

FUJITA'S VANISHING THEOREM (PRIVATE NOTE)

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The following theorem is obtained by Takao Fujita (cf. [F1, Theorem (1)] and [F2, (5.1) Theorem]). See also [L, Theorem 1.4.35].

Theorem 1 (Fujita's vanishing theorem). *Let X be a projective scheme defined over a field k and let H be an ample Cartier divisor on X . Given any coherent sheaf $\mathcal{F}$ on X , there exists an integer $m(\mathcal{F}, H)$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for all $i > 0$, $m \geq m(\mathcal{F}, H)$, and any nef Cartier divisor D on X .

Proof. Without loss of generality, we may assume that k is algebraically closed. By replacing X with $\text{Supp} \mathcal{F}$, we may assume that $X = \text{Supp} \mathcal{F}$.

Remark 2. Let $\mathcal{F}$ be a coherent sheaf on X . We always define a subscheme structure on $\text{Supp} \mathcal{F}$ by the $\mathcal{O}_X$ -ideal $\text{Ker}(\mathcal{O}_X \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}))$.

We use the induction on dimension.

Step 1. When $\dim X = 0$, Theorem 1 obviously holds.

From now on, we assume that Theorem 1 holds in the lower dimensional case.

Step 2. We can reduce the proof to the case when X is reduced.

Proof. We assume that Theorem 1 holds for reduced schemes. Let $\mathcal{N}$ be the nilradical of $\mathcal{O}_X$, so that $\mathcal{N}^r = 0$ for some $r > 0$. Consider the filtration

$$\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \supset \cdots \supset \mathcal{N}^r \cdot \mathcal{F} = 0.$$

The quotients $\mathcal{N}^i \mathcal{F} / \mathcal{N}^{i+1} \mathcal{F}$ are coherent $\mathcal{O}_{X_{\text{red}}}$ -modules, and therefore, by the assumption,

$$H^j(X, (\mathcal{N}^i \mathcal{F} / \mathcal{N}^{i+1} \mathcal{F}) \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{N}^i \mathcal{F} / \mathcal{N}^{i+1} \mathcal{F}, H)$ thanks to the amplitude of $\mathcal{O}_{X_{\text{red}}}(H)$. Twisting the exact sequences

$$0 \rightarrow \mathcal{N}^{i+1} \mathcal{F} \rightarrow \mathcal{N}^i \mathcal{F} \rightarrow \mathcal{N}^i \mathcal{F} / \mathcal{N}^{i+1} \mathcal{F} \rightarrow 0$$

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by $\mathcal{O}_X(mH + D)$ and taking cohomology, we then find by decreasing induction on i that

$$H^j(X, \mathcal{N}^i \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{N}^i \mathcal{F}, H)$. When $i = 0$ this gives the desired vanishings. $\square$

From now on, we assume that X is reduced.

Step 3. We can reduce the proof to the case when X is irreducible.

Proof. We assume that Theorem 1 holds for reduced and irreducible schemes. Let $X = X_1 \cup \cdots \cup X_k$ be its decomposition into irreducible components and let $\mathcal{I}$ be the ideal sheaf of X_1 in X . We consider the exact sequence

$$0 \rightarrow \mathcal{I} \cdot \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{I} \cdot \mathcal{F} \rightarrow 0.$$

The outer terms of the above exact sequence are supported on $X_2 \cup \cdots \cup X_k$ and X_1 respectively. So by induction on the number of irreducible components, we may assume that

$$H^j(X, \mathcal{I}\mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{I}\mathcal{F}, H|_{X_2 \cup \cdots \cup X_k})$ and

$$H^j(X, (\mathcal{F}/\mathcal{I}\mathcal{F}) \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{F}/\mathcal{I}\mathcal{F}, H|_{X_1})$. It then follows from the above exact sequence that

$$H^j(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

when $j > 0$ and

$$m \geq m(\mathcal{F}, H) := \max\{m(\mathcal{I}\mathcal{F}, H|_{X_2 \cup \cdots \cup X_k}), m(\mathcal{F}/\mathcal{I}\mathcal{F}, H|_{X_1})\},$$

as required. $\square$

From now on, we assume that X is reduced and irreducible.

Step 4. We can reduce the proof to the case when H is very ample.

Proof. Let l be a positive integer such that lH is very ample. We assume that Theorem 1 holds for lH . Apply Theorem 1 to $\mathcal{F} \otimes \mathcal{O}_X(nH)$ for $0 \leq n \leq l-1$ with lH . Then we obtain $m(\mathcal{F} \otimes \mathcal{O}_X(nH), lH)$ for $0 \leq n \leq l-1$. We put

$$m(\mathcal{F}, H) = l \left(\max_n m(\mathcal{F} \otimes \mathcal{O}_X(nH), lH) + 1 \right).$$

Then we can easily check that $m(\mathcal{F}, H)$ satisfies the desired property. $\square$

From now on, we assume that H is very ample.

Step 5. It is sufficient to find $m(\mathcal{F}, H)$ such that

$$H^1(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for all $m \geq m(\mathcal{F}, H)$ and any nef Cartier divisor D on X .

Proof. We take a general member A of $|H|$ and consider the exact sequence

$$0 \rightarrow \mathcal{F} \otimes \mathcal{O}_X(-A) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_A \rightarrow 0.$$

Since $\dim \text{Supp} \mathcal{F}_A < \dim X$, we can find $m(\mathcal{F}_A, H|_A)$ such that

$$H^i(A, \mathcal{F}_A \otimes \mathcal{O}_A(mH + D)) = 0$$

for all $i > 0$ and $m \geq m(\mathcal{F}_A, H|_A)$ by the induction. Therefore,

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X((m-1)H + D)) = H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D))$$

for every $i \geq 2$ and $m \geq m(\mathcal{F}_A, H|_A)$. By Serre's vanishing theorem, we obtain

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X((m-1)H + D)) = 0$$

for every $i \geq 2$ and $m \geq m(\mathcal{F}_A, H|_A)$. $\square$

Step 6. We can reduce the proof to the case when $\mathcal{F} = \mathcal{O}_X$.

Proof. We assume that Theorem 1 holds for $\mathcal{F} = \mathcal{O}_X$. There is an injective homomorphism

$$\alpha : \mathcal{O}_X \rightarrow \mathcal{F} \otimes \mathcal{O}_X(aH)$$

for some large integer a . We consider the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F} \otimes \mathcal{O}_X(aH) \rightarrow \text{Coker} \alpha \rightarrow 0$$

and use the induction on $\text{rank } \mathcal{F}$. Then we can find $m(\mathcal{F}, H)$. $\square$

From now on, we assume $\mathcal{F} = \mathcal{O}_X$.

Step 7. If the characteristic of k is zero, then Theorem 1 holds.

Proof. Let $f : Y \rightarrow X$ be a resolution. Then we obtain the following exact sequence

$$0 \rightarrow f_*\omega_Y \rightarrow \mathcal{O}_X(bH) \rightarrow \mathcal{C} \rightarrow 0$$

for some integer b , where $\dim \text{Supp } \mathcal{C} < \dim X$. Note that $f_*\omega_Y$ is torsion-free and $\text{rank } f_*\omega_Y$ is one. On the other hand,

$$H^j(X, f_*\omega_Y \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m > 0$ and $j > 0$ by Kollár's vanishing theorem. Therefore,

$$H^j(X, \mathcal{O}_X((b+m)H + D)) = 0$$

for every positive integer $m \geq m(\mathcal{C}, H)$ and $j > 0$. $\square$

Step 8. We can reduce the proof to the case when $\mathcal{F} = \omega_X$, where ω_X is the dualizing sheaf of X .

Remark 3. The dualizing sheaf ω_X is denoted by ω_X° in [H, Chapter III §7]. We know that $\omega_X^\circ \simeq \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^N}}^{N-\dim X}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ when $X \subset \mathbb{P}^N$. For details, see the proof of Proposition 7.5 in [H, Chapter III §7].

Proof. We assume that Theorem 1 holds for $\mathcal{F} = \omega_X$. There is an injective homomorphism

$$\beta : \omega_X \rightarrow \mathcal{O}_X(cH)$$

for some positive integer c . Note that ω_X is torsion-free. We consider the exact sequence

$$0 \rightarrow \omega_X \rightarrow \mathcal{O}_X(cH) \rightarrow \text{Coker } \beta \rightarrow 0.$$

We note that $\dim \text{Supp Coker } \beta < \dim X$ because

$$\text{rank } \omega_X = \text{rank } \mathcal{O}_X(cH) = 1.$$

Therefore, we can find $m(\mathcal{O}_X, H)$ by the induction on dimension and Theorem 1 for ω_X . $\square$

From now on, we assume that $\mathcal{F} = \omega_X$ and that the characteristic of k is positive.

Step 9. Theorem 1 holds when the characteristic of k is positive.

Proof. Let $X \rightarrow \mathbb{P}^N$ be the embedding induced by H . Let

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ \mathbb{P}^N & \xrightarrow{F} & \mathbb{P}^N \end{array}$$

be the commutative diagram of the Frobenius morphisms. By taking $R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(_, \omega_{\mathbb{P}^N}^\bullet)$ to $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$, we obtain

$$R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(F_*\mathcal{O}_X, \omega_{\mathbb{P}^N}^\bullet) \rightarrow R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}^\bullet).$$

By the Grothendieck duality,

$$R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(F_*\mathcal{O}_X, \omega_{\mathbb{P}^N}^\bullet) \simeq F_*R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}^\bullet).$$

Therefore, we obtain

$$\gamma : F_*\omega_X \rightarrow \omega_X.$$

Note that $\omega_X = \mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^N}}^{N-\dim X}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$. Let U be a non-empty Zariski open set of X such that U is smooth. We can easily check that

$$\gamma : F_*\omega_X \rightarrow \omega_X$$

is surjective on U . Note that the cokernel $\mathcal{A}$ of $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ is locally free on U . Then $\mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^N}}^k(\mathcal{A}, \omega_{\mathbb{P}^N}) = 0$ for $k > N - \dim X$ on U . We consider the exact sequences

$$0 \rightarrow \text{Ker}\gamma \rightarrow F_*\omega_X \rightarrow \text{Im}\gamma \rightarrow 0$$

and

$$0 \rightarrow \text{Im}\gamma \rightarrow \omega_X \rightarrow \mathcal{C} \rightarrow 0.$$

Then $\dim \text{Supp } \mathcal{C} < \dim X$. Note that there is an integer m_1 such that

$$H^2(X, \text{Ker}\gamma \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m \geq m_1$ by Step 5. By applying the induction on dimension to $\mathcal{C}$, we obtain some positive integer m_0 such that

$$H^1(X, F_*\omega_X \otimes \mathcal{O}_X(mH + D)) \rightarrow H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D))$$

is surjective for every $m \geq m_0$. We note that

$$H^1(X, F_*\omega_X \otimes \mathcal{O}_X(mH + D)) \simeq H^1(X, \omega_X \otimes \mathcal{O}_X(p(mH + D)))$$

by the projection formula, where p is the characteristic of k . By repeating the above process, we obtain that

$$H^1(X, \omega_X \otimes \mathcal{O}_X(p^e(mH + D))) \rightarrow H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D))$$

is surjective for every $e > 0$ and $m \geq m_0$. Note that m_0 is independent of the nef divisor D . Therefore, by Serre's vanishing theorem, we obtain

$$H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m \geq m_0$. □

We finish the proof of Theorem 1. □

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