

FUJITA-TYPE FREENESS FOR QUASI-LOG CANONICAL CURVES AND SURFACES

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ABSTRACT. We prove Fujita-type basepoint-freeness for projective quasi-log canonical curves and surfaces.

1. INTRODUCTION

Fujita's freeness conjecture is very famous and is still open for higher-dimensional varieties. Now we know that it holds true in dimension ≤ 5 (for the details, see [YZ] and the references therein).

Conjecture 1.1 (Fujita's freeness conjecture). *Let X be a smooth projective variety of dimension n . Let L be an ample Cartier divisor. Then the complete linear system $|K_X + (n + 1)L|$ is basepoint-free.*

In this paper, we treat a generalization of Fujita's freeness conjecture for highly singular varieties. More precisely, we are mainly interested in quasi-log canonical pairs. A quasi-log canonical pair may be reducible and is not necessarily equidimensional. The union of some log canonical centers of a given log canonical pair is a typical example of quasi-log canonical pairs. We think that it is worth formulating and studying various conjectures for quasi-log canonical pairs in order to solve the original conjecture by some inductive arguments on the dimension.

Conjecture 1.2 (Fujita-type freeness for quasi-log canonical pairs). *Let $[X, \omega]$ be a projective quasi-log canonical pair of dimension n . Let M be a Cartier divisor on X . We put $N = M - \omega$. Assume that $N^{\dim X_i} \cdot X_i > (\dim X_i)^{\dim X_i}$ for every positive-dimensional irreducible component X_i of X . For every positive-dimensional subvariety Z which is not an irreducible component of X , we put*

$$n_Z = \min_i \{ \dim X_i \mid X_i \text{ is an irreducible component of } X \text{ with } Z \subset X_i \}$$

and assume that $N^{\dim Z} \cdot Z \geq n_Z^{\dim Z}$. Then the complete linear system $|M|$ is basepoint-free.

If $N^{\dim X_i} \cdot X_i > (\frac{1}{2}n(n+1))^{\dim X_i}$ and $N^{\dim Z} \cdot Z > (\frac{1}{2}n(n+1))^{\dim Z}$ hold in Conjecture 1.2, then we have already known that the complete linear system $|M|$ is basepoint-free by the second author's theorem (see [L, Theorem 1.1] for the precise statement). It is a generalization of Angehrn–Siu's theorem (see [AS]). When $\dim X = 1$, we can easily check that Conjecture 1.2 holds true.

Theorem 1.3 (Theorem 3.1). *Conjecture 1.2 holds true for $n = 1$.*

The main technical result of this paper is the following theorem.

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Theorem 1.4 (Theorem 3.2). *Let $[X, \omega]$ be a quasi-log canonical pair such that X is a normal projective irreducible surface. Let M be a Cartier divisor on X . We put $N = M - \omega$. We assume that $N^2 > 4$ and $N \cdot C \geq 2$ for every curve C on X . Let P be any closed point of X that is not included in $\text{Nqklt}(X, \omega)$, the union of all qlc centers of $[X, \omega]$. Then there exists $s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))$ such that $s(P) \neq 0$, where $\mathcal{I}_{\text{Nqklt}(X, \omega)}$ is the defining ideal sheaf of $\text{Nqklt}(X, \omega)$ on X .*

The proof of Theorem 1.4 in Section 3 heavily depends on the first author's new result obtained in [F6] (see Theorem 2.12 below), which comes from the theory of variations of mixed Hodge structure on cohomology with compact support. By combining Theorems 1.3 and 1.4 with our result on the normalization of quasi-log canonical pairs (see Theorem 2.11 below), we prove Conjecture 1.2 for $n = 2$ in full generality.

Corollary 1.5 (Corollary 3.3). *Conjecture 1.2 holds true for $n = 2$.*

We note that we can recover the main theorem of [F4] by combining Theorem 1.3 and Corollary 1.5 with the main result of [F2].

Corollary 1.6 ([F4, Theorem 1.3]). *Let (X, Δ) be a projective semi-log canonical pair of dimension n . Let M be a Cartier divisor on X . We put $N = M - (K_X + \Delta)$. Assume that $N^n \cdot X_i > n^n$ for every irreducible component X_i of X and that $N^k \cdot Z \geq n^k$ for every subvariety Z with $0 < \dim Z = k < n$. We further assume that $n = 1$ or 2 . Then the complete linear system $|M|$ is basepoint-free.*

Let us quickly explain our strategy to prove Conjecture 1.2. From now on, we will use the same notation as in Conjecture 1.2. We take an arbitrary closed point P of X . Then it is sufficient to find $s \in H^0(X, \mathcal{O}_X(M))$ with $s(P) \neq 0$. Let X_i be an irreducible component of X such that $P \in X_i$. By adjunction (see Theorem 2.8 (i)), $[X_i, \omega|_{X_i}]$ is a quasi-log canonical pair. By the vanishing theorem (see Theorem 2.8 (ii)), the natural restriction map $H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(X_i, \mathcal{O}_{X_i}(M))$ is surjective. Therefore, by replacing X with X_i , we may assume that X is irreducible. By adjunction again, $[\text{Nqklt}(X, \omega), \omega|_{\text{Nqklt}(X, \omega)}]$ is a quasi-log canonical pair. By the vanishing theorem, the natural restriction map $H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(\text{Nqklt}(X, \omega), \mathcal{O}_{\text{Nqklt}(X, \omega)}(M))$ is surjective. Therefore, if $P \in \text{Nqklt}(X, \omega)$, then we can use induction on the dimension. Thus we may further assume that $P \notin \text{Nqklt}(X, \omega)$. In this situation, we know that X is normal at P . Let $\nu : \tilde{X} \rightarrow X$ be the normalization. Then, by Theorem 2.11, $[\tilde{X}, \nu^*\omega]$ is a quasi-log canonical pair with $\nu_*\mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} = \mathcal{I}_{\text{Nqklt}(X, \omega)}$. Therefore, it is sufficient to find $\tilde{s} \in H^0(\tilde{X}, \mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} \otimes \mathcal{O}_{\tilde{X}}(\nu^*M))$ with $\tilde{s}(\tilde{P}) \neq 0$, where $\tilde{P} = \nu^{-1}(P)$. By replacing X with \tilde{X} , we may assume that X is a normal irreducible variety. By using Theorem 2.12, we can take a boundary \mathbb{R} -divisor Δ , that is, an effective \mathbb{R} -divisor Δ with $\Delta = \Delta^{\leq 1}$, on X such that $K_X + \Delta \sim_{\mathbb{R}} \omega + \varepsilon N$ for $0 < \varepsilon \ll 1$ and $\mathcal{J}(X, \Delta) = \mathcal{I}_{\text{Nqklt}(X, \omega)}$. Note that $\mathcal{J}(X, \Delta)$ is the multiplier ideal sheaf of (X, Δ) . Since $\mathcal{J}(X, \Delta) = \mathcal{I}_{\text{Nqklt}(X, \omega)}$, (X, Δ) is klt in a neighborhood of P . Anyway, it is sufficient to find $s \in H^0(X, \mathcal{J}(X, \Delta) \otimes \mathcal{O}_X(M))$ with $s(P) \neq 0$. In this paper, we will carry out the above strategy in dimension two.

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We will work over \mathbb{C} , the complex number field, throughout this paper. A scheme means a separated scheme of finite type over \mathbb{C} . A variety means a reduced scheme, that is, a reduced separated scheme of finite type over \mathbb{C} . We sometimes assume that a variety is irreducible without mentioning it explicitly if there is no risk of confusion. We will

freely use the standard notation of the minimal model program and the theory of quasi-log schemes as in [F1] and [F5]. For the details of semi-log canonical pairs, see [F2].

2. PRELIMINARIES

In this section, we collect some basic definitions and explain some results on quasi-log schemes.

Definition 2.1 (\mathbb{R} -divisors). Let X be an equidimensional variety, which is not necessarily regular in codimension one. Let D be an \mathbb{R} -divisor, that is, D is a finite formal sum $\sum_i d_i D_i$, where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is a real number for every i such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad D^{>1} = \sum_{d_i > 1} d_i D_i, \quad \text{and} \quad D^=1 = \sum_{d_i = 1} D_i.$$

We also put

$$[D] = \sum_i [d_i] D_i \quad \text{and} \quad \lfloor D \rfloor = -[-D],$$

where $[d_i]$ is the integer defined by $d_i \leq [d_i] < d_i + 1$. When $D = D^{\leq 1}$ holds, we usually say that D is a *subboundary* \mathbb{R} -divisor.

Let B_1 and B_2 be \mathbb{R} -Cartier divisors on X . Then $B_1 \sim_{\mathbb{R}} B_2$ means that B_1 is \mathbb{R} -linearly equivalent to B_2 .

Let us quickly recall singularities of pairs for the reader's convenience. We recommend the reader to see [F5, Section 2.3] for the details.

Definition 2.2 (Singularities of pairs). Let X be a normal variety and let Δ be an \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a projective birational morphism from a smooth variety Y . Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta) E,$$

where $a(E, X, \Delta) \in \mathbb{R}$ and E is a prime divisor on Y . By taking $f : Y \rightarrow X$ suitably, we can define $a(E, X, \Delta)$ for any prime divisor E over X and call it the *discrepancy* of E with respect to (X, Δ) . If $a(E, X, \Delta) > -1$ (resp. $a(E, X, \Delta) \geq -1$) holds for any prime divisor E over X , then we say that (X, Δ) is *sub klt* (resp. *sub log canonical*). If (X, Δ) is sub klt (resp. sub log canonical) and Δ is effective, then we say that (X, Δ) is *klt* (resp. *log canonical*). If (X, Δ) is log canonical and $a(E, X, \Delta) > -1$ for any prime divisor E that is exceptional over X , then we say that (X, Δ) is *plt*.

If there exist a projective birational morphism $f : Y \rightarrow X$ from a smooth variety Y and a prime divisor E on Y such that $a(E, X, \Delta) = -1$ and (X, Δ) is log canonical in a neighborhood of the generic point of $f(E)$, then $f(E)$ is called a *log canonical center* of (X, Δ) .

Definition 2.3 (Multiplier ideal sheaves). Let X be a normal variety and let Δ be an effective \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a projective birational morphism from a smooth variety such that

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

and $\text{Supp } \Delta_Y$ is a simple normal crossing divisor on Y . We put

$$\mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor)$$

and call it the *multiplier ideal sheaf* of (X, Δ) . We can easily check that $\mathcal{J}(X, \Delta)$ is a well-defined ideal sheaf on X . The closed subscheme defined by $\mathcal{J}(X, \Delta)$ is denoted by $\text{Nklt}(X, \Delta)$.

The notion of *globally embedded simple normal crossing pairs* plays a crucial role in the theory of quasi-log schemes described in [F5, Chapter 6].

Definition 2.4 (Globally embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and let B be an \mathbb{R} -divisor on M such that Y and B have no common irreducible components and that the support of $Y + B$ is a simple normal crossing divisor on M . In this situation, (Y, B_Y) , where $B_Y := B|_Y$, is called a *globally embedded simple normal crossing pair*. A *stratum* of (Y, B_Y) means a log canonical center of $(M, Y + B)$ included in Y .

Let us recall the notion of *quasi-log schemes*, which was first introduced by Florin Ambro (see [A]). The following definition is slightly different from the original one. For the details, see [F3, Appendix A]. In this paper, we will use the framework of quasi-log schemes established in [F5, Chapter 6].

Definition 2.5 (Quasi-log schemes). A *quasi-log scheme* is a scheme X endowed with an \mathbb{R} -Cartier divisor (or \mathbb{R} -line bundle) ω on X , a closed subscheme $\text{Nqlc}(X, \omega) \subsetneq X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there exists a proper morphism $f : (Y, B_Y) \rightarrow X$ from a globally embedded simple normal crossing pair (Y, B_Y) satisfying the following properties:

- (1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
- (2) The natural map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y(\lceil -(B_Y^{\leq 1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{\text{Nqlc}(X, \omega)} \xrightarrow{\sim} f_*\mathcal{O}_Y(\lceil -(B_Y^{\leq 1}) \rceil - \lfloor B_Y^{\geq 1} \rfloor),$$

where $\mathcal{I}_{\text{Nqlc}(X, \omega)}$ is the defining ideal sheaf of $\text{Nqlc}(X, \omega)$.

- (3) The collection of subvarieties $\{C\}$ coincides with the images of (Y, B_Y) -strata that are not included in $\text{Nqlc}(X, \omega)$.

We simply write $[X, \omega]$ to denote the above data

$$(X, \omega, f : (Y, B_Y) \rightarrow X)$$

if there is no risk of confusion. We note that the subvarieties C are called the *qlc strata* of $(X, \omega, f : (Y, B_Y) \rightarrow X)$ or simply of $[X, \omega]$. If C is a qlc stratum of $[X, \omega]$ but is not an irreducible component of X , then C is called a *qlc center* of $[X, \omega]$. The union of all qlc centers of $[X, \omega]$ is denoted by $\text{Nqklt}(X, \omega)$.

If B_Y is a subboundary \mathbb{R} -divisor, then $[X, \omega]$ in Definition 2.5 is called a *quasi-log canonical pair*.

Definition 2.6 (Quasi-log canonical pairs). Let $(X, \omega, f : (Y, B_Y) \rightarrow X)$ be a quasi-log scheme as in Definition 2.5. We say that $(X, \omega, f : (Y, B_Y) \rightarrow X)$ or simply $[X, \omega]$ is a *quasi-log canonical pair* (*qlc pair*, for short) if $\text{Nqlc}(X, \omega) = \emptyset$. Note that the condition $\text{Nqlc}(X, \omega) = \emptyset$ is equivalent to $B_Y^{\geq 1} = 0$, that is, $B_Y = B_Y^{\leq 1}$.

The following example is very important. Precisely speaking, the notion of quasi-log schemes was originally introduced by Florin Ambro (see [A]) in order to establish the cone and contraction theorem for *generalized log varieties*. Note that a generalized log variety (X, Δ) means that X is a normal variety and Δ is an effective \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier as in Example 2.7 below.

Example 2.7. Let X be a normal irreducible variety and let Δ be an effective \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : Y \rightarrow X$ be a projective birational morphism from a smooth variety Y . We define Δ_Y by

$$K_Y + \Delta_Y = f^*(K_X + \Delta).$$

We may assume that $\text{Supp } \Delta_Y$ is a simple normal crossing divisor on Y by taking $f : Y \rightarrow X$ suitably. We put $M = Y \times \mathbb{C}$ and consider $Y \simeq Y \times \{0\} \hookrightarrow Y \times \mathbb{C} = M$. Then we can see (Y, Δ_Y) as a globally embedded simple normal crossing pair. We put $\omega := K_X + \Delta$ and

$$\mathcal{I}_{\text{Nqlc}(X, \omega)} := f_* \mathcal{O}_Y([- (\Delta_Y^{\leq 1})] - [\Delta_Y^{\geq 1}]) \subset \mathcal{O}_X.$$

Then $(X, \omega, f : (Y, \Delta_Y) \rightarrow X)$ is a quasi-log scheme. In this case, C is a qlc center of $[X, \omega]$ if and only if C is a log canonical center of (X, Δ) . If C is a qlc stratum but is not a qlc center of $[X, \omega]$, then C is nothing but X .

One of the most important results in the theory of quasi-log schemes is the following theorem.

Theorem 2.8. *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $\text{Nqlc}(X, \omega)$ with a (possibly empty) union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

- (i) (Adjunction). *Assume that $X' \neq \text{Nqlc}(X, \omega)$. Then $[X', \omega']$ is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $\text{Nqlc}(X', \omega') = \text{Nqlc}(X, \omega)$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X' .*
- (ii) (Vanishing theorem). *Assume that $\pi : X \rightarrow S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$, that is, $L - \omega$ is π -nef and $(L - \omega)|_C$ is π -big for every qlc stratum C of $[X, \omega]$. Then $R^i \pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X .*

For the proof of Theorem 2.8, see, for example, [F5, Theorem 6.3.5]. We note that we generalized Kollár's torsion-free and vanishing theorems in [F5, Chapter 5] by using the theory of mixed Hodge structures on cohomology with compact support in order to establish Theorem 2.8.

Let us quickly recall the definition of semi-log canonical pairs for the reader's convenience.

Definition 2.9 (Semi-log canonical pairs). Let X be an equidimensional variety that is normal crossing in codimension one and satisfies Serre's S_2 condition and let Δ be an effective \mathbb{R} -divisor on X such that the singular locus of X contains no irreducible components of $\text{Supp } \Delta$. Assume that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $\nu : \tilde{X} \rightarrow X$ be the normalization. We put $K_{\tilde{X}} + \Delta_{\tilde{X}} = \nu^*(K_X + \Delta)$, that is, $\Delta_{\tilde{X}}$ is the union of the inverse images of Δ and the conductor of X . If $(\tilde{X}, \Delta_{\tilde{X}})$ is log canonical, then (X, Δ) is called a *semi-log canonical pair*.

The theory of quasi-log schemes plays an important role for the study of semi-log canonical pairs by the following theorem: Theorem 2.10. For the precise statement and some related results, see [F2].

Theorem 2.10 ([F2, Theorem 1.2]). *Let (X, Δ) be a quasi-projective semi-log canonical pair. Then $[X, K_X + \Delta]$ is a quasi-log canonical pair.*

For the proof of Corollary 1.5, we will use Theorem 2.11 below.

Theorem 2.11 ([FL, Theorem 1.1]). *Let $[X, \omega]$ be a quasi-log canonical pair such that X is irreducible. Let $\nu : \tilde{X} \rightarrow X$ be the normalization. Then $[\tilde{X}, \nu^* \omega]$ naturally becomes a quasi-log canonical pair with the following properties:*

- (i) *if C is a qlc center of $[\tilde{X}, \nu^* \omega]$, then $\nu(C)$ is a qlc center of $[X, \omega]$, and*

(ii) $\mathrm{Nqklt}(\tilde{X}, \nu^*\omega) = \nu^{-1}(\mathrm{Nqklt}(X, \omega))$. More precisely, the equality

$$\nu_*\mathcal{I}_{\mathrm{Nqklt}(\tilde{X}, \nu^*\omega)} = \mathcal{I}_{\mathrm{Nqklt}(X, \omega)}$$

holds, where $\mathcal{I}_{\mathrm{Nqklt}(X, \omega)}$ and $\mathcal{I}_{\mathrm{Nqklt}(\tilde{X}, \nu^*\omega)}$ are the defining ideal sheaves of $\mathrm{Nqklt}(X, \omega)$ and $\mathrm{Nqklt}(\tilde{X}, \nu^*\omega)$ respectively.

The following theorem is a special case of [F6, Theorem 1.5]. It is a deep result based on the theory of variations of mixed Hodge structure on cohomology with compact support.

Theorem 2.12 ([F6, Theorem 1.5]). *Let $[X, \omega]$ be a quasi-log canonical pair such that X is a normal projective irreducible variety. Then there exists a projective birational morphism $p : X' \rightarrow X$ from a smooth projective variety X' such that*

$$K_{X'} + B_{X'} + M_{X'} = p^*\omega,$$

where $B_{X'}$ is a subboundary \mathbb{R} -divisor, that is, $B_{X'} = B_{X'}^{\leq 1}$, such that $\mathrm{Supp} B_{X'}$ is a simple normal crossing divisor and that $p_*B_{X'}$ is effective, and $M_{X'}$ is a nef \mathbb{R} -divisor on X' . Furthermore, we can make $B_{X'}$ satisfy $p(B_{X'}^{\leq 1}) = \mathrm{Nqklt}(X, \omega)$.

We close this section with an easy lemma, which is essentially contained in [F5, Chapter 6].

Lemma 2.13. *Let $[X, \omega]$ be a quasi-log canonical pair such that X is an irreducible curve. Let P be a smooth point of X such that P is not a qlc center of $[X, \omega]$. Then we can consider a natural quasi-log structure on $[X, \omega + tP]$ induced from $[X, \omega]$ for every $t \geq 0$. We put*

$$c = \max\{t \geq 0 \mid [X, \omega + tP] \text{ is quasi-log canonical}\}.$$

Then $0 < c \leq 1$ holds.

Proof. Since $[X, \omega]$ is a qlc pair, we can take a projective surjective morphism $f : (Y, B_Y) \rightarrow X$ from a globally embedded simple normal crossing pair (Y, B_Y) such that B_Y is a subboundary \mathbb{R} -divisor on Y and that the natural map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y([- (B_Y^{\leq 1})])$ is an isomorphism. By taking some blow-ups, we may further assume that $(Y, \mathrm{Supp} B_Y + \mathrm{Supp} f^*P)$ is a globally embedded simple normal crossing pair. Then it is easy to see that

$$(X, \omega + tP, f : (Y, B_Y + tf^*P) \rightarrow X)$$

is a quasi-log scheme for every $t \geq 0$. We assume that $c > 1$. Then $\mathrm{mult}_S(B_Y + f^*P) < 1$ for any irreducible component S of $\mathrm{Supp} f^*P$. Therefore, $f^*P \leq [- (B_Y^{\leq 1})]$ holds. Thus we have $\mathcal{O}_X \subsetneq \mathcal{O}_X(P) \subset f_*\mathcal{O}_Y([- (B_Y^{\leq 1})])$. This is a contradiction. This means that $c \leq 1$ holds. By definition, we can easily see that $0 < c$ holds. \square

3. PROOF

In this section, we prove the results in Section 1, that is, Theorems 1.3, 1.4, Corollaries 1.5, and 1.6.

First, we prove Theorem 1.3, that is, we prove Conjecture 1.2 when $\dim X = 1$.

Theorem 3.1 (Theorem 1.3). *Let $[X, \omega]$ be a projective quasi-log canonical pair of dimension one. Let M be a Cartier divisor on X . We put $N = M - \omega$. Assume that $N \cdot X_i > 1$ for every one-dimensional irreducible component X_i of X . Then the complete linear system $|M|$ is basepoint-free.*

Proof. Let P be an arbitrary closed point of X . If P is a qlc center of $[X, \omega]$, then $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(M)) = 0$ by Theorem 2.8 (ii), where \mathcal{I}_P is the defining ideal sheaf of P on X . Therefore, the natural restriction map

$$H^0(X, \mathcal{O}_X(M)) \rightarrow \mathcal{O}_X(M) \otimes \mathbb{C}(P)$$

is surjective. Thus, the complete linear system $|M|$ is basepoint-free in a neighborhood of P . From now on, we assume that P is not a qlc center of $[X, \omega]$. Let X_i be the unique irreducible component of X containing P . By Theorem 2.8 (ii), $H^1(X, \mathcal{I}_{X_i} \otimes \mathcal{O}_X(M)) = 0$, where \mathcal{I}_{X_i} is the defining ideal sheaf of X_i on X . We note that X_i is a qlc stratum of $[X, \omega]$. Thus, the restriction map

$$H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(X_i, \mathcal{O}_{X_i}(M))$$

is surjective. Therefore, by replacing X with X_i , we may assume that X is irreducible. By Lemma 2.13, we can take $c \in \mathbb{R}$ such that $0 < c \leq 1$ and that P is a qlc center of $[X, \omega + cP]$. Since $\deg(M - (\omega + cP)) > 1 - c \geq 0$, we have $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(M)) = 0$ by Theorem 2.8 (ii). Therefore, by the same argument as above, $|M|$ is basepoint-free in a neighborhood of P . Thus we obtain that the complete linear system $|M|$ is basepoint-free. \square

Next, we prove Theorem 1.4, which is the main technical result of this paper.

Theorem 3.2 (Theorem 1.4). *Let $[X, \omega]$ be a quasi-log canonical pair such that X is a normal projective irreducible surface. Let M be a Cartier divisor on X . We put $N = M - \omega$. We assume that $N^2 > 4$ and $N \cdot C \geq 2$ for every curve C on X . Let P be any closed point of X that is not included in $\text{Nqklt}(X, \omega)$. Then there exists $s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))$ such that $s(P) \neq 0$.*

Proof. By assumption and Nakai's ampleness criterion for \mathbb{R} -divisors (see [CP]), N is ample. In Step 1, we will prove Theorem 3.2 under the extra assumption that P is a smooth point of X . In Step 2, we will treat the case where P is a singular point of X .

Step 1. In this step, we assume that P is a smooth point of X . Since $N^2 > 4$, we can take an effective \mathbb{R} -divisor B on X such that $B \sim_{\mathbb{R}} N$ with $\text{mult}_P B = 2 + \alpha > 2$. By Theorem 2.12, there exists a projective birational morphism $p : X' \rightarrow X$ from a smooth projective surface X' such that $K_{X'} + B_{X'} + M_{X'} = p^*\omega$, where $B_{X'}$ is a subboundary \mathbb{R} -divisor such that $p_*B_{X'}$ is effective and $M_{X'}$ is a nef \mathbb{R} -divisor on X' . Let $\text{Exc}(p)$ denote the exceptional locus of p . By taking some more blow-ups, we may further assume that $p(B_{X'}^{-1}) = \text{Nqklt}(X, \omega)$ and that $\text{Supp } B_{X'} \cup \text{Supp } p_*^{-1}B \cup \text{Exc}(p)$ is contained in a simple normal crossing divisor Σ on X' (see Theorem 2.12).

Let ε be a small positive real number such that $(1 - \varepsilon)(2 + \alpha) > 2$. We can take an effective p -exceptional \mathbb{Q} -divisor E on X' such that $-E$ is p -ample and that $M_{X'} + \varepsilon(p^*N - E)$ is semi-ample for any $\varepsilon > 0$. For $0 < \varepsilon \ll 1$, we put $\Delta_\varepsilon := p_*(B_{X'} + \varepsilon E + G_\varepsilon)$ where G_ε is a general effective \mathbb{R} -divisor such that $G_\varepsilon \sim_{\mathbb{R}} M_{X'} + \varepsilon(p^*N - E)$, $\text{Supp } G_\varepsilon$ and $\text{Supp } \Sigma$ have no common irreducible components, $[G_\varepsilon] = 0$, and $\text{Supp}(\Sigma + G_\varepsilon)$ is a simple normal crossing divisor on X' . Since the effective part of $-[B_{X'} + \varepsilon E + G_\varepsilon]$ is p -exceptional and $p(B_{X'}^{-1}) = \text{Nqklt}(X, \omega)$, we obtain

$$\begin{aligned} \mathcal{J}(X, \Delta_\varepsilon) &= p_*\mathcal{O}_{X'}(-[B_{X'} + \varepsilon E + G_\varepsilon]) \\ &= p_*\mathcal{O}_{X'}(-[(B_{X'} + \varepsilon E + G_\varepsilon)^{\geq 1}]) \\ &= p_*\mathcal{O}_{X'}(-B_{X'}^{-1}) \\ &= \mathcal{I}_{\text{Nqklt}(X, \omega)}. \end{aligned}$$

We put $B_\varepsilon := (1 - \varepsilon)B$ and define

$$r_\varepsilon = \max\{t \geq 0 \mid (X, \Delta_\varepsilon + tB_\varepsilon) \text{ is log canonical at } P\}.$$

By construction, $\text{mult}_P B_\varepsilon > 2$ and Δ_ε is an effective \mathbb{R} -divisor on X . Therefore, we have $0 < r_\varepsilon < 1$. Note that (X, Δ_ε) is klt at P . By construction again, there is an irreducible component S_ε of Σ such that

$$r_\varepsilon \text{mult}_{S_\varepsilon} p^*B_\varepsilon + \text{mult}_{S_\varepsilon} B_{X'} + \varepsilon \text{mult}_{S_\varepsilon} E = 1.$$

Therefore,

$$0 < r_\varepsilon = \frac{1 - \text{mult}_{S_\varepsilon} B_{X'} - \varepsilon \text{mult}_{S_\varepsilon} E}{(1 - \varepsilon) \text{mult}_{S_\varepsilon} p^* B} < 1$$

holds. Since there are only finitely many components of Σ , we can take $\{\varepsilon_i\}_{i=1}^\infty$ and $\delta > 0$ such that $0 < \varepsilon_i \ll 1$, $\mathcal{J}(X, \Delta_{\varepsilon_i}) = \mathcal{I}_{\text{Nqklt}(X, \omega)}$, $(X, \Delta_{\varepsilon_i})$ is klt at P , $(X, \Delta_{\varepsilon_i} + r_{\varepsilon_i} B_{\varepsilon_i})$ is log canonical at P but is not klt at P with $\delta < r_{\varepsilon_i} < 1$ for every i , and $\varepsilon_i \searrow 0$ for $i \nearrow \infty$.

By $p : X' \rightarrow X$, we get a natural quasi-log structure on $[X, \omega_\varepsilon]$ with $\omega_\varepsilon := K_X + \Delta_\varepsilon + r_\varepsilon B_\varepsilon$ for any $\varepsilon = \varepsilon_i$ (see Example 2.7). Note that $[X, \omega_\varepsilon]$ is qlc in a neighborhood of P since $(X, \Delta_\varepsilon + r_\varepsilon B_\varepsilon)$ is log canonical around P . Let W_ε be the minimal qlc center of $[X, \omega_\varepsilon]$ passing through P , equivalently, let W_ε be the minimal log canonical center of $(X, \Delta_\varepsilon + r_\varepsilon B_\varepsilon)$ passing through P . Let V_ε be the union of all qlc centers of $[X, \omega_\varepsilon]$ contained in $\text{Nqklt}(X, \omega) = \text{Nklt}(X, \Delta_\varepsilon)$. We put $Z_\varepsilon = \text{Nqlc}(X, \omega_\varepsilon) \cup V_\varepsilon \cup W_\varepsilon$ and $Y_\varepsilon = \text{Nqlc}(X, \omega_\varepsilon) \cup V_\varepsilon$. Then $[Z_\varepsilon, \omega_\varepsilon|_{Z_\varepsilon}]$ and $[Y_\varepsilon, \omega_\varepsilon|_{Y_\varepsilon}]$ have natural quasi-log structures induced from $[X, \omega_\varepsilon]$ by adjunction (see Theorem 2.8 (i)). Since

$$M - \omega_\varepsilon = M - (K_X + \Delta_\varepsilon + r_\varepsilon B_\varepsilon) \sim_{\mathbb{R}} (1 - r_\varepsilon)(1 - \varepsilon)N,$$

which is still ample, the restriction map

$$(3.1) \quad H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(Z_\varepsilon, \mathcal{O}_{Z_\varepsilon}(M))$$

is surjective by Theorem 2.8 (ii).

Case 1. If $\dim W_\varepsilon = 0$, then $W_\varepsilon = P$ is isolated in Z_ε by construction. Thus Z_ε is the disjoint union of P and Y_ε . Therefore, by (3.1), the restriction map

$$H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(Y_\varepsilon, \mathcal{O}_{Y_\varepsilon}(M)) \oplus H^0(P, \mathcal{O}_P(M))$$

is surjective. This means that there exists $s \in H^0(X, \mathcal{O}_X(M))$ such that $s(P) \neq 0$ and $s \in H^0(X, \mathcal{I}_{Y_\varepsilon} \otimes \mathcal{O}_X(M)) \subset H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))$. Note that $\mathcal{I}_{Y_\varepsilon}$ is the defining ideal sheaf of Y_ε on X and the natural inclusion $\mathcal{I}_{Y_\varepsilon} \subset \mathcal{I}_{\text{Nqklt}(X, \omega)}$ holds by construction. This is what we wanted.

Case 2. By Case 1, we may assume that $\dim W_\varepsilon = 1$ for any $\varepsilon = \varepsilon_i$. By construction, P is not a qlc center of $[Z_\varepsilon, \omega_\varepsilon|_{Z_\varepsilon}]$. Therefore, Z_ε is smooth at P since $\dim W_\varepsilon = 1$ (see, for example, [F5, Theorem 6.3.11 (ii)]). Let us consider $[Z_\varepsilon, \omega_\varepsilon|_{Z_\varepsilon} + c_\varepsilon P]$ where c_ε is the minimum positive real number such that P is a qlc center of $[Z_\varepsilon, \omega_\varepsilon|_{Z_\varepsilon} + c_\varepsilon P]$ (see Lemma 2.13 and its proof). We write $\Delta_\varepsilon + r_\varepsilon B_\varepsilon = W_\varepsilon + \Delta'_\varepsilon$. We put $\text{mult}_P \Delta'_\varepsilon = \beta_\varepsilon \geq 0$. Then

$$\beta_\varepsilon = \text{mult}_P \Delta_\varepsilon + r_\varepsilon(1 - \varepsilon)(2 + \alpha) - 1 \geq r_\varepsilon(1 - \varepsilon)(2 + \alpha) - 1.$$

We note that

$$\beta_\varepsilon \leq \text{mult}_P(\Delta'_\varepsilon|_{W_\varepsilon}) < 1$$

holds because $(X, W_\varepsilon + \Delta'_\varepsilon)$ is plt in a neighborhood of P . We note that

$$(X, W_\varepsilon + \Delta'_\varepsilon + (1 - \text{mult}_P(\Delta'_\varepsilon|_{W_\varepsilon}))H)$$

is log canonical but is not plt in a neighborhood of P , where H is a general smooth curve passing through P . Therefore,

$$c_\varepsilon = 1 - \text{mult}_P(\Delta'_\varepsilon|_{W_\varepsilon}) \leq 1 - \beta_\varepsilon \leq 2 - r_\varepsilon(1 - \varepsilon)(2 + \alpha).$$

In this situation,

$$\begin{aligned} \deg((M - \omega_\varepsilon)|_{W_\varepsilon} - c_\varepsilon P) &= (1 - r_\varepsilon)(1 - \varepsilon)N \cdot W_\varepsilon - c_\varepsilon \\ &\geq 2(1 - r_\varepsilon)(1 - \varepsilon) - 2 + r_\varepsilon(1 - \varepsilon)(2 + \alpha) \\ &= (1 - \varepsilon)r_\varepsilon\alpha - 2\varepsilon. \end{aligned}$$

Here we used the assumption $N \cdot W_\varepsilon \geq 2$. We note that $(1 - \varepsilon_i)r_{\varepsilon_i}\alpha - 2\varepsilon_i > 0$ for every $i \gg 0$ since $\varepsilon_i \searrow 0$ for $i \nearrow \infty$ and $r_{\varepsilon_i} > \delta > 0$ for every i by construction. Therefore, if we choose $0 < \varepsilon = \varepsilon_i \ll 1$, then

$$\deg(M|_{W_\varepsilon} - (\omega_\varepsilon|_{W_\varepsilon} + c_\varepsilon P)) > 0.$$

Thus, we see that the restriction map

$$(3.2) \quad H^0(Z_\varepsilon, \mathcal{O}_{Z_\varepsilon}(M)) \rightarrow H^0(Y_\varepsilon, \mathcal{O}_{Y_\varepsilon}(M)) \oplus H^0(P, \mathcal{O}_P(M))$$

is surjective by considering the quasi-log structure of $[Z_\varepsilon, \omega_\varepsilon|_{Z_\varepsilon} + c_\varepsilon P]$ with the aid of Theorem 2.8. By combining (3.2) with (3.1), the restriction map

$$H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(Y_\varepsilon, \mathcal{O}_{Y_\varepsilon}(M)) \oplus H^0(P, \mathcal{O}_P(M))$$

is surjective. As in Case 1, we get $s \in H^0(X, \mathcal{O}_X(M))$ such that $s(P) \neq 0$ and $s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))$.

Anyway, we can construct $s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))$ such that $s(P) \neq 0$ when P is a smooth point of X .

Step 2. In this step, we assume that P is a singular point of X . Let $\pi : Y \rightarrow X$ be the minimal resolution of P . Then we have the following commutative diagram

$$\begin{array}{ccc} X' & & \\ q \downarrow & \searrow p & \\ Y & \xrightarrow{\pi} & X, \end{array}$$

where $p : X' \rightarrow X$ is a projective birational morphism from a smooth surface X' constructed in Step 1 by using Theorem 2.12. Let Δ_ε be an effective \mathbb{R} -divisor on X as in Step 1. We put $\pi^*(K_X + \Delta_\varepsilon) = K_Y + \Delta_\varepsilon^Y$. We note that Δ_ε^Y is effective since π is the minimal resolution of P . By construction, π is an isomorphism outside $\pi^{-1}(P)$. In particular, π is an isomorphism over some open neighborhood of $\text{Nqklt}(X, \omega)$. Therefore, $\mathcal{J}(Y, \Delta_\varepsilon^Y) = \mathcal{I}_{\pi^{-1}(\text{Nqklt}(X, \omega))}$ holds since $\mathcal{J}(X, \Delta_\varepsilon) = \mathcal{I}_{\text{Nqklt}(X, \omega)}$, where $\mathcal{I}_{\pi^{-1}(\text{Nqklt}(X, \omega))}$ is the defining ideal sheaf of $\pi^{-1}(\text{Nqklt}(X, \omega))$. Since $(\pi^*N)^2 > 4$, we can take an effective \mathbb{R} -divisor B on X such that $B \sim_{\mathbb{R}} N$ and $\text{mult}_Q D > 2$, where $D = \pi^*B$, for some $Q \in \pi^{-1}(P)$. We put $D_\varepsilon := (1 - \varepsilon)D$ and $B_\varepsilon := (1 - \varepsilon)B$ and define

$$s_\varepsilon = \max\{t \geq 0 \mid (Y, \Delta_\varepsilon^Y + tD_\varepsilon) \text{ is log canonical at any point of } \pi^{-1}(P)\}.$$

Then we have $0 < s_\varepsilon < 1$ since $\text{mult}_Q D_\varepsilon > 2$ for $0 < \varepsilon \ll 1$. Therefore, we can take $Q_\varepsilon \in \pi^{-1}(P)$ such that $(Y, \Delta_\varepsilon^Y + s_\varepsilon D_\varepsilon)$ is log canonical but is not klt at Q_ε . As in Step 1, we may assume that $\text{Supp } B_{X'} \cup \text{Supp } p_*^{-1}B \cup \text{Exc}(p)$ is contained in a simple normal crossing divisor Σ on X' . By the same argument as in Step 1, we can take some point R on $\pi^{-1}(P)$, $\{\varepsilon_i\}_{i=1}^\infty$, and $\delta > 0$ such that $0 < \varepsilon_i \ll 1$, $\varepsilon_i \searrow 0$ for $i \nearrow \infty$, $\mathcal{J}(Y, \Delta_{\varepsilon_i}^Y) = \mathcal{I}_{\pi^{-1}(\text{Nqklt}(X, \omega))}$, $(Y, \Delta_{\varepsilon_i}^Y + s_{\varepsilon_i} D_{\varepsilon_i})$ is log canonical at R but is not klt at R with $\delta < s_{\varepsilon_i} < 1$ for every i since there are only finitely many components of Σ . By $q : X' \rightarrow Y$, we have a natural quasi-log structure on $[Y, \omega_\varepsilon^Y]$ with $\omega_\varepsilon^Y := K_Y + \Delta_\varepsilon^Y + s_\varepsilon D_\varepsilon$ for any $\varepsilon = \varepsilon_i$ (see Example 2.7). If there is a one-dimensional qlc center C of $[Y, \omega_\varepsilon^Y]$ for some ε with $(\pi^*M - \omega_\varepsilon^Y) \cdot C = 0$, then $C \subset \pi^{-1}(P)$. This is because

$$(\pi^*M - \omega_\varepsilon^Y) \cdot C = (1 - s_\varepsilon)(1 - \varepsilon)N \cdot \pi_*C = 0.$$

This means that P is a qlc center of $[X, \omega_\varepsilon]$, where $\omega_\varepsilon := K_X + \Delta_\varepsilon + s_\varepsilon B_\varepsilon$. In this case, we can use Case 1 in Step 1. Therefore, for any $\varepsilon = \varepsilon_i$, we may assume that $(\pi^*M - \omega_\varepsilon^Y) \cdot C > 0$ for every one-dimensional qlc center C of $[Y, \omega_\varepsilon^Y]$. Now we can apply the arguments for $[X, \omega_\varepsilon]$ and M in Step 1 to $[Y, \omega_\varepsilon^Y]$ and π^*M here. We note that $\pi^*M - \omega_\varepsilon^Y$ is not ample but is nef and log big with respect to $[Y, \omega_\varepsilon^Y]$. Thus we can use Theorem 2.8 (ii). Then we

obtain $s^Y \in H^0(Y, \mathcal{I}_{\pi^{-1}(\text{Nqklt}(X, \omega))} \otimes \mathcal{O}_Y(\pi^*M))$ such that $s^Y(R) \neq 0$. Therefore, we have $s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))$ such that $\pi^*s = s^Y$. In particular, $s(P) \neq 0$. This is what we wanted.

Anyway, we finish the proof of Theorem 3.2. \square

Now the proof of Corollary 1.5 is easy.

Corollary 3.3 (Corollary 1.5). *Conjecture 1.2 is true in dimension two.*

Proof. Let P be an arbitrary closed point of X and let W be the unique minimal qlc stratum of $[X, \omega]$ passing through P . Note that W is irreducible by definition. By adjunction (see Theorem 2.8 (i)), $[W, \omega|_W]$ is an irreducible quasi-log canonical pair. By Theorem 2.8 (ii), the natural restriction map

$$(3.3) \quad H^0(X, \mathcal{O}_X(M)) \rightarrow H^0(W, \mathcal{O}_W(M))$$

is surjective. From now on, we will see that $|M|$ is basepoint-free in a neighborhood of P . If $W = P$, that is, P is a qlc center of $[X, \omega]$, then the complete linear system $|M|$ is obviously basepoint-free in a neighborhood of P by the surjection (3.3). Let us consider the case where $\dim W = 1$. We put $M' = M|_W$ and $N' = N|_W = M' - \omega|_W$. Then $\deg N' = N \cdot W > 1$ by assumption. Therefore, by Theorem 3.1, $|M'|$ is basepoint-free at P because $[W, \omega|_W]$ is an irreducible projective quasi-log canonical curve. Therefore, by the surjection (3.3), we see that $|M|$ is basepoint-free in a neighborhood of P . Thus we may assume that $\dim W = \dim X = 2$ and X is irreducible by replacing X with W since the restriction map (3.3) is surjective. Therefore, we can assume that X is irreducible and that X is the unique qlc stratum of $[X, \omega]$ passing through P . In particular, X is normal at P (see, for example, [F5, Theorem 6.3.11 (ii)]). Let $\nu: \tilde{X} \rightarrow X$ be the normalization. Note that $[\tilde{X}, \nu^*\omega]$ is a qlc pair by Theorem 2.11. We put $\tilde{M} = \nu^*M$ and $\tilde{N} = \nu^*N = \tilde{M} - \nu^*\omega$. It is obvious that \tilde{M} is Cartier. Moreover, we have $(\tilde{N})^2 = N^2 > 4$ and $\tilde{N} \cdot Z \geq N \cdot \nu(Z) \geq 2$ for every curve Z on \tilde{X} . Note that $\dim \nu(Z) = \dim Z = 1$ since ν is finite. We also note that $P' := \nu^{-1}(P)$ is a point since $\nu: \tilde{X} \rightarrow X$ is an isomorphism over some open neighborhood of P . This is because X is normal at P . Now the assumptions of Theorem 3.2 are all satisfied. Therefore, there is a section $s' \in H^0(\tilde{X}, \mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} \otimes \mathcal{O}_{\tilde{X}}(\tilde{M}))$ such that $s'(P') \neq 0$. We note that the non-normal part of X is contained in $\text{Nqklt}(X, \omega)$ (see, for example, [F5, Theorem 6.3.11 (ii)]) and that the equality

$$\nu_* \mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} = \mathcal{I}_{\text{Nqklt}(X, \omega)}$$

holds by Theorem 2.11. Therefore, we have

$$H^0(\tilde{X}, \mathcal{I}_{\text{Nqklt}(\tilde{X}, \nu^*\omega)} \otimes \mathcal{O}_{\tilde{X}}(\tilde{M})) \simeq H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M)).$$

Thus we can descend the section s' on \tilde{X} to a section $s \in H^0(X, \mathcal{I}_{\text{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(M))$ with $s(P) \neq 0$. Therefore, by this section $s \in H^0(X, \mathcal{O}_X(M))$, we see that $|M|$ is basepoint-free in a neighborhood of P . This is what we wanted. \square

We close this section with the proof of Corollary 1.6.

Proof of Corollary 1.6. Let (X, Δ) be a projective semi-log canonical pair. Then, by Theorem 2.10, $[X, K_X + \Delta]$ is a quasi-log canonical pair. Therefore, Corollary 1.6 is a direct consequence of Theorem 1.3 and Corollary 1.5. \square

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