S_Z}(\Gamma - (B_Z^{\leq 1})^\vee)$ (cf. Proposition 8.2), where $\mathcal{I}_S \simeq \mathcal{O}_Y(-S)$ is the defining ideal sheaf of S . We note that

$$R^1 g_* \mathcal{O}_Z(\Gamma - (B_Z^{\leq 1})^\vee - S_Z) = 0$$

by Theorem 6.3 (i) since g is an isomorphism at the generic point of any stratum of $(Z, \{B_Z\} + B_Z^{\leq 1} - S_Z)$ and that $\Gamma - (B_Z^{\leq 1})^\vee$ is effective and g -exceptional. Therefore, $\mathcal{O}_W \simeq h_* \mathcal{O}_{S_Z} \simeq f_* g_* \mathcal{O}_{S_Z} \simeq f_* \mathcal{O}_S$. \square

§11. Vanishing theorem for minimal lc centers

In this section, we prove a vanishing theorem on minimal lc centers. It is very powerful and will play crucial roles in the proof of Theorem 12.1. We note that a key point of Theorem 11.1 is in its formulation which is best suited for our subsequent applications.

Theorem 11.1 (Vanishing theorem for minimal lc centers). *Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let W be a minimal lc center of (X, B) such that W is disjoint from $\text{Nlc}(X, B)$. Let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Let D be a Cartier divisor on W such that $D - (K_X + B)|_W$ is π -ample. Then*

$$R^i \pi_* \mathcal{O}_W(D) = 0$$

for every $i > 0$.

Proof. Without loss of generality, we can assume that S is quasi-projective. We shrink X around W and assume that (X, B) is log canonical. By Theorem 10.4, we can make a projective birational morphism $f : Y \rightarrow X$ such that $K_Y + B_Y = f^*(K_X + B)$ and (Y, B_Y) is dlt. We take an lc center V of (Y, B_Y) such that $f(V) = W$ and put $K_V + B_V = (K_Y + B_Y)|_V$. Then (V, B_V) is dlt by Lemma 10.2 and $K_V + B_V \sim_{\mathbb{R}} f^*((K_X + B)|_W)$. Let $g : Z \rightarrow V$ be a resolution such that $K_Z + B_Z = g^*(K_V + B_V)$ and $\text{Supp} B_Z$ is simple normal crossing. Then we have $K_Z + B_Z \sim_{\mathbb{R}} h^*((K_X + B)|_W)$, where $h = f \circ g$. Since

$$h^*(D - (K_X + B)|_W) \sim_{\mathbb{R}} h^*D + \ulcorner -(B_Z^{\leq 1})^\urcorner - (K_Z + B_Z^{-1} + \{B_Z\}),$$

we obtain

$$R^i \pi_* h_* \mathcal{O}_Z(h^*D + \ulcorner -(B_Z^{\leq 1})^\urcorner) = 0$$

for every $i > 0$ by Theorem 6.3 (ii). We note that

$$h_* \mathcal{O}_Z(h^*D + \ulcorner -(B_Z^{\leq 1})^\urcorner) \simeq f_* \mathcal{O}_V(f^*D)$$

by the projection formula since $\ulcorner -(B_Z^{\leq 1})^\urcorner$ is effective and g -exceptional. We note that $\mathcal{O}_W(D)$ is a direct summand of $f_* \mathcal{O}_V(f^*D) \simeq \mathcal{O}_W(D) \otimes f_* \mathcal{O}_V$ since W is normal (cf. Theorem 9.1 (4)). Therefore, we have $R^i \pi_* \mathcal{O}_W(D) = 0$ for every $i > 0$. \square

We close this section with a very important remark.

Remark 11.2. The short proof of Theorem 11.1 given in this section depends on Theorem 6.3 (ii), Theorem 9.1 (4), and Theorem 10.4 which is a corollary to [BCHM]. However Theorem 11.1, a special case of [A1, Theorem 4.4], is independent of [BCHM] since it can be proved without using Theorem 10.4. We refer the reader to [F11, Theorem 3.39] as for the independent proof of Theorem 11.1, which heavily depends on the theory of mixed Hodge structures on compact support cohomology groups of *reducible* varieties (cf. [F11, Chapter 2]).

§12. Non-vanishing theorem

In this section, we prove the non-vanishing theorem, which is a generalization of the main theorem of [F16]. In [A1], Ambro does not discuss any generalization of Shokurov's non-vanishing theorem. Therefore, the result in this section is one of the main differences between the theory of quasi-log varieties and our new framework.

Theorem 12.1 (Non-vanishing theorem). *Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let $\pi : X \rightarrow S$ be*

a projective morphism onto a variety S and let L be a π -nef Cartier divisor on X . Assume that

- (i) $aL - (K_X + B)$ is π -ample for some real number $a > 0$, and
- (ii) $\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is $\pi|_{\mathrm{Nlc}(X,B)}$ -generated for $m \gg 0$.

Then the relative base locus $\mathrm{Bs}_\pi|mL|$ contains no lc centers of (X, B) and is disjoint from $\mathrm{Nlc}(X, B)$ for $m \gg 0$.

Proof. Without loss of generality, we can assume that S is affine.

Step 1. In this step, we will prove that $\mathcal{O}_X(mL)$ is π -generated on an open neighborhood of $\mathrm{Nlc}(X, B)$ for $m \gg 0$.

By the assumption, $\pi^*\pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL) \rightarrow \mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is surjective for $m \gg 0$. On the other hand, $\pi_*\mathcal{O}_X(mL) \rightarrow \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is surjective for $m \geq a$ since

$$R^1\pi_*(\mathcal{J}_{\mathrm{Nlc}}(X, B) \otimes \mathcal{O}_X(mL)) = 0$$

for $m \geq a$ by Theorem 8.1. Therefore, for every large integer m , $\pi^*\pi_*\mathcal{O}_X(mL) \rightarrow \mathcal{O}_X(mL)$ is surjective on an open neighborhood of $\mathrm{Nlc}(X, B)$. See the following commutative diagram.

$$\begin{array}{ccccc} \pi^*\pi_*\mathcal{O}_X(mL) & \longrightarrow & \pi^*\pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{\mathrm{Nlc}(X,B)}(mL) & \longrightarrow & 0, \end{array}$$

Let W be a minimal lc center of (X, B) . Then it is sufficient to see that W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$.

Step 2. If $W \cap \mathrm{Nlc}(X, B) \neq \emptyset$, then $\mathrm{Bs}|mL|$ does not contain W by Step 1. So, from now on, we can assume that $W \cap \mathrm{Nlc}(X, B) = \emptyset$.

Step 3. We assume that $L|_{W_\eta}$ is numerically trivial, where W_η is the generic fiber of $W \rightarrow \pi(W)$. In this case,

$$\begin{aligned} h^0(W_\eta, \mathcal{O}_{W_\eta}(L)) &= \chi(W_\eta, \mathcal{O}_{W_\eta}(L)) \\ &= \chi(W_\eta, \mathcal{O}_{W_\eta}) = h^0(W_\eta, \mathcal{O}_{W_\eta}) > 0 \end{aligned}$$

by [K1, Chapter II §2 Theorem 1] and the vanishing theorem: Theorem 11.1. On the other hand,

$$\pi_*\mathcal{O}_X(mL) \rightarrow \pi_*\mathcal{O}_W(mL) \oplus \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$$

is surjective for every $m \geq a$ by Theorem 8.1. In particular, the restriction map $\pi_* \mathcal{O}_X(mL) \rightarrow \pi_* \mathcal{O}_W(mL)$ is surjective for every $m \geq a$. Thus, $\text{Bs}|mL|$ does not contain W for every $m \geq a$.

Step 4. We assume that $L|_{W_\eta}$ is not numerically trivial. We take a general subvariety V of W such that $V \rightarrow \pi(W)$ is generically finite. If l is a positive large integer, then we can write

$$lL - (K_X + B) = N_1 + a_2 N_2 + \cdots + a_k N_k$$

with the following properties.

(a) N_1 is a π -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor on X such that

$$((N_1|_W)|_F)^{\dim F} > d(\text{codim}_W V)^{\dim F},$$

where d is the mapping degree of $V \rightarrow \pi(W)$ and F is a general fiber of $W \rightarrow \pi(W)$.

(b) a_i is a positive real number and N_i is a π -very ample Cartier divisor on X for every $i \geq 2$.

By Lemma 12.2, we can find an effective \mathbb{Q} -divisor D_1 on W such that $D_1 \sim_{\mathbb{Q}} N_1|_W$ with $\text{mult}_V D_1 > \text{codim}_W V$. If b is sufficiently large and divisible, then $bD_1 \sim bN_1|_W$, $\mathcal{I}_W \otimes \mathcal{O}_X(bN_1)$ is π -generated, and $R^1 \pi_*(\mathcal{I}_W \otimes \mathcal{O}_X(bN_1)) = 0$ since N_1 is π -ample, where \mathcal{I}_W is the defining ideal sheaf of W . By using the following short exact sequence

$$0 \rightarrow \pi_*(\mathcal{I}_W \otimes \mathcal{O}_X(bN_1)) \rightarrow \pi_* \mathcal{O}_X(bN_1) \rightarrow \pi_* \mathcal{O}_W(bN_1) \rightarrow 0,$$

we can find an effective \mathbb{Q} -divisor M_1 on X with the following properties.

- (1) $M_1|_W = D_1$.
- (2) $M_1 \sim_{\mathbb{Q}} N_1$.
- (3) $(X, B + M_1)$ is lc outside $W \cup \text{Nlc}(X, B)$.
- (4) $\mathcal{J}_{NLC}(X, B + M_1) = \mathcal{J}_{NLC}(X, B)$ outside W .

Let M_i be a general member of $|N_i|$ for every $i \geq 2$. We put $M = M_1 + a_2 M_2 + \cdots + a_k M_k$. Then we have

- (i) $M|_W \geq D_1$.
- (ii) $M \sim_{\mathbb{R}} lL - (K_X + B)$.
- (iii) $(X, B + M)$ is lc outside $W \cup \text{Nlc}(X, B)$.
- (iv) $\mathcal{J}_{NLC}(X, B + M) = \mathcal{J}_{NLC}(X, B)$ outside W .

We take the log canonical threshold c of (X, B) with respect to M outside $\text{Nlc}(X, B)$. By the above construction, we have $0 < c < 1$. More precisely, we see $0 < c$ since M contains no lc centers of (X, B) . The inequality $c < 1$ follows from the fact that $M|_W \geq D_1$ and $\text{mult}_V D_1 > \text{codim}_W V$. We note that

$$(a - ac + cl)L - (K_X + B + cM) \sim_{\mathbb{R}} (1 - c)(aL - (K_X + B))$$

is π -ample. Moreover, we can find a smaller lc center W' of $(X, B + cM)$ contained in W (cf. Theorem 9.1 (2)). Therefore, we replace (X, B) with $(X, B + cM)$, a with $a - ac + cl$, and consider the new lc center W' . By repeating this process, we reach the situation where $L|_{W_\eta}$ is numerically trivial.

Anyway, we proved that $\text{Bs}|mL|$ contains no lc centers of (X, B) for $m \gg 0$. \square

The following lemma is a relative version of Shokurov's concentration method. We used it in the proof of Theorem 12.1.

Lemma 12.2. *Let $f : Y \rightarrow Z$ be a projective morphism from a normal variety Y onto an affine variety Z . Let V be a general closed subvariety of Y such that $f : V \rightarrow Z$ is generically finite. Let M be an f -ample \mathbb{R} -divisor on Y . Assume that*

$$(M|_F)^d > km^d,$$

where F is a general fiber of $f : Y \rightarrow Z$, $d = \dim F$, and k is the mapping degree of $f : V \rightarrow Z$. Then we can find an effective \mathbb{R} -divisor D on Y such that

$$D \sim_{\mathbb{R}} M$$

and that $\text{mult}_V D > m$. If M is a \mathbb{Q} -divisor, then we can make D a \mathbb{Q} -divisor with $D \sim_{\mathbb{Q}} M$.

Proof. We can write

$$M = M_1 + a_2 M_2 + \cdots + a_l M_l,$$

where M_1 is an f -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor such that $(M_1|_F)^d > km^d$, a_i is a positive real number, and M_i is an f -ample Cartier divisor for every i . If M is a \mathbb{Q} -divisor, then we can assume that $l = 2$ and a_2 is rational. Let \mathcal{I}_V be the defining ideal sheaf of V on Y . We consider the following exact sequence

$$\begin{aligned} 0 &\rightarrow f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm}) \rightarrow f_*\mathcal{O}_Y(pM_1) \\ &\rightarrow f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{O}_Y/\mathcal{I}_V^{pm}) \rightarrow \cdots \end{aligned}$$

for a sufficiently large and divisible integer p . By restricting the above sequence to a general fiber F of f , we can check that the rank of $f_*\mathcal{O}_Y(pM_1)$ is greater than

that of $f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{O}_Y/\mathcal{I}_V^{pm})$ by the usual estimates (see Lemma 12.3 below). Therefore, $f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm}) \neq 0$. Let D_1 be a member of

$$H^0(Z, f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm})) = H^0(Y, \mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm})$$

and let D_i be an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor such that $D_i \sim_{\mathbb{Q}} M_i$ for $i \geq 2$. We can take D_2 with $\text{mult}_V D_2 > 0$. Then $D = (1/p)D_1 + a_2 D_2 + \cdots + a_l D_l$ satisfies the desired properties. \square

We close this section with the following well-known lemma. The proof is obvious.

Lemma 12.3. *Let X be a normal projective variety with $\dim X = d$ and let A be an ample \mathbb{Q} -divisor on X such that rA is Cartier for some positive integer r . Then*

$$\begin{aligned} h^0(X, \mathcal{O}_X(trA)) &= \chi(X, \mathcal{O}_X(trA)) \\ &= \frac{(trA)^d}{d!} + (\text{lower terms in } t) \end{aligned}$$

by the Riemann-Roch formula and the Serre vanishing theorem for $t \gg 0$.

Let $P \in X$ be a smooth point. Then

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{O}_X/m_P^\alpha &= \binom{\alpha - 1 + d}{d} \\ &= \frac{\alpha^d}{d!} + (\text{lower terms in } \alpha) \end{aligned}$$

for all $\alpha \geq 1$, where m_P is the maximal ideal associated to P .

§13. Base point free theorem

The base point free theorem is one of the most important theorems in the log minimal model program. Since we have already established the non-vanishing theorem (Theorem 12.1) in our framework, there are no difficulties in obtaining the base point free theorem (Theorem 13.1). Our approach is simpler than [A1], though Theorem 13.1 is a special case of the base point free theorem for quasi-log varieties obtained by Ambro (cf. [A1, Theorem 5.1] and [F11, Theorem 3.66]). Indeed in the approach of [A1] it is necessary to treat reducible non-equidimensional quasi-log varieties even for the proof of the base point free theorem for log canonical pairs.

Theorem 13.1 (Base point free theorem). *Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let $\pi : X \rightarrow S$ be*

a projective morphism onto a variety S and let L be a π -nef Cartier divisor on X . Assume that

- (i) $aL - (K_X + B)$ is π -ample for some real number $a > 0$, and
- (ii) $\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is $\pi|_{\mathrm{Nlc}(X,B)}$ -generated for $m \gg 0$.

Then $\mathcal{O}_X(mL)$ is π -generated for $m \gg 0$.

We will prove the base point free theorem for \mathbb{R} -divisors in Section 17 as an application of the cone theorem: Theorem 16.6.

Proof. We can assume that S is affine.

Step 1. We assume that (X, B) is klt and that L_η is numerically trivial, where $L_\eta = L|_{X_\eta}$ and X_η is the generic fiber of $\pi : X \rightarrow S$. Then we have

$$\begin{aligned} h^0(X_\eta, \mathcal{O}_{X_\eta}(L_\eta)) &= \chi(X_\eta, \mathcal{O}_{X_\eta}(L_\eta)) \\ &= \chi(X_\eta, \mathcal{O}_{X_\eta}) = h^0(X_\eta, \mathcal{O}_{X_\eta}) > 0 \end{aligned}$$

by [Kl, Chapter II §2 Theorem 1] and the vanishing theorem. Here, the Kawamata–Viehweg vanishing theorem is sufficient. Therefore, $|L| \neq \emptyset$. Let D be a member of $|L|$. If $D = 0$, then it is obvious that $|mL|$ is free for every m . Thus, we can assume that $D \neq 0$. Let c be the log canonical threshold of (X, B) with respect to D . We replace (X, B) with $(X, B + cD)$, a with $a + c$. Then we can assume that (X, B) is lc but not klt. This case will be treated in Step 3.

Step 2. We assume that (X, B) is klt and that L_η is not numerically trivial. We take a general subvariety V on X such that $\pi : V \rightarrow S$ is generically finite. By Lemma 12.2, we can find an effective \mathbb{R} -divisor D on X such that

$$D \sim_{\mathbb{R}} lL - (K_X + B)$$

for some large l and that $\mathrm{mult}_V D > \mathrm{codim}_X V$. Let c be the log canonical threshold of (X, B) with respect to D . By the above construction, we obtain $0 < c < 1$. We replace (X, B) with $(X, B + cD)$, a with $a - ac + cl$ and can assume that (X, B) is lc but not klt. We note that

$$(a - ac + cl)L - (K_X + B + cD) \sim_{\mathbb{R}} (1 - c)(aL - (K_X + B)).$$

So, the problem is reduced to the case when (X, B) is lc but not klt. It will be treated in Step 3.

Step 3. We assume that (X, B) is not klt. Let p be a prime integer. We will prove that $\mathrm{Bs}|p^m L| = \emptyset$ for some positive integer m .

By Theorem 12.1, $|p^{m_1}L| \neq \emptyset$ for some positive integer m_1 . If $\text{Bs}|p^{m_1}L| = \emptyset$, then there are nothing to prove. So, we can assume that $\text{Bs}|p^{m_1}L| \neq \emptyset$. We take general members $D_1, \dots, D_{n+1} \in |p^{m_1}L|$, where $n = \dim X$. Since D_1, \dots, D_{n+1} are general, $(X, B + D_1 + \dots + D_{n+1})$ is lc outside $\text{Bs}|p^{m_1}L| \cup \text{Nlc}(X, B)$. It is easy to see that $(X, B + D)$, where $D = D_1 + \dots + D_{n+1}$, is not lc at the generic point of every irreducible component of $\text{Bs}|p^{m_1}L|$ (see Lemma 13.2 below). Let c be the log canonical threshold of (X, B) with respect to D outside $\text{Nlc}(X, B)$. Then $(X, B + cD)$ is lc but not klt outside $\text{Nlc}(X, B)$, $0 < c < 1$, and $\mathcal{J}_{NLC}(X, B + cD) = \mathcal{J}_{NLC}(X, B)$ (see Proposition 7.5). We note that

$$(c(n+1)p^{m_1} + a)L - (K_X + B + cD) \sim_{\mathbb{R}} aL - (K_X + B)$$

is f -ample. By the construction, there exists an lc center of $(X, B + cD)$ contained in $\text{Bs}|p^{m_1}L|$. By Theorem 12.1, we can find $m_2 > m_1$ such that $\text{Bs}|p^{m_2}L| \subsetneq \text{Bs}|p^{m_1}L|$. By the noetherian induction, there exists m such that $\text{Bs}|p^mL| = \emptyset$.

Step 4. Let p' be a prime integer such that $p' \neq p$. Then, by Step 3 again, we can find a positive integer m' such that $\text{Bs}|p'^{m'}L| = \emptyset$. So, there exists a positive integer m_0 such that $|kL|$ is free for every $k \geq m_0$ by $\text{Bs}|p^mL| = \emptyset$ and $\text{Bs}|p'^{m'}L| = \emptyset$.

This completes the proof. \square

We close this section with the following lemma. We used it in the proof of Theorem 13.1.

Lemma 13.2. *Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let P be a closed point of X and let $P \in D_i$ be a Cartier divisor for every i . If $(X, B + \sum_{i=1}^k D_i)$ is log canonical at P , then $k \leq \dim X$.*

Proof. The proof is by the induction on $\dim X$. The assertion is clear if $\dim X = 1$. We put $S = D_1$. Let $\nu : S^\nu \rightarrow S$ be the normalization and let B_{S^ν} be the *different* of $(X, S + B)$ on S^ν (see Section 14 below). So, we have $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$. Since $(X, B + S + \sum_{i=2}^k D_i)$ is log canonical at P , $(S^\nu, B_{S^\nu} + \sum_{i=2}^k \nu^*D_i)$ is log canonical at $Q \in \nu^{-1}(P)$. Thus, $k - 1 \leq \dim S^\nu$ by induction. This means that $k \leq \dim X$. \square

§14. Shokurov's differentials

Let us recall the definition and basic properties of Shokurov's *differentials* following [S2, §3] and [A2, 9.2.1].

14.1. Let X be a normal variety and let $S + B$ be an \mathbb{R} -divisor on X such that $K_X + S + B$ is \mathbb{R} -Cartier. Assume that S is reduced and that S and B have no common irreducible components. Let $f : Y \rightarrow X$ be a resolution such that

$$K_Y + S_Y + B_Y = f^*(K_X + S + B)$$

and $\text{Supp}(S_Y + B_Y)$ is simple normal crossing and S_Y is smooth, where S_Y is the strict transform of S on Y . Let $\nu : S^\nu \rightarrow S$ be the normalization. Then $f : S_Y \rightarrow S$ can be decomposed as

$$f : S_Y \xrightarrow{\pi} S^\nu \xrightarrow{\nu} S.$$

We define $B_{S_Y} = B_Y|_{S_Y}$. Then we obtain

$$(K_Y + S_Y + B_Y)|_{S_Y} = K_{S_Y} + B_{S_Y}$$

by adjunction. We put $B_{S^\nu} = \pi_* B_{S_Y}$. Then we have

$$K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B).$$

The \mathbb{R} -divisor B_{S^ν} on S^ν is called the *different* of $(X, S + B)$ on S^ν . We can easily check that B_{S^ν} is independent of the resolution $f : Y \rightarrow X$. So, B_{S^ν} is a well-defined \mathbb{R} -divisor on S^ν . We can check the following properties.

- (i) $K_{S^\nu} + B_{S^\nu}$ is \mathbb{R} -Cartier and $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$.
- (ii) If B is a \mathbb{Q} -divisor, then so is B_{S^ν} .
- (iii) B_{S^ν} is effective if B is effective in a neighborhood of S .
- (iv) (S^ν, B_{S^ν}) is log canonical if $(X, S + B)$ is log canonical in a neighborhood of S .
- (v) Let D be an \mathbb{R} -Cartier \mathbb{R} -divisor on X such that S and D have no common irreducible components. Then we have

$$(B + D)_{S^\nu} = B_{S^\nu} + \nu^* D.$$

We sometimes write $D|_{S^\nu} = \nu^* D$ for simplicity.

The properties except (iii) are obvious by the definition. We give a proof of (iii) for the reader's convenience.

Proof of (iii). By shrinking X , we can assume that X is quasi-projective and B is effective. By taking hyperplane cuts, we can also assume that X is a surface. Run the log minimal model program over X with respect to $K_Y + S_Y$. Let C be a curve on Y such that $(K_Y + S_Y) \cdot C < 0$ and $f(C)$ is a point. Then $K_Y \cdot C < 0$ because S_Y is the strict transform of S . Therefore, each step of the log minimal model program over X with respect to $K_Y + S_Y$ is a contraction of a (-1) -curve

E with $(K_Y + S_Y) \cdot E < 0$. So, by replacing (Y, S_Y) with the output of the above log minimal model program, we can assume that Y is smooth, (Y, S_Y) is plt, and $K_Y + S_Y$ is f -nef. We note that S_Y is a smooth curve since (Y, S_Y) is plt (cf. [KM, Proposition 5.51]). By the negativity lemma (see Lemma 4.16) and the assumption that B is effective, B_Y is effective. We note the following equality

$$-B_Y = K_Y + S_Y - f^*(K_X + S + B).$$

By adjunction, we obtain

$$(K_Y + S_Y + B_Y)|_{S_Y} = K_{S_Y} + B_Y|_{S_Y}.$$

It is obvious that $B_Y|_{S_Y}$ is effective. This implies that $B_{S^\nu} = B_Y|_{S_Y}$ is effective. \square

When X is singular, B_{S^ν} is not necessarily zero even if $B = 0$.

§15. Rationality theorem

In this section, we prove the following rationality theorem, though it is a special case of [A1, Theorem 5.9] (see also [F11, Theorem 3.68]). In the traditional X-method, the rationality theorem for klt pairs is proved by the Kawamata–Viehweg vanishing theorem, Hironaka’s resolution theorem, and Shokurov’s non-vanishing theorem (see, for example, [KM, §3.4]). Our proof of the rationality theorem given below only uses the vanishing theorem: Theorem 8.1. We do not need the non-vanishing theorem (cf. Theorem 12.1) nor Hironaka’s resolution theorem in this section.

Theorem 15.1 (Rationality theorem). *Let X be a normal variety and let B be an effective \mathbb{Q} -divisor on X such that $K_X + B$ is \mathbb{Q} -Cartier. Let $\pi : X \rightarrow S$ be a projective morphism and let H be a π -ample Cartier divisor on X . Assume that $K_X + B$ is not π -nef and that r is a positive number such that*

- (1) $H + r(K_X + B)$ is π -nef but not π -ample, and
- (2) $(H + r(K_X + B))|_{\text{Nlc}(X, B)}$ is $\pi|_{\text{Nlc}(X, B)}$ -ample.

Then r is a rational number, and in reduced form, r has denominator at most $a(\dim X + 1)$, where $a(K_X + B)$ is a Cartier divisor on X .

Before the proof of Theorem 15.1, we recall the following lemmas.

Lemma 15.2 (cf. [KM, Lemma 3.19]). *Let $P(x, y)$ be a non-trivial polynomial of degree $\leq n$ and assume that P vanishes for all sufficiently large integral solutions*

of $0 < ay - rx < \varepsilon$ for some fixed positive integer a and positive ε for some $r \in \mathbb{R}$. Then r is rational, and in reduced form, r has denominator $\leq a(n+1)/\varepsilon$.

Proof. We assume that r is irrational. Then an infinite number of integral points in the (x, y) -plane on each side of the line $ay - rx = 0$ are closer than $\varepsilon/(n+2)$ to that line. So there is a large integral solution (x', y') with $0 < ay' - rx' < \varepsilon/(n+2)$. In this case, we see that

$$(2x', 2y'), \dots, ((n+1)x', (n+1)y')$$

are also solutions by hypothesis. So $(y'x - x'y)$ divides P , since P and $(y'x - x'y)$ have $(n+1)$ common zeroes. We choose a smaller ε and repeat the argument. We do this $n+1$ times to get a contradiction.

Now we assume that $r = u/v$ in lowest terms. For given j , let (x', y') be a solution of $ay - rx = aj/v$. Note that an integral solution exists for every j . Then we have $a(y' + ku) - r(x' + akv) = aj/v$ for all k . So, as above, if $aj/v < \varepsilon$, $(ay - rx) - (aj/v)$ must divide P . So we can have at most n such values of j . Thus $a(n+1)/v \geq \varepsilon$. \square

Lemma 15.3. *Let C be a projective variety and let D_1 and D_2 be Cartier divisors on X . Consider the Hilbert polynomial*

$$P(u_1, u_2) = \chi(C, \mathcal{O}_C(u_1D_1 + u_2D_2)).$$

If D_1 is ample, then $P(u_1, u_2)$ is a non-trivial polynomial of total degree $\leq \dim C$. It is because $P(u_1, 0) = h^0(C, \mathcal{O}_C(u_1D_1)) \neq 0$ if u_1 is sufficiently large.

Proof of Theorem 15.1. Let m be a positive integer such that $H' = mH$ is π -very ample. If $H' + r'(K_X + B)$ is π -nef but not π -ample, and $(H' + r'(K_X + B))|_{\text{Nlc}(X, B)}$ is $\pi|_{\text{Nlc}(X, B)}$ -ample, then we have

$$H + r(K_X + B) = \frac{1}{m}(H' + r'(K_X + B)).$$

This gives $r = \frac{1}{m}r'$. Thus, r is rational if and only if r' is rational. Assume furthermore that r' has denominator v . Then r has denominator dividing mv . Since m can be arbitrary sufficiently large integer, this implies that r has denominator dividing v . Therefore, by replacing H with mH , we can assume that H is very ample over S .

For each $(p, q) \in \mathbb{Z}^2$, let $L(p, q)$ denote the relative base locus of the linear system $M(p, q)$ on X (with the reduced scheme structure), that is,

$$L(p, q) = \text{Supp}(\text{Coker}(\pi^* \pi_* \mathcal{O}_X(M(p, q)) \rightarrow \mathcal{O}_X(M(p, q)))),$$

where $M(p, q) = pH + qa(K_X + B)$. By the definition, $L(p, q) = X$ if and only if $\pi_* \mathcal{O}_X(M(p, q)) = 0$.

Claim 1 (cf. [KM, Claim 3.20]). *Let ε be a positive number. For (p, q) sufficiently large and $0 < aq - rp < \varepsilon$, $L(p, q)$ is the same subset of X . We call this subset L_0 . Let $I \subset \mathbb{Z}^2$ be the set of (p, q) for which $0 < aq - rp < 1$ and $L(p, q) = L_0$. We note that I contains all sufficiently large (p, q) with $0 < aq - rp < 1$.*

Proof. We fix $(p_0, q_0) \in \mathbb{Z}^2$ such that $p_0 > 0$ and $0 < aq_0 - rp_0 < 1$. Since H is π -very ample, there exists a positive integer m_0 such that $\mathcal{O}_X(mH + ja(K_X + B))$ is π -generated for every $m > m_0$ and every $0 \leq j \leq q_0 - 1$. Let M be the round-up of

$$\left(m_0 + \frac{1}{r}\right) / \left(\frac{a}{r} - \frac{p_0}{q_0}\right).$$

If $(p', q') \in \mathbb{Z}^2$ such that $0 < aq' - rp' < 1$ and $q' \geq M + q_0 - 1$, then we can write

$$p'H + q'a(K_X + B) = k(p_0H + q_0a(K_X + B)) + (lH + ja(K_X + B))$$

for some $k \geq 0$, $0 \leq j \leq q_0 - 1$ with $l > m_0$. It is because we can uniquely write $q' = kq_0 + j$ with $0 \leq j \leq q_0 - 1$. Thus, we have $kq_0 \geq M$. So, we obtain

$$l = p' - kp_0 > \frac{a}{r}q' - \frac{1}{r} - (kq_0)\frac{p_0}{q_0} \geq \left(\frac{a}{r} - \frac{p_0}{q_0}\right)M - \frac{1}{r} \geq m_0.$$

Therefore, $L(p', q') \subset L(p_0, q_0)$. By the noetherian induction, we obtain the desired closed subset $L_0 \subset X$ and $I \subset \mathbb{Z}^2$. \square

Claim 2. *We have $L_0 \cap \text{Nlc}(X, B) = \emptyset$.*

Proof of Claim 2. We take $(\alpha, \beta) \in \mathbb{Q}^2$ such that $\alpha > 0$, $\beta > 0$, and $\beta a / \alpha > r$ is sufficiently close to r . Then $(\alpha H + \beta a(K_X + B))|_{\text{Nlc}(X, B)}$ is $\pi|_{\text{Nlc}(X, B)}$ -ample because $(H + r(K_X + B))|_{\text{Nlc}(X, B)}$ is $\pi|_{\text{Nlc}(X, B)}$ -ample. If $0 < aq - rp < 1$ and $(p, q) \in \mathbb{Z}^2$ is sufficiently large, then

$$M(p, q) = mM(\alpha, \beta) + (M(p, q) - mM(\alpha, \beta))$$

such that $M(p, q) - mM(\alpha, \beta)$ is π -very ample and that

$$m(\alpha H + \beta a(K_X + B))|_{\text{Nlc}(X, B)}$$

is also $\pi|_{\text{Nlc}(X, B)}$ -very ample. It can be checked by the same argument as in the proof of Claim 1. Therefore, $\mathcal{O}_{\text{Nlc}(X, B)}(M(p, q))$ is π -very ample. Since

$$\pi_* \mathcal{O}_X(M(p, q)) \rightarrow \pi_* \mathcal{O}_{\text{Nlc}(X, B)}(M(p, q))$$

is surjective by the vanishing theorem: Theorem 8.1, we obtain $L(p, q) \cap \text{Nlc}(X, B) = \emptyset$. We note that

$$M(p, q) - (K_X + B) = pH + (qa - 1)(K_X + B)$$

is π -ample because (p, q) is sufficiently large and $aq - rp < 1$. By Claim 1, we have $L_0 \cap \text{Nlc}(X, B) = \emptyset$. \square

Claim 3. *We assume that r is not rational or that r is rational and has denominator $> a(n + 1)$ in reduced form, where $n = \dim X$. Then, for (p, q) sufficiently large and $0 < aq - rp < 1$, $\mathcal{O}_X(M(p, q))$ is π -generated at the generic point of every lc center of (X, B) .*

Proof of Claim 3. We note that

$$M(p, q) - (K_X + B) = pH + (qa - 1)(K_X + B).$$

If $aq - rp < 1$ and (p, q) is sufficiently large, then $M(p, q) - (K_X + B)$ is π -ample. Let C be an lc center of (X, B) . We note that we can assume $C \cap \text{Nlc}(X, B) = \emptyset$ by Claim 2. Then $P_{C_\eta}(p, q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$ is a non-zero polynomial of degree at most $\dim C_\eta \leq \dim X$ by Lemma 15.3. Note that C_η is the generic fiber of $C \rightarrow \pi(C)$. By Lemma 15.2, there exists (p, q) such that $P_{C_\eta}(p, q) \neq 0$, (p, q) sufficiently large, and $0 < aq - rp < 1$. By the π -ampleness of $M(p, q) - (K_X + B)$,

$$P_{C_\eta}(p, q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$$

and

$$\pi_* \mathcal{O}_X(M(p, q)) \rightarrow \pi_* \mathcal{O}_C(M(p, q))$$

is surjective by Theorem 8.1. We note that $C \cap \text{Nlc}(X, B) = \emptyset$. Therefore, we see that $\mathcal{O}_X(M(p, q))$ is π -generated at the generic point of C . By combining this with Claim 1, $\mathcal{O}_X(M(p, q))$ is π -generated at the generic point of every lc center of (X, B) if (p, q) is sufficiently large with $0 < aq - rp < 1$. So, we obtain Claim 3. \square

Note that $\mathcal{O}_X(M(p, q))$ is not π -generated for $(p, q) \in I$ because $M(p, q)$ is not π -nef. Therefore, $L_0 \neq \emptyset$. We shrink S to an affine open subset intersecting $\pi(L_0)$. Let D_1, \dots, D_{n+1} be general members of

$$\pi_* \mathcal{O}_X(M(p_0, q_0)) = H^0(X, \mathcal{O}_X(M(p_0, q_0)))$$

with $(p_0, q_0) \in I$. We can check that $K_X + B + \sum_{i=1}^{n+1} D_i$ is not lc at the generic point of every irreducible component of L_0 by Lemma 13.2. On the other hand, $K_X + B + \sum_{i=1}^{n+1} D_i$ is lc outside $L_0 \cup \text{Nlc}(X, B)$ since D_i is a general member of

$|M(p_0, q_0)|$ for every i . Let $0 < c < 1$ be the log canonical threshold of (X, B) with respect to $D = \sum_{i=1}^{n+1} D_i$ outside $\text{Nlc}(X, B)$. Note that $c > 0$ by Claim 3. Thus, the pair $(X, B + cD)$ has some lc centers contained in L_0 . Let C be an lc center contained in L_0 . We note that $\mathcal{J}_{NLC}(X, B + cD) = \mathcal{J}_{NLC}(X, B)$ by Proposition 7.5 and that $C \cap \text{Nlc}(X, B + cD) = C \cap \text{Nlc}(X, B) = \emptyset$. We consider

$$K_X + B + cD = c(n+1)p_0H + (1 + c(n+1)q_0a)(K_X + B).$$

Thus we have

$$\begin{aligned} pH + qa(K_X + B) - (K_X + B + cD) \\ = (p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0a))(K_X + B). \end{aligned}$$

If p and q are large enough and $0 < aq - rp \leq aq_0 - rp_0$, then

$$pH + qa(K_X + B) - (K_X + B + cD)$$

is π -ample. It is because

$$\begin{aligned} (p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0a))(K_X + B) \\ = (p - (1 + c(n+1)p_0)H + (qa - (1 + c(n+1)q_0a))(K_X + B) \\ + p_0H + (q_0a - 1)(K_X + B). \end{aligned}$$

Suppose that r is not rational. There must be arbitrarily large (p, q) such that $0 < aq - rp < \varepsilon = aq_0 - rp_0$ and $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) \neq 0$ by Lemma 15.2 because $P_{C_\eta}(p, q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$ is a non-trivial polynomial of degree at most $\dim C_\eta$ by Lemma 15.3. Since $M(p, q) - (K_X + B + cD)$ is π -ample by $0 < aq - rp < aq_0 - rp_0$, we have $h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) \neq 0$ by the vanishing theorem: Theorem 8.1. By the vanishing theorem: Theorem 8.1,

$$\pi_* \mathcal{O}_X(M(p, q)) \rightarrow \pi_* \mathcal{O}_C(M(p, q))$$

is surjective because $M(p, q) - (K_X + B + cD)$ is π -ample. We note that $C \cap \text{Nlc}(X, B + cD) = \emptyset$. Thus C is not contained in $L(p, q)$. Therefore, $L(p, q)$ is a proper subset of $L(p_0, q_0) = L_0$, giving the desired contradiction. So now we know that r is rational.

We next suppose that the assertion of the theorem concerning the denominator of r is false. We choose $(p_0, q_0) \in I$ such that $aq_0 - rp_0$ is the maximum, say it is equal to d/v . If $0 < aq - rp \leq d/v$ and (p, q) is sufficiently large, then $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$ since $M(p, q) - (K_X + B + cD)$ is π -ample. There exists sufficiently large (p, q) in the strip $0 < aq - rp < 1$ with $\varepsilon = 1$ for which $h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q))) \neq 0$ by Lemma 15.2

since $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p, q)))$ is a non-trivial polynomial of degree at most $\dim C_\eta$ by Lemma 15.3. Note that $aq - rp \leq d/v = aq_0 - rp_0$ holds automatically for $(p, q) \in I$. Since

$$\pi_* \mathcal{O}_X(M(p, q)) \rightarrow \pi_* \mathcal{O}_C(M(p, q))$$

is surjective by the π -ampleness of $M(p, q) - (K_X + B + cD)$, we obtain the desired contradiction by the same reason as above. So, we finish the proof of the rationality theorem. \square

We close this section with an important remark, which is indispensable for the proof of the cone theorem: Theorem 16.6.

Remark 15.4. In Theorem 15.1, it is sufficient to assume that B is an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -linearly equivalent to a \mathbb{Q} -Cartier \mathbb{Q} -divisor ω on X with the condition that $a\omega$ is Cartier. All we have to do is to replace $a(K_X + B)$ with $a\omega$ in the proof of Theorem 15.1. We need this generalization in the proof of the cone theorem: Theorem 16.6.

§16. Cone theorem

The main theorem of this section is the cone theorem. Before we state the main theorem, let us fix the notation.

Definition 16.1. Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier. Let $\pi : X \rightarrow S$ be a projective morphism. We put

$$\overline{NE}(X/S)_{\text{Nlc}(X, B)} = \text{Im}(\overline{NE}(\text{Nlc}(X, B)/S) \rightarrow \overline{NE}(X/S)).$$

For an \mathbb{R} -Cartier \mathbb{R} -divisor D , we define

$$D_{\geq 0} = \{z \in N_1(X/S) \mid D \cdot z \geq 0\}.$$

Similarly, we can define $D_{> 0}$, $D_{\leq 0}$, and $D_{< 0}$. We also define

$$D^\perp = \{z \in N_1(X/S) \mid D \cdot z = 0\}.$$

We use the following notation

$$\overline{NE}(X/S)_{D \geq 0} = \overline{NE}(X/S) \cap D_{\geq 0},$$

and similarly for > 0 , ≤ 0 , and < 0 .

Definition 16.2. An *extremal face* of $\overline{NE}(X/S)$ is a non-zero subcone $F \subset \overline{NE}(X/S)$ such that $z, z' \in F$ and $z + z' \in F$ imply that $z, z' \in F$. Equivalently,

$F = \overline{NE}(X/S) \cap H^\perp$ for some π -nef \mathbb{R} -divisor H , which is called a *supporting function* of F . An *extremal ray* is a one-dimensional extremal face.

- (1) An extremal face F is called $(K_X + B)$ -*negative* if

$$F \cap \overline{NE}(X/S)_{K_X+B \geq 0} = \{0\}.$$

- (2) An extremal face F is called *rational* if we can choose a π -nef \mathbb{Q} -divisor H as a support function of F .
 (3) An extremal face F is called *relatively ample at* $\mathrm{Nlc}(X, B)$ if

$$F \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X, B)} = \{0\}.$$

Equivalently, $H|_{\mathrm{Nlc}(X, B)}$ is $\pi|_{\mathrm{Nlc}(X, B)}$ -ample for every supporting function H of F .

- (4) An extremal face F is called *contractible at* $\mathrm{Nlc}(X, B)$ if it has a rational supporting function H such that $H|_{\mathrm{Nlc}(X, B)}$ is $\pi|_{\mathrm{Nlc}(X, B)}$ -semi-ample.

Remark 16.3. If X is complete but non-projective, then it sometimes happens that $\overline{NE}(X) = N_1(X)$ even when X is smooth (cf. [FP]). Therefore, the projectivity is crucial for the log minimal model program.

The following theorem is a direct consequence of Theorem 13.1.

Theorem 16.4 (Contraction theorem). *Let X be a normal variety, let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier, and let $\pi : X \rightarrow S$ be a projective morphism. Let H be a π -nef Cartier divisor such that $F = H^\perp \cap \overline{NE}(X/S)$ is $(K_X + B)$ -negative and contractible at $\mathrm{Nlc}(X, B)$. Then there exists a projective morphism $\varphi_F : X \rightarrow Y$ over S with the following properties.*

- (1) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*
 (2) $\mathcal{O}_Y \simeq (\varphi_F)_* \mathcal{O}_X$.
 (3) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Assume that $L^{\otimes m}|_{\mathrm{Nlc}(X, B)}$ is $\varphi_F|_{\mathrm{Nlc}(X, B)}$ -generated for $m \gg 0$. Then there is a line bundle L_Y on Y such that $L \simeq \varphi_F^* L_Y$.*

Proof. By the assumption, $qH - (K_X + B)$ is π -ample for some positive integer q and we can assume that $H|_{\mathrm{Nlc}(X, B)}$ is $\pi|_{\mathrm{Nlc}(X, B)}$ -semi-ample. By Theorem 13.1, $\mathcal{O}_X(mH)$ is π -generated for some $m > 0$. We take the Stein factorization of the associated morphism. Then, we have the contraction morphism $\varphi_F : X \rightarrow Y$ with the properties (1) and (2).

We consider $\varphi_F : X \rightarrow Y$ and $\overline{NE}(X/Y)$. Then $\overline{NE}(X/Y) = F$, L is numerically trivial over Y , and $-(K_X + B)$ is φ_F -ample. Applying the base point free theorem (cf. Theorem 13.1) over Y , both $L^{\otimes m}$ and $L^{\otimes(m+1)}$ are pull-backs of line bundles on Y . Their difference gives a line bundle L_Y such that $L \simeq \varphi_F^* L_Y$. \square

Example 16.5. Let S be a cone over a smooth cubic curve and let $\pi : X \rightarrow S$ be the blow-up at the vertex of S . Then $K_X + E = \pi^* K_S$, where E is the π -exceptional curve. We put $B = 2E$ and consider the pair (X, B) . In this case, $\varphi_F = \pi : X \rightarrow Y = S$ with $F = 0^\perp \cap \overline{NE}(X/S) = \overline{NE}(X/S) = \mathbb{R}_{\geq 0}[E]$ is an example of contraction morphisms in Theorem 16.4.

The time is ripe to state one of the main theorems in this paper.

Theorem 16.6 (Cone theorem). *Let X be a normal variety, let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier, and let $\pi : X \rightarrow S$ be a projective morphism. Then we have the following properties.*

- (1) $\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+B \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)} + \sum R_j$, where R_j 's are the $(K_X + B)$ -negative extremal rays of $\overline{NE}(X/S)$ that are rational and relatively ample at $\text{Nlc}(X, B)$. In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point.
- (2) Let H be a π -ample \mathbb{R} -divisor on X . Then there are only finitely many R_j 's included in $(K_X + B + H)_{< 0}$. In particular, the R_j 's are discrete in the half-space $(K_X + B)_{< 0}$.
- (3) Let F be a $(K_X + B)$ -negative extremal face of $\overline{NE}(X/S)$ that is relatively ample at $\text{Nlc}(X, B)$. Then F is a rational face. In particular, F is contractible at $\text{Nlc}(X, B)$.

Proof. First, we assume that $K_X + B$ is \mathbb{R} -linearly equivalent to a \mathbb{Q} -Cartier \mathbb{Q} -divisor on X (see Remark 15.4). We can assume that $\dim_{\mathbb{R}} N_1(X/S) \geq 2$ and $K_X + B$ is not π -nef. Otherwise, the theorem is obvious.

Step 1. We have

$$\overline{NE}(X/S) = \overline{\overline{NE}(X/S)_{K_X+B \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)} + \sum_F F},$$

where F 's vary among all rational proper $(K_X + B)$ -negative faces that are relatively ample at $\text{Nlc}(X, B)$ and $\overline{\quad}$ denotes the closure with respect to the real topology.

Proof. We put

$$\mathfrak{B} = \overline{\overline{NE}(X/S)_{K_X+B \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)} + \sum_F F}.$$

It is clear that $\overline{NE}(X/S) \supset \mathfrak{B}$. We note that each F is spanned by curves on X mapped to points on S by Theorem 16.4 (1). Supposing $\overline{NE}(X/S) \neq \mathfrak{B}$, we shall derive a contradiction. There is a separating function M which is Cartier and is not a multiple of $K_X + B$ in $N^1(X/S)$ such that $M > 0$ on $\mathfrak{B} \setminus \{0\}$ and $M \cdot z_0 < 0$ for some $z_0 \in \overline{NE}(X/S)$. Let C be the dual cone of $\overline{NE}(X/S)_{K_X+B \geq 0}$, that is,

$$C = \{D \in N^1(X/S) \mid D \cdot z \geq 0 \text{ for } z \in \overline{NE}(X/S)_{K_X+B \geq 0}\}.$$

Then we see that C is generated by π -nef divisors and $K_X + B$. Since $M > 0$ on $\overline{NE}(X/S)_{K_X+B \geq 0} \setminus \{0\}$, M is in the interior of C , and hence there exists a π -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor A such that $M - A = L' + p(K_X + B)$ in $N^1(X/S)$, where L' is a π -nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X and p is a non-negative rational number. Therefore, M is expressed in the form $M = H + p(K_X + B)$ in $N^1(X/S)$, where $H = A + L'$ is a π -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. The rationality theorem (see Theorem 15.1) implies that there exists a positive rational number $r < p$ such that $L = H + r(K_X + B)$ is π -nef but not π -ample, and $L|_{\text{Nlc}(X,B)}$ is $\pi|_{\text{Nlc}(X,B)}$ -ample. Note that $L \neq 0$ in $N^1(X/S)$, since M is not a multiple of $K_X + B$. Thus the extremal face F_L associated to the supporting function L is contained in \mathfrak{B} , which implies $M > 0$ on F_L . Therefore, $p < r$. It is a contradiction. This completes the proof of our first claim. \square

Step 2. In the equality of Step 1, we can assume that every extremal face F is one-dimensional.

Proof. Let F be a rational proper $(K_X + B)$ -negative extremal face that is relatively ample at $\text{Nlc}(X, B)$, and assume that $\dim F \geq 2$. Let $\varphi_F : X \rightarrow W$ be the associated contraction. Note that $-(K_X + B)$ is φ_F -ample. By Step 1, we obtain

$$F = \overline{NE}(X/W) = \overline{\sum_G G},$$

where the G 's are the rational proper $(K_X + B)$ -negative extremal faces of the cone $\overline{NE}(X/W)$. We note that $\overline{NE}(X/W)_{\text{Nlc}(X,B)} = 0$ because φ_F embeds $\text{Nlc}(X, B)$ into W . The G 's are also $(K_X + B)$ -negative extremal faces of $\overline{NE}(X/S)$ that are ample at $\text{Nlc}(X, B)$, and $\dim G < \dim F$. By induction, we obtain

$$(\clubsuit) \quad \overline{NE}(X/S) = \overline{\overline{NE}(X/S)_{K_X+B \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)} + \sum R_j},$$

where the R_j 's are $(K_X + B)$ -negative rational extremal rays. Note that each R_j does not intersect $\overline{NE}(X/S)_{\text{Nlc}(X,B)}$. \square

Step 3. The contraction theorem (cf. Theorem 16.4) guarantees that for each extremal ray R_j there exists a reduced irreducible curve C_j on X such that $[C_j] \in R_j$. Let $\psi_j : X \rightarrow W_j$ be the contraction morphism of R_j , and let A be a π -ample Cartier divisor. We set

$$r_j = -\frac{A \cdot C_j}{(K_X + B) \cdot C_j}.$$

Then $A + r_j(K_X + B)$ is ψ_j -nef but not ψ_j -ample, and $(A + r_j(K_X + B))|_{\text{Nlc}(X,B)}$ is $\psi_j|_{\text{Nlc}(X,B)}$ -ample. By the rationality theorem (see Theorem 15.1), expressing $r_j = u_j/v_j$ with $u_j, v_j \in \mathbb{Z}_{>0}$ and $(u_j, v_j) = 1$, we have the inequality $v_j \leq a(\dim X + 1)$.

Step 4. Now take π -ample Cartier divisors $H_1, H_2, \dots, H_{\rho-1}$ such that $K_X + B$ and the H_i 's form a basis of $N^1(X/S)$, where $\rho = \dim_{\mathbb{R}} N^1(X/S)$. By Step 3, the intersection of the extremal rays R_j with the hyperplane

$$\{z \in N_1(X/S) \mid a(K_X + B) \cdot z = -1\}$$

in $N_1(X/S)$ lie on the lattice

$$\Lambda = \{z \in N_1(X/S) \mid a(K_X + B) \cdot z = -1, H_i \cdot z \in (a(a(\dim X + 1))!)^{-1}\mathbb{Z}\}.$$

This implies that the extremal rays are discrete in the half space

$$\{z \in N_1(X/S) \mid (K_X + B) \cdot z < 0\}.$$

Thus we can omit the closure sign --- from the formula (\clubsuit) and this completes the proof of (1) when $K_X + B$ is \mathbb{R} -linearly equivalent to a \mathbb{Q} -Cartier \mathbb{Q} -divisor.

Step 5. Let H be a π -ample \mathbb{R} -divisor on X . We choose $0 < \varepsilon_i \ll 1$ for $1 \leq i \leq \rho - 1$ such that $H - \sum_{i=1}^{\rho-1} \varepsilon_i H_i$ is π -ample. Then the R_j 's included in $(K_X + B + H)_{<0}$ correspond to some elements of the above lattice Λ for which $\sum_{i=1}^{\rho-1} \varepsilon_i H_i \cdot z < 1/a$. Therefore, we obtain (2).

Step 6. Let F be a $(K_X + B)$ -negative extremal face as in (3). The vector space $V = F^\perp \subset N^1(X/S)$ is defined over \mathbb{Q} because F is generated by some of the R_j 's. There exists a π -ample \mathbb{R} -divisor H such that F is contained in $(K_X + B + H)_{<0}$. Let $\langle F \rangle$ be the vector space spanned by F . We put

$$W_F = \overline{NE}(X/S)_{K_X+B+H \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)} + \sum_{R_j \not\subset F} R_j.$$

Then W_F is a closed cone, $\overline{NE}(X/S) = W_F + F$, and $W_F \cap \langle F \rangle = \{0\}$. The supporting functions of F are the elements of V that are positive on $W_F \setminus \{0\}$.

This is a non-empty open set and thus it contains a rational element that, after scaling, gives a π -nef Cartier divisor L such that $F = L^\perp \cap \overline{NE}(X/S)$. Therefore, F is rational. So, we have (3).

From now on, $K_X + B$ is \mathbb{R} -Cartier.

Step 7. Let H be a π -ample \mathbb{R} -divisor on X . We shall prove (2). We assume that there are infinitely many R_j 's in $(K_X + B + H)_{<0}$ and get a contradiction. There exists an affine open subset U of S such that $\overline{NE}(\pi^{-1}(U)/U)$ has infinitely many $(K_X + B + H)$ -negative extremal rays. So, we shrink S and can assume that S is affine. We can write $H = E + H'$ such that H' is π -ample, $\mathcal{J}_{NLC}(X, B + E) = \mathcal{J}_{NLC}(X, B)$, and $K_X + B + E$ is \mathbb{R} -linearly equivalent to a \mathbb{Q} -Cartier \mathbb{Q} -divisor. Since $K_X + B + H = K_X + B + E + H'$, we have

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+B+H \geq 0} + \overline{NE}(X/S)_{\text{Nlc}(X,B)} + \sum_{\text{finite}} R_j.$$

It is a contradiction. Thus, we obtain (2). The statement (1) is a direct consequence of (2). Of course, (3) holds by Step 6 once we obtain (1).

So, we complete the proof of the cone theorem. \square

We close this section with the following elementary example.

Example 16.7. We consider $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_i : Y \rightarrow \mathbb{P}^1$ be the i -th projection for $i = 1, 2$. Let F_i be a fiber of π_i for $i = 1, 2$. We put $P = F_1 \cap F_2$ and consider the blow-up $f : X \rightarrow Y$ at P . Let E be the exceptional curve of f and $C_i = f_*^{-1}F_i$ for $i = 1, 2$. In this situation, we can check that $-K_X$ is ample, $\rho(X) = 3$, and

$$\overline{NE}(X) = \mathbb{R}_{\geq 0}[C_1] + \mathbb{R}_{\geq 0}[C_2] + \mathbb{R}_{\geq 0}[E].$$

We put

$$B = \frac{3}{2}E + \frac{1}{2}C_1 + C_2.$$

Then we have

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B \geq 0} + \overline{NE}(X)_{\text{Nlc}(X,B)} + \mathbb{R}_{\geq 0}[C_2],$$

where

$$\overline{NE}(X)_{\text{Nlc}(X,B)} = \mathbb{R}_{\geq 0}[E], \quad \overline{NE}(X)_{K_X+B \geq 0} = \mathbb{R}_{\geq 0}[C_1],$$

and

$$C_2 \cdot (K_X + B) < 0.$$

§17. Base point free theorem revisited

This section is a supplement to the base point free theorem: Theorem 13.1. In the recent log minimal model program (cf. [S3], [BCHM], and so on), we frequently use \mathbb{R} -divisors. Therefore, the following theorem is useful.

Theorem 17.1 (Base point free theorem for \mathbb{R} -divisors). *Let X be a normal variety and let B be an effective \mathbb{R} -divisor on X such that (X, B) is log canonical, and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Let D be a π -nef \mathbb{R} -Cartier \mathbb{R} -divisor on X such that $aD - (K_X + B)$ is π -ample for some real number $a > 0$. Then D is π -semi-ample.*

Proof. We can assume that $a = 1$ by replacing D with aD . We put

$$F = \{z \in \overline{NE}(X/S) \mid D \cdot z = 0\}.$$

Then F is a face of $\overline{NE}(X/S)$ and $(K_X + B) \cdot z < 0$ for $z \in F$. We claim that F contains only finitely many $(K_X + B)$ -negative extremal rays R_1, \dots, R_k of $\overline{NE}(X/S)$. If F contains infinitely many $(K_X + B)$ -negative extremal rays of $\overline{NE}(X/S)$, then it also holds after shrinking S suitably. Therefore, we can assume that S is affine. In this situation, X is quasi-projective. We take a general small π -ample \mathbb{Q} -divisor A on X such that $D - (K_X + B + A)$ is π -ample and that $(X, B + A)$ is log canonical. Let R be a $(K_X + B)$ -negative extremal ray such that $R \subset F$. Then R is a $(K_X + B + A)$ -negative extremal ray since $D \cdot R = 0$ and $D - (K_X + B + A)$ is π -ample. On the other hand, there are only finitely many $(K_X + B + A)$ -negative extremal rays in $\overline{NE}(X/S)$ by Theorem 16.6 (2). It is a contradiction. Therefore, F is spanned by the extremal rays R_1, \dots, R_k . We consider the finite dimensional real vector space $V = \bigoplus_j \mathbb{R}D_j$, where $\sum_j D_j = \text{Supp}D$ is the irreducible decomposition.

Then

$$\mathcal{R} = \{E \in V \mid E \text{ is } \mathbb{R}\text{-Cartier and } E \cdot z = 0 \text{ for every } z \in F\}$$

is a rational affine subspace of V and $D \in \mathcal{R}$. Thus, we can find positive real numbers r_1, r_2, \dots, r_m and π -nef \mathbb{Q} -Cartier \mathbb{Q} -divisors E_1, E_2, \dots, E_m such that $D = \sum_{i=1}^m r_i E_i$ and that $E_i - (K_X + B)$ is π -ample for every i (cf. Step 6 in the proof of Theorem 16.6). By Theorem 13.1, E_i is a π -semi-ample \mathbb{Q} -Cartier \mathbb{Q} -divisor for every i . Therefore, D is π -semi-ample. \square

§18. Lengths of extremal rays

In this section, we discuss estimates of lengths of extremal rays. It is indispensable for the log minimal model program with scaling (see, for example, [BCHM]). Some results in this section have already been obtained in [Ko2], [Ko3], [Ka2], [S3], [S4],

and [B1] with some extra assumptions. We note that the formulation of the main theorem of this section (cf. Theorem 18.10) is new.

Let us recall the following easy lemma.

Lemma 18.1 (cf. [S4, Lemma 1]). *Let (X, B) be a log canonical pair, where B is an \mathbb{R} -divisor. Then there are positive real numbers r_i and effective \mathbb{Q} -divisors B_i for $1 \leq i \leq l$ and a positive integer m such that $\sum_{i=1}^l r_i = 1$, $K_X + B = \sum_{i=1}^l r_i(K_X + B_i)$, (X, B_i) is lc for every i , and $m(K_X + B_i)$ is Cartier for every i .*

Proof. Let $\sum_k D_k$ be the irreducible decomposition of $\text{Supp} B$. We consider the finite dimensional real vector space $V = \bigoplus_k \mathbb{R}D_k$. We put

$$\mathcal{Q} = \{D \in V \mid K_X + D \text{ is } \mathbb{R}\text{-Cartier}\}.$$

Then it is easy to see that \mathcal{Q} is an affine subspace of V defined over \mathbb{Q} . We put

$$\mathcal{P} = \{D \in \mathcal{Q} \mid K_X + D \text{ is log canonical}\}.$$

Thus by the definition of log canonicity, it is also easy to check that \mathcal{P} is a closed convex rational polytope in V . We note that \mathcal{P} is compact in the classical topology of V . By the assumption, $B \in \mathcal{P}$. Therefore, we can find the desired \mathbb{Q} -divisors $B_i \in \mathcal{P}$ and positive real numbers r_i . \square

The next result is essentially due to [Ka2] and [S4, Proposition 1]. We will prove a more general result in Theorem 18.10 whose proof depends on Theorem 18.2.

Theorem 18.2. *Let (X, B) be an lc pair and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Let R be a $(K_X + B)$ -negative extremal ray of $\overline{NE}(X/S)$. Then we can find a rational curve C on X such that $[C] \in R$ and*

$$0 < -(K_X + B) \cdot C \leq 2 \dim X.$$

Proof. By shrinking S , we can assume that S is quasi-projective. By replacing $\pi : X \rightarrow S$ with the extremal contraction $\varphi_R : X \rightarrow Y$ over S , we can assume that the relative Picard number $\rho(X/S) = 1$. In particular, $-(K_X + B)$ is π -ample. Let $K_X + B = \sum_{i=1}^l r_i(K_X + B_i)$ be as in Lemma 18.1. Without loss of generality, we can assume that $-(K_X + B_1)$ is π -ample and $-(K_X + B_i) = -s_i(K_X + B_1)$ in $N^1(X/S)$ with $s_i \leq 1$ for every $i \geq 2$. Thus, it is sufficient to find a rational curve C such that $\pi(C)$ is a point and that $-(K_X + B_1) \cdot C \leq 2 \dim X$. So, we can assume that $K_X + B$ is \mathbb{Q} -Cartier and lc. By Theorem 10.4, there is a birational

morphism $f : (V, B_V) \rightarrow (X, B)$ such that $K_V + B_V = f^*(K_X + B)$, V is \mathbb{Q} -factorial, and (V, B_V) is dlt. By [Ka2, Theorem 1] and [Ma, Theorem 10-2-1], we can find a rational curve C' on V such that $-(K_V + B_V) \cdot C' \leq 2 \dim V = 2 \dim X$ and that C' spans a $(K_V + B_V)$ -negative extremal ray. By the projection formula, the f -image of C' is a desired rational curve. So, we finish the proof. \square

Remark 18.3. It is conjectured that the estimate $\leq 2 \dim X$ in Theorem 18.2 should be replaced by $\leq \dim X + 1$. When X is smooth projective, it is true by Mori's famous result (cf. [Mo]). See, for example, [KM, Theorem 1.13]. When X is a toric variety, it is also true by [F2] and [F5].

Remark 18.4. In the proof of Theorem 18.2, we need Kawamata's estimate on the length of an extremal rational curve (cf. [Ka2, Theorem 1] and [Ma, Theorem 10-2-1]). It depends on Mori's bend and break technique to create rational curves. So, we need the mod p reduction technique there.

Remark 18.5. We give a remark on [BCHM]. We use the same notation as in [BCHM, 3.8]. In the proof of [BCHM, Corollary 3.8.2], we can assume that $K_X + \Delta$ is klt by [BCHM, Lemma 3.7.4]. By perturbing the coefficients of B slightly, we can further assume that B is a \mathbb{Q} -divisor. By applying the usual cone theorem to the klt pair (X, B) , we obtain that there are only finitely many $(K_X + \Delta)$ -negative extremal rays of $\overline{NE}(X/U)$. We note that [BCHM, Theorem 3.8.1] is only used in the proof of [BCHM, Corollary 3.8.2]. Therefore, we do not need the estimate of lengths of extremal rays in [BCHM]. In particular, we do not need mod p reduction arguments for the proof of the main results in [BCHM].

By the proof of Theorem 18.2, we have the following corollary.

Corollary 18.6. *Let (X, B) be a log canonical pair and let*

$$K_X + B = \sum_{i=1}^l r_i (K_X + B_i)$$

and m be as in Lemma 18.1. Let $\varphi : X \rightarrow Y$ be a projective surjective morphism with connected fibers such that the relative Picard number $\rho(X/Y) = 1$. Then we can find a curve C on X such that C spans $N_1(X/Y)$ and

$$-(K_X + B_i) \cdot C = \frac{n_i}{m}$$

with $n_i \leq 2m \dim X$ for every i . Of course, we have

$$-(K_X + B) \cdot C = \sum_i \frac{r_i n_i}{m} \leq 2 \dim X.$$

If $-(K_X + B_i)$ is φ -ample for some i , then we can find a rational curve C in the above statement. We note that φ is not necessarily assumed to be a $(K_X + B)$ -negative extremal contraction.

The following important lemma is a very special case of [S3, 6.2. First Main Theorem].

Lemma 18.7. *Let (X, B) be a log canonical pair and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . We take $\sum_k D_k$ such that $\text{Supp} B \subset \sum_k D_k$, where D_i is an irreducible Weil divisor for every i and $D_i \neq D_j$ for every $i \neq j$. We put*

$$\mathcal{P} = \left\{ \sum_k d_k D_k ; 0 \leq d_k \leq 1 \text{ for all } k \text{ and } K_X + \sum_k d_k D_k \text{ is lc} \right\}.$$

Then \mathcal{P} is a closed convex rational polytope.

Let $\{R_j\}$ be any set of $(K_X + B)$ -negative extremal rays of the lc pair (X, B) over S . We put

$$\mathcal{N} = \bigcap_j \left\{ \sum_k d_k D_k \in \mathcal{P}; (K_X + \sum_k d_k D_k) \cdot R_j \geq 0 \right\}.$$

Then \mathcal{N} is a closed convex subset of \mathcal{P} .

We take $B' \in \mathcal{P}$. Let \mathcal{F} be the minimal face of \mathcal{P} containing B' . Assume that $(K_X + B') \cdot R_j > 0$ for every j . Then there is an open subset U of \mathcal{F} in the classical topology such that $B' \in U \subset \mathcal{N} \cap \mathcal{F}$. In particular, we can write

$$K_X + B' = \sum_{i=1}^{d+1} r'_i (K_X + B'_i)$$

with the following properties.

- (a) $d = \dim \mathcal{F}$.
- (b) $B'_i \in \mathcal{F}$ for every i .
- (c) $m'(K_X + B'_i)$ is Cartier for some positive integer m' for every i .
- (d) $\sum_{i=1}^{d+1} r'_i = 1$ and $0 \leq r'_i \leq 1$ for every i .
- (e) $(K_X + B'_i) \cdot R_j > 0$ for every i and j .

Proof. It is obvious that \mathcal{P} is a closed convex rational polytope (see the proof of Lemma 18.1). By the definition, \mathcal{N} is a closed convex subset of \mathcal{P} . Since \mathcal{F} is a face of \mathcal{P} and contains B' , we can take a d -dimensional rational simplex spanned by Δ_i for $1 \leq i \leq d+1$ in \mathcal{F} containing B' inside it. Thus, we can write

$$K_X + B' = \sum_{i=1}^{d+1} r_i (K_X + \Delta_i)$$

such that $\sum_{i=1}^{d+1} r_i = 1$ and $0 < r_i < 1$ for every i , and $m(K_X + \Delta_i)$ is Cartier for every i , where m is a positive integer.

We take an extremal ray R_j . By Corollary 18.6, we can find a curve C_j on X such that C_j spans R_j and that $m(K_X + \Delta_i) \cdot C_j = n_{ij}$ with $n_{ij} \geq -2m \dim X$ for every i . By the assumption, we have

$$(K_X + B') \cdot C_j = \sum_i \frac{r_i n_{ij}}{m} > 0.$$

We define

$$\alpha = \inf \left\{ \sum_i \frac{r_i n_i}{m} > 0 ; n_i \geq -2m \dim X \text{ and } n_i \in \mathbb{Z} \text{ for every } i \right\}.$$

Then we obtain $\alpha > 0$. We put

$$c = \frac{\alpha}{2 \dim X + \alpha + 1} > 0.$$

It is obvious that

$$B' + c(\Delta_i - B') \in \mathcal{F}$$

for every i since $0 < c < 1$ and that

$$(K_X + B' + c(\Delta_i - B')) \cdot C_j > 0$$

for every i and j by the definition of c . Thus, the d -dimensional simplex spanned by $B' + c(\Delta_i - B')$ for $1 \leq i \leq d+1$ is contained in $\mathcal{N} \cap \mathcal{F}$ and contains B' in its interior. So, the interior of the above simplex is a desired open set contained in $\mathcal{N} \cap \mathcal{F}$. Thus, we can write

$$K_X + B' = \sum_{i=1}^{d+1} r'_i (K_X + B'_i)$$

with the required properties. \square

Remark 18.8. In [S3, 6.2. First Main Theorem], it is proved that \mathcal{N} is a closed convex rational polytope. We recommend the reader to see [B2, Section 3] for details. The arguments in [B2, Section 3] work for lc pairs by Theorem 18.2 (see, for example, [F11]).

By Corollary 18.6 and Lemma 18.7, Lemma 2.6 in [B1] holds for lc pairs. It may be useful for the log minimal model program with scaling. We follow Birkar's proof in [BP].

Theorem 18.9 (cf. [B1, Lemma 2.6]). *Let (X, B) be an lc pair, let B be an \mathbb{R} -divisor, and let $\pi : X \rightarrow S$ be a projective morphism between algebraic varieties.*

Let H be an effective \mathbb{R} -Cartier \mathbb{R} -divisor on X such that $K_X + B + H$ is π -nef and $(X, B + H)$ is lc. Then, either $K_X + B$ is also π -nef or there is a $(K_X + B)$ -negative extremal ray R such that $(K_X + B + \lambda H) \cdot R = 0$, where

$$\lambda := \inf\{t \geq 0 \mid K_X + B + tH \text{ is } \pi\text{-nef}\}.$$

Of course, $K_X + B + \lambda H$ is π -nef.

Proof. Assume that $K_X + B$ is not π -nef. Let $\{R_j\}$ be the set of $(K_X + B)$ -negative extremal rays over S . Let C_j be the rational curve spanning R_j with the estimate as in Corollary 18.6 for every j . We put $\mu = \sup_j \{\mu_j\}$, where

$$\mu_j = \frac{-(K_X + B) \cdot C_j}{H \cdot C_j}.$$

Obviously, $\lambda = \mu$ and $0 < \mu \leq 1$. So, it is sufficient to prove that $\mu = \mu_l$ for some l . By Corollary 18.6, there are positive real numbers r_1, \dots, r_l and a positive integer m , which are independent of j , such that

$$-(K_X + B) \cdot C_j = \sum_{i=1}^l \frac{r_i n_{ij}}{m} > 0,$$

where n_{ij} is an integer with $n_{ij} \leq 2m \dim X$ for every i and j . If $(K_X + B + H) \cdot R_l = 0$ for some l , then there are nothing to prove since $\lambda = 1$ and $(K_X + B + H) \cdot R = 0$ with $R = R_l$. Thus, we assume that $(K_X + B + H) \cdot R_j > 0$ for every j . Therefore, we can apply Lemma 18.7 and obtain

$$K_X + B + H = \sum_{p=1}^q r'_p (K_X + \Delta_p),$$

where r'_1, \dots, r'_q are positive real numbers, (X, Δ_p) is lc for every p , $m'(K_X + \Delta_p)$ is Cartier for some positive integer m' and every p , and $(K_X + \Delta_p) \cdot C_j > 0$ for every p and j . So, we obtain

$$(K_X + B + H) \cdot C_j = \sum_{p=1}^q \frac{r'_p n'_{pj}}{m'}$$

with $0 < n'_{pj} = m'(K_X + \Delta_p) \cdot C_j \in \mathbb{Z}$. Note that m' and r'_p are independent of j for every p . We also note that

$$\begin{aligned} \frac{1}{\mu_j} &= \frac{H \cdot C_j}{-(K_X + B) \cdot C_j} = \frac{(K_X + B + H) \cdot C_j}{-(K_X + B) \cdot C_j} + 1 \\ &= \frac{m \sum_{p=1}^q r'_p n'_{pj}}{m' \sum_{i=1}^l r_i n_{ij}} + 1. \end{aligned}$$

Since

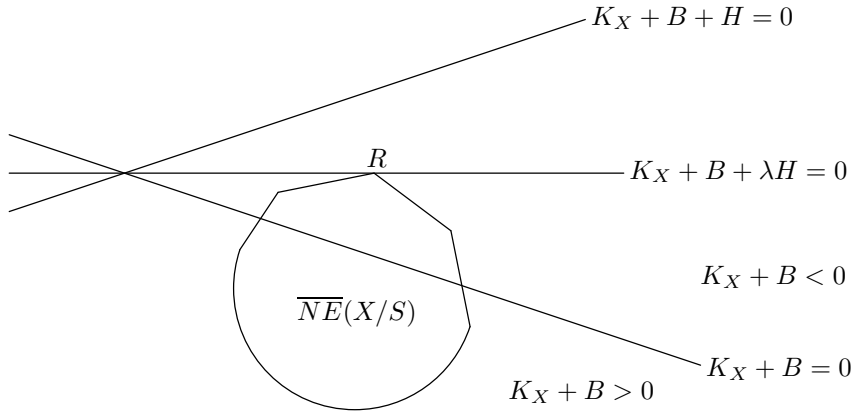
$$\sum_{i=1}^l \frac{r_i n_{ij}}{m} > 0$$

for every j and $n_{ij} \leq 2m \dim X$ with $n_{ij} \in \mathbb{Z}$ for every i and j , the number of the set $\{n_{ij}\}_{i,j}$ is finite. Thus,

$$\inf_j \left\{ \frac{1}{\mu_j} \right\} = \frac{1}{\mu_l}$$

for some l . Therefore, we obtain $\mu = \mu_l$. We finish the proof. \square

The following picture helps the reader to understand Theorem 18.9.



The main result of this section is an estimate of lengths of extremal rays which are relatively ample at non-lc loci (cf. [Ko2], [Ko3]).

Theorem 18.10. *Let X be a normal variety, let B be an effective \mathbb{R} -divisor on X such that $K_X + B$ is \mathbb{R} -Cartier, and let $\pi : X \rightarrow S$ be a projective morphism onto a variety S . Let R be a $(K_X + B)$ -negative extremal ray of $\overline{NE}(X/S)$ which is relatively ample at $\text{Nlc}(X, B)$. Then we can find a rational curve C on X such that $[C] \in R$ and*

$$0 < -(K_X + B) \cdot C \leq 2 \dim X.$$

Proof. By shrinking S , we can assume that S is quasi-projective. By replacing $\pi : X \rightarrow S$ with the extremal contraction $\varphi_R : X \rightarrow Y$ over S (cf. Theorem 16.6 (3)), we can assume that the relative Picard number $\rho(X/S) = 1$ and that π is an isomorphism in a neighborhood of $\text{Nlc}(X, B)$. In particular, $-(K_X + B)$ is π -ample. By Theorem 10.4, there is a projective birational morphism $f : Y \rightarrow X$ such that

(i) $K_Y + B_Y = f^*(K_X + B) + \sum_{a(E, X, B) < -1} (a(E, X, B) + 1)E$, where

$$B_Y = f_*^{-1}B + \sum_{E: f\text{-exceptional}} E,$$

(ii) (Y, B_Y) is a \mathbb{Q} -factorial dlt pair, and

(iii) $D = B_Y + F$ with $F = -\sum_{a(E, X, B) < -1} (a(E, X, B) + 1)E \geq 0$.

We note that $K_Y + D = f^*(K_X + B)$. Therefore, we have

$$f_*(\overline{NE}(Y/S)_{K_Y + D \geq 0}) \subseteq \overline{NE}(X/S)_{K_X + B \geq 0} = \{0\}.$$

We also note that

$$f_*(\overline{NE}(Y/S)_{\text{Nlc}(Y, D)}) = \{0\}.$$

Thus, there is a $(K_Y + D)$ -negative extremal ray R' of $\overline{NE}(Y/S)$ which is relatively ample at $\text{Nlc}(Y, D)$. By Theorem 16.6 (1), R' is spanned by a curve C^\dagger . Since $-(K_Y + D) \cdot C^\dagger > 0$, we see that $f(C^\dagger)$ is a curve. If $C^\dagger \subset \text{Supp}F$, then $f(C^\dagger) \subset \text{Nlc}(X, B)$. It is a contradiction because $\pi \circ f(C^\dagger)$ is a point. Thus, $C^\dagger \not\subset \text{Supp}F$. Since $-(K_Y + B_Y) = -(K_Y + D) + F$, we can see that R' is a $(K_Y + B_Y)$ -negative extremal ray of $\overline{NE}(Y/S)$. Therefore, we can find a rational curve C' on Y such that C' spans R' and that

$$0 < -(K_Y + B_Y) \cdot C' \leq 2 \dim X$$

by Theorem 18.2. By the above argument, we can easily see that $C' \not\subset \text{Supp}F$. Therefore, we obtain

$$\begin{aligned} 0 < -(K_Y + D) \cdot C' &= -(K_Y + B_Y) \cdot C' - F \cdot C' \\ &\leq -(K_Y + B_Y) \cdot C' \leq 2 \dim X. \end{aligned}$$

Since $K_Y + D = f^*(K_X + B)$, $C = f(C')$ is a rational curve on X such that $\pi(C)$ is a point and $0 < -(K_X + B) \cdot C \leq 2 \dim X$. \square

Remark 18.11. In Theorem 18.10, we can easily prove $0 < -(K_X + B) \cdot C \leq \dim X + 1$ when $\dim X \leq 2$. For details, see [F17, Proposition 3.7].

§19. Ambro's theory of quasi-log varieties

In this section, we make some comments on Ambro's theory of quasi-log varieties. We strongly recommend the reader to see [F15] for an introduction to the theory of quasi-log varieties.

In the acknowledgements in [A1], Ambro wrote “The motivation behind this work is his (Professor Shokurov’s) idea that log varieties and their LCS loci should be treated on an equal footing.” So, in the theory of quasi-log varieties, we have to treat highly reducible non-equidimensional varieties (see Example 19.2 below). Therefore, our approach explained in this paper is completely different from the theory of quasi-log varieties. We recommend the reader to compare our proof of the base point free theorem for projective lc surfaces in Section 2 with Ambro’s proof (see, for example, [F15, Section 4]).

Let us explain some results of the theory of quasi-log varieties which can not be covered by our approach.

19.1. Let (X, B) be a projective log canonical pair and let $\{C_i\}$ be any set of lc centers of the pair (X, B) . We put $W = \bigcup C_i$ with the reduced scheme structure. Then $[W, \omega]$ is a *qlc pair*, where $\omega = (K_X + B)|_W$. For the definition of *qlc pairs*, see [F11, Definition 3.29] or [F15, Definition 3.1].

Example 19.2. Let V be a projective toric variety and let D be the complement of the big torus. Then (V, D) is log canonical and $K_V + D \sim 0$. In this case, every torus invariant closed subvariety W of V with $\omega = 0$ is a qlc pair. In particular, W is not necessarily pure-dimensional (cf. [F7, §5]).

We can prove the cone theorem for $[W, \omega]$.

Theorem 19.3 (Cone theorem). *We have*

$$\overline{NE}(W) = \overline{NE}(W)_{\omega \geq 0} + \sum_j R_j.$$

For the details, see [F11, 3.3.3 Cone Theorem]. We can also prove the base point free theorem.

Theorem 19.4 (Base point free theorem). *Let L be a nef Cartier divisor on W such that $aL - \omega$ is ample for some $a > 0$. Then $|mL|$ is base point free for $m \gg 0$.*

See, for example, [F11, 3.3.1 Base point free theorem]. By these theorems, we have the following statement.

Theorem 19.5 (Contraction theorem). *Let F be an ω -negative extremal face of $\overline{NE}(W)$. Then there is a contraction morphism $\varphi_F : W \rightarrow V$ with the following properties.*

- (i) *Let C be an integral curve on W . Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*

- (ii) $\mathcal{O}_V \simeq (\varphi_F)_* \mathcal{O}_W$.
- (iii) *Let L be a line bundle on W such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_V on V such that $L \simeq \varphi_F^* L_V$.*

For the details of the theory of quasi-log varieties, see [F11]. The book [F11] treats some various other topics which can not be covered by this paper.

§20. Related topics

In this final section, we briefly explain some related topics obtained by the author for the reader's convenience.

In this paper, we did not describe the notion of singularities of pairs. However, it is very important when we read some papers on the log minimal model program. We think that [F6] helps the reader to understand the subtlety of the notion of dlt pairs.

The reader can find that all the injectivity, vanishing, and torsion-free theorems in this paper are discussed in full generality in [F11, Sections 2 and 3]. They heavily depend on the theory of mixed Hodge structures on compact support cohomology groups of reducible varieties.

We omitted the explanation of the log minimal model program for log canonical pairs. It is because the framework is the same as for klt pairs. The reader can find it in [F11, Section 3]. We note that the existence problem of log canonical flips is still open in dimension ≥ 5 and the termination of log canonical flips follows from the termination of klt flips. For the details, see [F11, Section 3].

In [F8], we prove an effective version of the base point free theorem for log canonical pairs. It is a log canonical version of Kollár's effective freeness. In [F9], the Angehrn–Siu type effective base point free theorems are proved for log canonical pairs. The reader can find that the proof of our non-vanishing theorem (cf. Theorem 12.1 and [F16, Theorem 1.1]) grew out from the arguments in [F8] and [F9].

In [F10], we systematically treat the basic properties of non-lc ideal sheaves, especially, the restriction theorem of non-lc ideal sheaves for normal divisors. It is a generalization of Kawakita's inversion of adjunction on log canonicity. See also [FST] for further discussions on various analogues of non-lc ideal sheaves.

In [F13], we prove the finite generation of the log canonical ring for log canonical pairs in dimension four and discuss related topics. It induces the existence of fourfold log canonical flips.

In [F17], we discuss the minimal model theory for log surfaces. The results in [F17] are obtained under much weaker assumptions than everybody expected.

The paper [F17] is an ultimate application of our new approach to the log minimal model program.

Acknowledgements

The author was partially supported by The Inamori Foundation and by the Grant-in-Aid for Young Scientists (A) #20684001 from JSPS. He thanks Natsuo Saito for drawing a beautiful picture of a Kleiman-Mori cone. He also thanks Takeshi Abe for useful discussions and Yoshinori Gongyo for some questions. Finally, he thanks Professor Shigefumi Mori for useful comments, information, and warm encouragement.

References

- [A1] F. Ambro, Quasi-log varieties, *Tr. Mat. Inst. Steklova* **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebr, 220–239; translation in *Proc. Steklov Inst. Math.* 2003, no. 1 (240), 214–233.
- [A2] ———, Non-plt techniques, *Flips for 3-folds and 4-folds*, 163–170, Oxford Lecture Ser. Math. Appl., **35**, Oxford Univ. Press, Oxford, 2007.
- [B1] C. Birkar, On existence of log minimal models, *Compos. Math.* **146** (2010), no. 4, 919–928.
- [B2] ———, On existence of log minimal models II, to appear in *J. Reine Angew. Math.*
- [BCHM] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type, *J. Amer. Math. Soc.* **23**, no. 2, 405–468.
- [BP] C. Birkar, M. Păun, Minimal models, flips and finite generation: a tribute to V. V. Shokurov and Y.-T. Siu, preprint (2009).
- [D] P. Deligne, Théorie de Hodge. II, *Inst. Hautes Études Sci. Publ. Math.* No. **40** (1971), 5–57.
- [EV] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, **20**. Birkhäuser Verlag, Basel, 1992.
- [F1] O. Fujino, Abundance theorem for semi log canonical threefolds, *Duke Math. J.* **102** (2000), no. 3, 513–532.
- [F2] ———, Notes on toric varieties from Mori theoretic viewpoint, *Tohoku Math. J.* **55** (2003), 551–564.
- [F3] ———, Higher direct images of log canonical divisors, *J. Differential Geom.* **66** (2004), no. 3, 453–479.
- [F4] ———, On the Kleiman-Mori cone, *Proc. Japan Acad. Ser. A Math. Sci.* **81** (2005), no. 5, 80–84.
- [F5] ———, Equivariant completions of toric contraction morphisms, *Tohoku Math. J.* **58** (2006), 303–321.
- [F6] ———, What is log terminal?, in *Flips for 3-folds and 4-folds* (Alessio Corti, ed.), 29–62, Oxford University Press, 2007.
- [F7] ———, Vanishing theorems for toric polyhedra, RIMS Kokyuroku Bessatsu, B9, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008, 81–95.
- [F8] ———, Effective base point free theorem for log canonical pairs—Kollár type theorem, *Tohoku Math. J.* **61** (2009), 475–481.

- [F9] ———, Effective base point free theorem for log canonical pairs, II. Angehrn–Siu type theorems, *Michigan Math. J.* **59** (2010), 303–312.
- [F10] ———, Theory of non-lc ideal sheaves: basic properties, *Kyoto Journal of Mathematics*, Vol. **50**, No. 2 (2010), 225–245.
- [F11] ———, Introduction to the log minimal model program for log canonical pairs, preprint (2009). arXiv:0907.1506v1
- [F12] ———, On injectivity, vanishing and torsion-free theorems for algebraic varieties, *Proc. Japan Acad. Ser. A Math. Sci.* **85** (2009), no. 8, 95–100.
- [F13] ———, Finite generation of the log canonical ring in dimension four, *Kyoto Journal of Mathematics*, Vol. **50**, No. 4 (2010), 671–684.
- [F14] ———, On Kawamata’s theorem, to appear in the proceeding of the “Classification of Algebraic Varieties” conference, Schiermonnikoog, Netherlands, May 10–15, 2009.
- [F15] ———, Introduction to the theory of quasi-log varieties, to appear in the proceeding of the “Classification of Algebraic Varieties” conference, Schiermonnikoog, Netherlands, May 10–15, 2009.
- [F16] ———, Non-vanishing theorem for log canonical pairs, to appear in *Journal of Algebraic Geometry*.
- [F17] ———, Minimal model theory for log surfaces, preprint (2010).
- [FG] O. Fujino, Y. Gongyo, On canonical bundle formulae and subadjunctions, preprint (2010).
- [FP] O. Fujino, S. Payne, Smooth complete toric threefolds with no nontrivial nef line bundles, *Proc. Japan Acad. Ser. A Math. Sci.* **81** (2005), no. 10, 174–179 (2006).
- [FST] O. Fujino, K. Schwede, and S. Takagi, Supplements to non-lc ideal sheaves, preprint (2010).
- [G] Y. Gongyo, On weak Fano varieties with log canonical singularities, preprint (2009).
- [HM] C. Hacon, J. McKernan, Extension theorems and the existence of flips, *Flips for 3-folds and 4-folds*, 76–110, *Oxford Lecture Ser. Math. Appl.*, **35**, Oxford Univ. Press, Oxford, 2007.
- [Ka1] Y. Kawamata, The cone of curves of algebraic varieties, *Ann. of Math. (2)* **119** (1984), no. 3, 603–633.
- [Ka2] ———, On the length of an extremal rational curve, *Invent. Math.* **105** (1991), no. 3, 609–611.
- [KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the Minimal Model Problem, in *Algebraic Geometry, Sendai 1985*, *Advanced Studies in Pure Math.* **10**, (1987) Kinokuniya and North-Holland, 283–360.
- [Kl] S. L. Kleiman, Toward a numerical theory of ampleness, *Ann. of Math. (2)* **84** (1966), 293–344.
- [Ko1] J. Kollár, The cone theorem, *Ann. of Math. (2)* **120** (1984), no. 1, 1–5.
- [Ko2] ———, Cone theorems and cyclic covers, *Algebraic geometry and analytic geometry (Tokyo, 1990)*, 101–110, *ICM-90 Satell. Conf. Proc.*, Springer, Tokyo, 1991.
- [Ko3] ———, Cone theorems and bug-eyed covers, *J. Algebraic Geom.* **1** (1992), no. 2, 293–323.
- [Ko4] ———, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1995.
- [KK] J. Kollár, S. Kovács, Log canonical singularities are Du Bois, *J. Amer. Math. Soc.* **23**, no. 3, 791–813.
- [KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, *Cambridge Tracts in*

- Mathematics, Vol. **134**, 1998.
- [L] R. Lazarsfeld, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], **49**. Springer-Verlag, Berlin, 2004.
 - [Ma] K. Matsuki, *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, 2002.
 - [Mo] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math. (2) **116** (1982), no. 1, 133–176.
 - [R] M. Reid, Projective morphisms according to Kawamata, preprint (1983).
 - [S1] V. V. Shokurov, The nonvanishing theorem, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 635–651.
 - [S2] ———, Three-dimensional log perestroikas (Russian) With an appendix in English by Yujiro Kawamata, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 1, 105–203; translation in Russian Acad. Sci. Izv. Math. **40** (1993), no. 1, 95–202.
 - [S3] ———, 3-fold log models, Algebraic geometry, 4. J. Math. Sci. **81** (1996), no. 3, 2667–2699.
 - [S4] ———, Letters of a bi-rationalist: VII. Ordered termination, *Multidimensional Algebraic Geometry*, Collected papers. Dedicated to the Memory of Vasilii Alekseevich Iskovskikh, Corresponding Member of the Russian Academy of Sciences, Tr. Mat. Inst. Steklova, **264**, MAIK Nauka/Interperiodica, Moscow, 2009, 184–208.