FUJITA'S VANISHING THEOREM (PRIVATE NOTE)

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Contents

1. Fujita's vanishing theorem

The following theorem is obtained by Takao Fujita (cf. [F1, Theorem (1)] and [F2, (5.1) Theorem]). See also [L, Theorem 1.4.35].

Theorem 1.1 (Fujita's vanishing theorem)**.** *Let X be a projective scheme defined over a field k and let H be an ample Cartier divisor on X. Given any coherent sheaf $\mathcal{F}$ on X, there exists an integer $m(\mathcal{F}, H)$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for all $i > 0$, $m \geq m(\mathcal{F}, H)$, and any nef Cartier divisor D on X.

Proof. Without loss of generality, we may assume that k is algebraically closed. By replacing X with $\mathrm{Supp}\mathcal{F}$, we may assume that $X = \mathrm{Supp}\mathcal{F}$.

Remark 1.2. Let $\mathcal{F}$ be a coherent sheaf on X. In the proof of Theorem 1.1, we always define a subscheme structure on $\mathrm{Supp}\mathcal{F}$ by the $\mathcal{O}_X$-ideal $\mathrm{Ker}(\mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}))$.

We use the induction on dimension.

Step 1. When $\dim X = 0$, Theorem 1.1 obviously holds.

From now on, we assume that Theorem 1.1 holds in the lower dimensional case.

Step 2. We can reduce the proof to the case when X is reduced.

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Proof. We assume that Theorem 1.1 holds for reduced schemes. Let $\mathcal{N}$ be the nilradical of $\mathcal{O}_X$, so that $\mathcal{N}^r = 0$ for some $r > 0$. Consider the filtration

$$\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \supset \cdots \supset \mathcal{N}^r \cdot \mathcal{F} = 0.$$

The quotients $\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}$ are coherent $\mathcal{O}_{X_{\mathrm{red}}}$-modules, and therefore, by the assumption,

$$H^j(X, (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}, H)$ thanks to the amplitude of $\mathcal{O}_{X_{\mathrm{red}}}(H)$. Twisting the exact sequences

$$0 \to \mathcal{N}^{i+1}\mathcal{F} \to \mathcal{N}^i\mathcal{F} \to \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \to 0$$

by $\mathcal{O}_X(mH + D)$ and taking cohomology, we then find by decreasing induction on i that

$$H^j(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{N}^i\mathcal{F}, H)$. When $i = 0$ this gives the desired vanishings. □

From now on, we assume that X is reduced.

Step 3. We can reduce the proof to the case when X is irreducible.

Proof. We assume that Theorem 1.1 holds for reduced and irreducible schemes. Let $X = X_1 \cup \cdots \cup X_k$ be its decomposition into irreducible components and let $\mathcal{I}$ be the ideal sheaf of X_1 in X. We consider the exact sequence

$$0 \to \mathcal{I} \cdot \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I} \cdot \mathcal{F} \to 0.$$

The outer terms of the above exact sequence are supported on $X_2 \cup \cdots \cup X_k$ and X_1 respectively. So by induction on the number of irreducible components, we may assume that

$$H^j(X, \mathcal{I}\mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{I}\mathcal{F}, H|_{X_2 \cup \cdots \cup H_k})$ and

$$H^j(X, (\mathcal{F}/\mathcal{I}\mathcal{F}) \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{F}/\mathcal{I}\mathcal{F}, H|_{X_1})$. It then follows from the above exact sequence that

$$H^j(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

when $j > 0$ and

$$m \geq m(\mathcal{F}, H) := \max\{m(\mathcal{I}\mathcal{F}, H|_{X_2 \cup \cdots \cup H_k}), m(\mathcal{F}/\mathcal{I}\mathcal{F}, H|_{X_1})\},$$

as required. □

From now on, we assume that X is reduced and irreducible.

Step 4. We can reduce the proof to the case when H is very ample.

Proof. Let l be a positive integer such that lH is very ample. We assume that Theorem 1.1 holds for lH. Apply Theorem 1.1 to $\mathcal{F} \otimes \mathcal{O}_X(nH)$ for $0 \le n \le l-1$ with lH. Then we obtain $m(\mathcal{F}\otimes\mathcal{O}_X(nH), lH)$ for $0 \le n \le l-1$. We put

$$m(\mathcal{F}, H) = l\left(\max_n m(\mathcal{F} \otimes \mathcal{O}_X(nH), lH) + 1\right).$$

Then we can easily check that $m(\mathcal{F}, H)$ satisfies the desired property. □

From now on, we assume that H is very ample.

Step 5. It is sufficient to find $m(\mathcal{F}, H)$ such that

$$H^1(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for all $m \ge m(\mathcal{F}, H)$ and any nef Cartier divisor D on X.

Proof. We take a general member A of $|H|$ and consider the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(-A) \to \mathcal{F} \to \mathcal{F}_A \to 0.$$

Since $\dim \mathrm{Supp}\mathcal{F}_A < \dim X$, we can find $m(\mathcal{F}_A, H|_A)$ such that

$$H^i(A, \mathcal{F}_A \otimes \mathcal{O}_A(mH + D)) = 0$$

for all $i > 0$ and $m \ge m(\mathcal{F}_A, H|_A)$ by the induction. Therefore,

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X((m-1)H + D)) = H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D))$$

for every $i \ge 2$ and $m \ge m(\mathcal{F}_A, H|_A)$. By Serre's vanishing theorem, we obtain

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X((m-1)H + D)) = 0$$

for every $i \ge 2$ and $m \ge m(\mathcal{F}_A, H|_A)$. □

Step 6. We can reduce the proof to the case when $\mathcal{F} = \mathcal{O}_X$.

Proof. We assume that Theorem 1.1 holds for $\mathcal{F} = \mathcal{O}_X$. There is an injective homomorphism

$$\alpha : \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{O}_X(aH)$$

for some large integer a. We consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{O}_X(aH) \to \mathrm{Coker}\alpha \to 0$$

and use the induction on $\mathrm{rank}\,\mathcal{F}$. Then we can find $m(\mathcal{F}, H)$. □

From now on, we assume $\mathcal{F} = \mathcal{O}_X$.

Step 7. If the characteristic of k is zero, then Theorem 1.1 holds.

Proof. Let $f : Y \to X$ be a resolution. Then we obtain the following exact sequence

$$0 \to f_*\omega_Y \to \mathcal{O}_X(bH) \to \mathcal{C} \to 0$$

for some integer b, where $\dim \operatorname{Supp} \mathcal{C} < \dim X$. Note that $f_*\omega_Y$ is torsion-free and $\operatorname{rank} f_*\omega_Y$ is one. On the other hand,

$$H^j(X, f_*\omega_Y \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m > 0$ and $j > 0$ by Kollár's vanishing theorem. Therefore,

$$H^j(X, \mathcal{O}_X((b + m)H + D)) = 0$$

for every positive integer $m \geq m(\mathcal{C}, H)$ and $j > 0$. □

Step 8. We can reduce the proof to the case when $\mathcal{F} = \omega_X$, where ω_X is the dualizing sheaf of X.

Remark 1.3. The dualizing sheaf ω_X is denoted by ω°_X in [H, Chapter III §7]. We know that $\omega^\circ_X \simeq \mathcal{E}xt^{N-\dim X}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ when $X \subset \mathbb{P}^N$. For details, see the proof of Proposition 7.5 in [H, Chapter III §7].

Proof. We assume that Theorem 1.1 holds for $\mathcal{F} = \omega_X$. There is an injective homomorphism

$$\beta : \omega_X \to \mathcal{O}_X(cH)$$

for some positive integer c. Note that ω_X is torsion-free. We consider the exact sequence

$$0 \to \omega_X \to \mathcal{O}_X(cH) \to \operatorname{Coker} \beta \to 0.$$

We note that $\dim \operatorname{SuppCoker} \beta < \dim X$ because

$$\operatorname{rank} \omega_X = \operatorname{rank} \mathcal{O}_X(cH) = 1.$$

Therefore, we can find $m(\mathcal{O}_X, H)$ by the induction on dimension and Theorem 1.1 for ω_X. □

From now on, we assume that $\mathcal{F} = \omega_X$ and that the characteristic of k is positive.

Step 9. Theorem 1.1 holds when the characteristic of k is positive.

Proof. Let $X \to \mathbb{P}^N$ be the embedding induced by H. Let

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ \mathbb{P}^N & \xrightarrow[F]{} & \mathbb{P}^N \end{array}$$

be the commutative diagram of the Frobenius morphisms. By taking $R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\underline{\quad}, \omega^{\bullet}_{\mathbb{P}^N})$ to $\mathcal{O}_X \to F_*\mathcal{O}_X$, we obtain

$$R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(F_*\mathcal{O}_X, \omega^{\bullet}_{\mathbb{P}^N}) \to R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega^{\bullet}_{\mathbb{P}^N}).$$

By the Grothendieck duality,

$$R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(F_*\mathcal{O}_X, \omega^{\bullet}_{\mathbb{P}^N}) \simeq F_*R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega^{\bullet}_{\mathbb{P}^N}).$$

Therefore, we obtain

$$\gamma : F_*\omega_X \to \omega_X.$$

Note that $\omega_X = \mathcal{E}xt^{N-\dim X}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$. Let U be a non-empty Zariski open set of X such that U is smooth. We can easily check that

$$\gamma : F_*\omega_X \to \omega_X$$

is surjective on U. Note that the cokernel $\mathcal{A}$ of $\mathcal{O}_X \to F_*\mathcal{O}_X$ is locally free on U. Then $\mathcal{E}xt^k_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{A}, \omega_{\mathbb{P}^N}) = 0$ for $k > N - \dim X$ on U. We consider the exact sequences

$$0 \to \mathrm{Ker}\gamma \to F_*\omega_X \to \mathrm{Im}\gamma \to 0$$

and

$$0 \to \mathrm{Im}\gamma \to \omega_X \to \mathcal{C} \to 0.$$

Then $\dim \mathrm{Supp}\,\mathcal{C} < \dim X$. Note that there is an integer m_1 such that

$$H^2(X, \mathrm{Ker}\gamma \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m \geq m_1$ by Step 5. By applying the induction on dimension to $\mathcal{C}$, we obtain some positive integer m_0 such that

$$H^1(X, F_*\omega_X \otimes \mathcal{O}_X(mH + D)) \to H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D))$$

is surjective for every $m \geq m_0$. We note that

$$H^1(X, F_*\omega_X \otimes \mathcal{O}_X(mH + D)) \simeq H^1(X, \omega_X \otimes \mathcal{O}_X(p(mH + D)))$$

by the projection formula, where p is the characteristic of k. By repeating the above process, we obtain that

$$H^1(X, \omega_X \otimes \mathcal{O}_X(p^e(mH + D))) \to H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D))$$

is surjective for every $e > 0$ and $m \geq m_0$. Note that m_0 is independent of the nef divisor D. Therefore, by Serre's vanishing theorem, we obtain

$$H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m \geq m_0$. □

We finish the proof of Theorem 1.1. □

In Step 9, we can use the following elementary lemma to construct a generically surjective homomorphism $F_*\omega_X \to \omega_X$.

Lemma 1.4 (cf. [F2, (5.7) Corollary])**.** *Let $f : V \to W$ be a projective surjective morphism between projective varieties defined over an algebraically closed field k with $\dim V = \dim W = n$. Then there is a generically surjective homomorphism $\varphi : f_*\omega_V \to \omega_W$.*

Proof. By the definition (cf. [H, Chapter III §7]), $H^n(V, \omega_V) \neq 0$. We consider the Leray spectral sequence

$$E_2^{p,q} = H^p(W, R^q f_*\omega_W) \Rightarrow H^{p+q}(V, \omega_V).$$

Note that $\mathrm{Supp} R^q f_*\omega_V$ is contained in the set

$$W_q := \{w \in W \mid \dim f^{-1}(w) \geq q\}.$$

Since $\dim f^{-1}(W_q) < n$ for every $q > 0$, we have $\dim W_q < n - q$ for every $q > 0$. Therefore, $E_2^{n-q,q} = 0$ unless $q = 0$. Thus we obtain $E_2^{n,0} = H^n(W, f_*\omega_V) \neq 0$ since $H^n(V, \omega_V) \neq 0$. By the definition of ω_W, $\mathrm{Hom}(f_*\omega_V, \omega_W) \neq 0$. We take a non-zero element $\varphi \in \mathrm{Hom}(f_*\omega_V, \omega_W)$ and consider $\mathrm{Im}(\varphi) \subset \omega_W$. Since $\mathrm{Hom}(\mathrm{Im}(\varphi), \omega_W) \neq 0$, we have $H^n(W, \mathrm{Im}(\varphi)) \neq 0$ (see [H, Chapter III §7]). This implies that $\dim \mathrm{Supp}\mathrm{Im}(\varphi) = n$. Therefore, $\varphi : f_*\omega_V \to \omega_W$ is generically surjective since $\mathrm{rank}\, \omega_W = 1$. □

Remark 1.5. In Lemma 1.4, if $R^q f_*\omega_V = 0$ for every $q > 0$, then we obtain $H^n(W, f_*\omega_V) \simeq H^n(V, \omega_V)$. We note that $H^n(V, \omega_V) \simeq k$ since k is algebraically closed. Therefore, $\mathrm{Hom}(f_*\omega_V, \omega_W) \simeq k$. This means that, for any non-trivial homomorphism $\psi : f_*\omega_V \to \omega_W$, there is some $a \in k \setminus \{0\}$ such that $\psi = a\varphi$, where φ is given in Lemma 1.4. Note that $R^q f_*\omega_V = 0$ for every $q > 0$ if f is finite. We also note that $R^q f_*\omega_V = 0$ for every $q > 0$ if the characteristic of k is zero and V has only rational singularities by the Grauert–Riemenschneider vanishing theorem or by Kollár's torsion-free theorem (see also Lemma 1.6 below).

Although the following lemma is a special case of Kollár's torsion-freeness, it easily follows from the Kawamata–Viehweg vanishing theorem.

Lemma 1.6 (cf. [F2, (4.13) Proposition])**.** *Let $f : V \to W$ be a projective surjective morphism from a smooth projective variety V to a projective variety W, which is defined over an algebraically closed field k of characteristic zero. Then $R^q f_*\omega_V = 0$ for every $q > \dim V - \dim W$.*

Proof. Let A be a sufficiently ample Cartier divisor on W such that

$$H^0(W, R^q f_*\omega_V \otimes \mathcal{O}_W(A)) \simeq H^q(V, \omega_V \otimes \mathcal{O}_V(f^*A))$$

and that $R^q f_*\omega_V \otimes \mathcal{O}_W(A)$ is generated by global sections for every q. We note that the numerical dimension $\nu(V, f^*A)$ of f^*A is $\dim W$.

Therefore, we can easily check that

$$H^q(V, \omega_V \otimes \mathcal{O}_V(f^*A)) = 0$$

for $q > \dim V - \dim W = \dim V - \nu(V, f^*A)$ by the Kawamata–Viehweg vanishing theorem. Thus, we obtain $R^q f_*\omega_V = 0$ for $q > \dim V - \dim W$. □

Remark 1.7. In [F2, Section 4], Takao Fujita proves Lemma 1.6 for a proper surjective morphism $f : V \to W$ from a complex manifold V in Fujiki's class $\mathcal{C}$ to a projective variety W. His proof uses the theory of harmonic forms. For the details, see [F2, Section 4]. See also Theorem 1.8 below.

The following theorem is a weak generalization of Kodaira's vanishing theorem. We need no new ideas to prove Theorem 1.8. The proof of Kodaira's vanishing theorem based on Bochner's method works.

Theorem 1.8 (A weak generalization of Kodaira's vanishing theorem)**.** *Let X be an n-dimensional compact Kähler manifold and let $\mathcal{L}$ be a line bundle on X whose curvature form $\sqrt{-1}\Theta(\mathcal{L})$ is semi-positive and has at least k positive eigenvalues on a dense open subset of X. Then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for $i > n - k$.*

We note that $H^i(X, \omega_X \otimes \mathcal{L})$ is isomorphic to $\mathcal{H}^{n,i}(X, \mathcal{L})$, which is the space of $\mathcal{L}$-valued harmonic (n, i)-forms on X. By Nakano's formula, we can easily check that $\mathcal{H}^{n,i}(X, \mathcal{L}) = 0$ for $i + k \geq n + 1$.

We close this section with a slight generalization of Kollár's result (cf. [K, Proposition 7.6]), which is related to Lemma 1.4.

Proposition 1.9. *Let $f : V \to W$ be a proper surjective morphism between normal algebraic varieties with connected fibers, which is defined over an algebraically closed field k of characteristic zero. Assume that V and W have only rational singularities. Then $R^d f_*\omega_V \simeq \omega_W$ where $d = \dim V - \dim W$.*

Proof. We can construct a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[p]{} & W \end{array}$$

with the following properties.

(i) X and Y are smooth algebraic varieties.
(ii) π and p are projective birational.

(iii) g is projective, and smooth outside a simple normal crossing divisor Σ on Y.

We note that $R^j g_*\omega_X$ is locally free for every j. By the Grothendieck duality, we have

$$Rg_*\mathcal{O}_X \simeq R\mathcal{H}om_{\mathcal{O}_Y}(Rg_*\omega^\bullet_X, \omega^\bullet_Y).$$

Therefore, we have

$$\mathcal{O}_Y \simeq \mathcal{H}om_{\mathcal{O}_Y}(R^d g_*\omega_X, \omega_Y).$$

Thus, we obtain $R^d g_*\omega_X \simeq \omega_Y$. By applying p_*, we have $p_*R^d g_*\omega_X \simeq p_*\omega_Y \simeq \omega_W$. We note that $p_*R^d g_*\omega_X \simeq R^d(p\circ g)_*\omega_X$ since $R^i p_* R^d g_*\omega_X = 0$ for every $i > 0$. On the other hand,

$$R^d(p \circ g)_*\omega_X \simeq R^d(f \circ \pi)_*\omega_X \simeq R^d f_*\omega_V$$

since $R^i\pi_*\omega_X = 0$ for every $i > 0$ and $\pi_*\omega_X \simeq \omega_V$. Therefore, we obtain $R^d f_*\omega_V \simeq \omega_W$. □

2. Applications

In this section, we discuss some applications of Theorem 1.1. For more general statements and other applications, see [F2, Section 6].

Theorem 2.1 (cf. [F1, Theorem (4)] and [F2, (6.2) Theorem])**.** *Let $\mathcal{F}$ be a coherent sheaf on a scheme X which is proper over an algebraically closed field k. Let $\mathcal{L}$ be a nef line bundle on X. Then*

$$\dim H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q})$$

where $m = \dim \operatorname{Supp}\mathcal{F}$.

Proof. First, we assume that X is projective. We use the induction on q. Let H be an effective ample Cartier divisor on X such that $\mathcal{L} \otimes \mathcal{O}_X(H)$ is ample. Since

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \subset H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t} \otimes \mathcal{O}_X(tH))$$

for every positive integer t, we can assume that $\mathcal{L}$ is ample by replacing $\mathcal{L}$ with $\mathcal{L} \otimes \mathcal{O}_X(H)$. In this case, $\dim H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^m)$ because

$$\dim H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) = \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t})$$

for $t \gg 0$ by Serre's vanishing theorem. When $q > 0$, by Theorem 1.1, we have a very ample Cartier divisor A on X such that

$$H^q(X, \mathcal{F} \otimes \mathcal{O}_X(A) \otimes \mathcal{L}^{\otimes t}) = 0$$

for every $t \geq 0$. Let D be a general member of $|A|$ such that the induced homomorphism $\alpha : \mathcal{F} \otimes \mathcal{O}_X(-D) \to \mathcal{F}$ is injective. Then

$$\begin{aligned}\dim H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) &\leq \dim H^{q-1}(D, \mathrm{Coker}(\alpha) \otimes \mathcal{O}_D(A) \otimes \mathcal{L}^{\otimes t}) \\ &\leq O(t^{m-q})\end{aligned}$$

by the induction hypothesis. Therefore, we obtain the theorem when X is projective.

Next, we consider the general case. We use the Noetherian induction on $\mathrm{Supp}\mathcal{F}$. By the same arguments as in Step 2 and Step 3 in the proof of Theorem 1.1, we may assume that $X = \mathrm{Supp}\mathcal{F}$ is a variety, that is, X is reduced and irreducible. By Chow's lemma, there is a biratinal morphism $f : V \to X$ from a projective variety V. We put $\mathcal{G} = f^*\mathcal{F}$ and consider the natural homomorphism $\beta : \mathcal{F} \to f_*\mathcal{G}$. Since β is an isomorphism on a non-empty Zariski open subset of X. We consider the following short exact sequences

$$0 \to \mathrm{Ker}(\beta) \to \mathcal{F} \to \mathrm{Im}(\beta) \to 0$$

and

$$0 \to \mathrm{Im}(\beta) \to f_*\mathcal{G} \to \mathrm{Coker}(\beta) \to 0.$$

By the induction, we obtain

$$\dim H^q(X, \mathrm{Ker}(\beta) \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q})$$

and

$$\dim H^{q-1}(X, \mathrm{Coker}(\beta) \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

Therefore, it is sufficient to see that

$$\dim H^q(X, f_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

We consider the Leray spectral sequence

$$E_2^{i,j} = H^i(X, R^j f_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \Rightarrow H^{i+j}(V, \mathcal{G} \otimes (f^*\mathcal{L})^{\otimes t}).$$

Then we have

$$\begin{aligned}\dim H^q(X, f_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \leq &\sum_{j \geq 1} \dim H^{q-j-1}(X, R^j f_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \\ &+ \dim H^q(V, \mathcal{G} \otimes (f^*\mathcal{L})^{\otimes t}).\end{aligned}$$

Note that

$$\dim H^q(V, \mathcal{G} \otimes (f^*\mathcal{L})^{\otimes t}) \leq O(t^{m-q})$$

since V is projective. On the other hand, we have

$$\dim \mathrm{Supp} R^j f_*\mathcal{G} \leq \dim X - j - 1$$

for every $j \geq 1$ as in the proof of Lemma 1.4. Therefore,

$$\dim H^{q-j-1}(X, R^j f_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q})$$

by the induction hypothesis. Thus, we obtain

$$\dim H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

We complete the proof. □

As an application of Theorem 2.1, we can prove Fujita's numerical characterization of nef and big line bundles. We note that the characteristic of the base field is arbitrary in Corollary 2.2.

Corollary 2.2 (cf. [F1, Theorem (6)] and [F2, (6.5) Corollary]). *Let $\mathcal{L}$ be a nef line bundle on a proper algebraic variety V defined over an algebraically closed field k with $\dim V = n$. Then $\kappa(X, \mathcal{L}) = n$ if and only if the self-intersection number $\mathcal{L}^n$ is positive. We note that $\mathcal{L}$ is called* big *when $\kappa(V, \mathcal{L}) = n$.*

Proof. It is well known that

$$\chi(V, \mathcal{L}^{\otimes t}) - \frac{\mathcal{L}^n}{n!} t^n \leq O(t^{n-1}).$$

By Theorem 2.1, we have

$$\dim H^0(V, \mathcal{L}^{\otimes t}) - \chi(V, \mathcal{L}^{\otimes t}) \leq O(t^{n-1}).$$

Therefore, $\kappa(V, \mathcal{L}) = n$ if and only if $\mathcal{L}^n > 0$. Note that $\mathcal{L}^n \geq 0$ since $\mathcal{L}$ is nef. □

Corollary 2.3 (cf. [F1, Corollary (7)] and [F2, (6.7) Corollary]). *Let $\mathcal{L}$ be a nef nad big line bundle on a projective variety V defined over an algebraically closed field k with $\dim V = n$. Then, for any coherent sheaf $\mathcal{F}$ on V, we have*

$$\dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q-1})$$

for every $q \geq 1$. In particular, $H^n(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) = 0$ for $t \gg 0$.

Proof. Let A be an ample Cartier divisor such that

$$H^i(V, \mathcal{F} \otimes \mathcal{O}_V(A) \otimes \mathcal{L}^{\otimes t}) = 0$$

for every $i > 0$ and $t \geq 0$. Since $\mathcal{L}$ is big, there is a positive integer m such that $|\mathcal{L}^{\otimes m} \otimes \mathcal{O}_V(-A)| \neq \emptyset$. We take $D \in |\mathcal{L}^{\otimes m} \otimes \mathcal{O}_V(-A)|$ and consider the homomorphism $\gamma : \mathcal{F} \otimes \mathcal{O}_V(-D) \to \mathcal{F}$ induced by γ. Then we have

$$\begin{aligned}\dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq & \dim H^q(V, \operatorname{Coker}(\gamma) \otimes \mathcal{L}^{\otimes t}) \\ & + \dim H^q(V, \operatorname{Im}(\gamma) \otimes \mathcal{L}^{\otimes t}),\end{aligned}$$

and

$$\begin{aligned}\dim H^q(V, \mathrm{Im}(\gamma)\otimes \mathcal{L}^{\otimes t}) &\leq \dim H^q(V, \mathcal{F}\otimes \mathcal{O}_V(-D)\otimes \mathcal{L}^{\otimes t})\\ &\quad + \dim H^{q+1}(V, \mathrm{Ker}(\gamma)\otimes \mathcal{L}^{\otimes t})\\ &= \dim H^{q+1}(V, \mathrm{Ker}(\gamma)\otimes \mathcal{L}^{\otimes t})\end{aligned}$$

for every $t \geq m$. It is because

$$\begin{aligned}&H^q(V, \mathcal{F}\otimes \mathcal{O}_V(-D)\otimes \mathcal{L}^{\otimes t})\\ &\simeq H^q(V, \mathcal{F}\otimes \mathcal{O}_V(A)\otimes \mathcal{L}^{\otimes (t-m)}) = 0\end{aligned}$$

for every $t \geq m$. Note that

$$\dim H^q(V, \mathrm{Coker}(\gamma)\otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-1-q})$$

by Theorem 2.1 since $\mathrm{SuppCoker}(\gamma)$ is contained in D. On the other hand,

$$\dim H^{q+1}(V, \mathrm{Ker}(\gamma)\otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q-1})$$

by Theorem 2.1. By combining there estimates, we obtain the desired estimate. □

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