RIMS Kôkyûroku Bessatsu
B9 (2008), 81–95

Vanishing theorems for toric polyhedra

By

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Abstract

A toric polyhedron is a reduced closed subscheme of a toric variety that are partial unions of the orbits of the torus action. We prove vanishing theorems for toric polyhedra. We also give a proof of the E_1-degeneration of Hodge to de Rham type spectral sequence for toric polyhedra in any characteristic. Finally, we give a very powerful extension theorem for ample line bundles.

Contents

§ 1. Introduction

In this paper, we treat vanishing theorems for *toric polyhedra*. Section 2 is a continuation of my paper [F1], where we gave a very simple, characteristic-free approach

Received September 8, 2007. Revised February 4, 2008.
2000 Mathematics Subject Classification(s): Primary 14F17; Secondary 14F25.
Key Words: toric variety, Ishida's de Rham complex, vanishing theorem
The author was partially supported by the Inamori Foundation and the Grant-in-Aid for Young Scientists (A) 17684001 from JSPS.
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to vanishing theorems on toric varieties by using multiplication maps. Here, we give a generalization of Danilov's vanishing theorem on toric polyhedra.

Theorem 1.1 (Vanishing Theorem). *Let $Y = Y(\Phi)$ be a projective toric polyhedron defined over a field k of arbitrary characteristic. Then*

$$H^i(Y, \widetilde{\Omega}^a_Y \otimes L) = 0 \quad \textit{for} \quad i \neq 0$$

holds for every ample line bundle L on Y.

Note that a toric polyhedron is a reduced closed subscheme of a toric variety that are partial unions of the orbits of the torus action. Once we understand Ishida's de Rham complexes on toric polyhedra, then we can easily see that the arguments in [F1] works for toric polyhedra with only small modifications. Moreover, we give a proof of the E_1-degeneration of Hodge to de Rham type spectral sequence for toric polyhedra.

Theorem 1.2 (E_1-degeneration). *Let $Y = Y(\Phi)$ be a complete toric polyhedron defined over a field k of any characteristic. Then the spectral sequence*

$$E_1^{a,b} = H^b(Y, \widetilde{\Omega}^a_Y) \Rightarrow \mathbb{H}^{a+b}(Y, \widetilde{\Omega}^\bullet_Y)$$

degenerates at the E_1-term.

It seems to be new when the characteristic of the base field is positive. So, Section 2 supplements [BTLM], [D], and [I]. In Section 3, we will give the following two results supplementary to [F1].

Theorem 1.3 (cf. [F1, Theorem 1.1]). *Let X be a toric variety defined over a field k of any characteristic and let A and B be reduced torus invariant Weil divisors on X without common irreducible components. Let L be a line bundle on X. If $H^i(X, \widetilde{\Omega}^a_X(\log(A+B))(-A)\otimes L^{\otimes l}) = 0$ for some positive integer l, then $H^i(X, \widetilde{\Omega}^a_X(\log(A+B))(-A) \otimes L) = 0$.*

It is a slight generalization of [F1, Theorem 1.1].

Theorem 1.4. *Let X be a complete toric variety defined over a field k of any characteristic and let A and B be reduced torus invariant Weil divisors on X without common irreducible components. Then the spectral sequence*

$$E_1^{a,b} = H^b(X, \widetilde{\Omega}^a_X(\log(A+B))(-A)) \Rightarrow \mathbb{H}^{a+b}(X, \widetilde{\Omega}^\bullet_X(\log(A+B))(-A))$$

degenerates at the E_1-term.

One of the main results of this paper is the next theorem, which is a complete generalization of [M, Theorem 5.1]. For the precise statement, see Theorem 4.5 below. We will give a proof of Theorem 1.5 as an application of our new vanishing arguments in Section 4. The technique in Section 4 is very powerful and produces Kollár type vanishing theorem in the toric category, which is missing in [F1].

Theorem 1.5 (Extension Theorem)**.** *Let X be a projective toric variety defined over a field k of any characteristic and let L be an ample line bundle on X. Let Y be a toric polyhedron on X and let $\mathcal{I}_Y$ be the defining ideal sheaf of Y on X. Then $H^i(X, \mathcal{I}_Y \otimes L) = 0$ for any $i > 0$. In particular, the restriction map $H^0(X, L) \to H^0(Y, L)$ is surjective.*

We state a special case of the vanishing theorems in Section 4 for the reader's convenience.

Theorem 1.6 (cf. Theorem 4.3)**.** *Let $f : Z \to X$ be a toric morphism between projective toric varieties and let A and B be reduced torus invariant Weil divisors on Z without common irreducible components. Let L be an ample line bundle on X. Then $H^i(X, L \otimes R^j f_* \widetilde{\Omega}^a_Z(\log(A+B))(-A)) = 0$ for any $i > 0$, $j \geq 0$, and $a \geq 0$.*

In the final section: Section 5, we treat toric polyhdera as *quasi-log varieties* and explain the background and motivation of this work.

Acknowledgments. I was partially supported by the Grant-in-Aid for Young Scientists (A) ♯17684001 from JSPS. I was also supported by the Inamori Foundation. I am grateful to Professor Shigefumi Mori for his questions, comments, and warm encouragement. I thank Hiroshi Sato and the referee for their comments. I also thank Takeshi Abe for answering my question.

Notation. Let N be a free $\mathbb{Z}$-module of rank $n \geq 0$ and let M be its dual $\mathbb{Z}$-module. The natural pairing $\langle\ ,\ \rangle : N \times M \to \mathbb{Z}$ is extended to the bilinear form $\langle\ ,\ \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$, where $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. A non-empty subset σ of $N_{\mathbb{R}}$ is said to be a *cone* if there exists a finite subset $\{n_1, \cdots, n_s\}$ of N such that $\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_s$, where $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R}; r \geq 0\}$, and that $\sigma \cap (-\sigma) = \{0\}$, where $-\sigma = \{-a; a \in \sigma\}$. A subset ρ of a cone σ is said to be a *face* of σ and we denote $\rho \prec \sigma$ if there exists an element m of $M_{\mathbb{R}}$ such that $\langle a, m\rangle \geq 0$ for every $a \in \sigma$ and $\rho = \{a \in \sigma; \langle a, m\rangle = 0\}$. A set Δ of cones of $N_{\mathbb{R}}$ is said to be a *fan* if (1) $\sigma \in \Delta$ and $\rho \prec \sigma$ imply $\rho \in \Delta$, and (2) $\sigma, \tau \in \Delta$ and $\rho = \sigma \cap \tau$ imply $\rho \prec \sigma$ and $\rho \prec \tau$. We do not assume that Δ is *finite*, that is, Δ does not always consist of a finite number of cones. For a cone σ of $N_{\mathbb{R}}$, $\sigma^{\vee} = \{x \in M_{\mathbb{R}}; \langle a, x\rangle \geq 0$ for every $a \in \sigma\}$ and $\sigma^{\perp} = \{x \in M_{\mathbb{R}}; \langle a, x\rangle = 0$ for every $a \in \sigma\}$. Let $X = X(\Delta)$ be the *toric variety* associated to a fan Δ. Note that X is just *locally* of finite type over k in our notation, where k is the base field of $X(\Delta)$. Each cone σ of Δ uniquely defines an $(n - \dim \sigma)$-dimensional torus $T_{N(\sigma)} = \mathrm{Spec} k[M \cap \sigma^{\perp}]$ on $X(\Delta)$. The closure of $T_{N(\sigma)}$ in $X(\Delta)$ is denoted by $V(\sigma)$.

§ 2. Vanishing theorem and E_1-degeneration

We will work over a fixed field k of any characteristic throughout this section.

§ 2.1. Toric polyhedra

Let us recall the definition of *toric polyhedra*. See [I, Definition 3.5].

Definition 2.1.1. For a subset Φ of a fan Δ, we say that Φ is *star closed* if $\sigma \in \Phi$, $\tau \in \Delta$ and $\sigma \prec \tau$ imply $\tau \in \Phi$.

Definition 2.1.2 (Toric polyhedron)**.** For a star closed subset Φ of a fan Δ, we denote by $Y = Y(\Phi)$ the reduced subscheme $\bigcup_{\sigma \in \Phi} V(\sigma)$ of $X = X(\Delta)$, and we call it the *toric polyhedron associated to* Φ.

Example 2.1.3. Let $X = \mathbb{P}^2$ and let $T \subset \mathbb{P}^2$ be the big torus. We put $Y = \mathbb{P}^2 \setminus T$. Then Y is a toric polyhedron, which is a circle of three projective lines.

The above example is a special case of the following one.

Example 2.1.4. Let $X = X(\Delta)$ be an n-dimensional toric variety. We put $\Phi_m = \{\sigma \in \Delta; \dim \sigma \geq m\}$ for $0 \leq m \leq n$. Then Φ_m is a star closed subset of Δ and the toric polyhedron $Y_m = Y(\Phi_m)$ is pure $(n-m)$-dimensional.

Example 2.1.5. We consider $X = \mathbb{A}^3_k = \mathrm{Spec} k[x_1, x_2, x_3]$. Then the subvariety $Y = (x_1 = x_2 = 0) \cup (x_3 = 0) \simeq \mathbb{A}^1_k \cup \mathbb{A}^2_k$ of X is a toric polyhedron, which is not pure dimensional.

Remark 2.1.6. Let Y be a toric polyhedron. We do not know how to describe line bundles on Y by combinatorial data. Note that a line bundle L on Y can not necessarily be extended to a line bundle $\mathcal{L}$ on X.

In [I], Ishida defined the de Rham complex $\widetilde{\Omega}^{\bullet}_Y$ of a toric polyhedron Y. When Y is a toric variety, Ishida's de Rham complex is nothing but Danilov's de Rham complex (see [D, Chapter I. §4]). For the details, see [I]. Here, we quickly review $\widetilde{\Omega}^{\bullet}_Y$ when X is affine.

2.1.7 (Ishida's de Rham complex). We put $\Delta = \{\pi,\ \text{its faces}\}$, where π is a cone in $N_{\mathbb{R}}$. Then $X = X(\Delta)$ is an affine toric variety Spec$k[M\cap\pi^{\vee}]$. Let Φ be a star closed subset of Δ and let Y be the toric polyhedron associated to Φ. In this case, $\widetilde{\Omega}^a_Y$ is an $\mathcal{O}_Y = k[M\cap\pi^{\vee}]/k[M\cap(\pi^{\vee}\setminus(\cup_{\sigma\in\Phi}\sigma^{\perp}))]$-module generated by $x^m\otimes m_{\alpha_1}\wedge\cdots\wedge m_{\alpha_a}$, where $m\in M\cap(\pi^{\vee}\cap(\cup_{\sigma\in\Phi}\sigma^{\perp}))$ and $m_{\alpha_1},\cdots,m_{\alpha_a}\in M[\rho(m)]$, for any $a\geq 0$. Note that $\rho(m)=\pi\cap m^{\perp}$ is a face of π when $m\in M\cap\pi^{\vee}$, and that $M[\rho(m)] = M\cap\rho(m)^{\perp}\subset M$.

§ 2.2. Multiplication maps

In this subsection, let us quickly review the multiplication maps in [F1, Section 2].

2.2.1 (Multiplication maps). For a fan Δ in $N_{\mathbb{R}}$, we have the associated toric variety $X = X(\Delta)$. We put $N' = \frac{1}{l}N$ and $M' = \mathrm{Hom}_{\mathbb{Z}}(N',\mathbb{Z})$ for any positive integer l. We note that $M' = lM$. Since $N_{\mathbb{R}} = N'_{\mathbb{R}}$, Δ is also a fan in $N'_{\mathbb{R}}$. We write Δ' to express the fan Δ in $N'_{\mathbb{R}}$. Let $X' = X(\Delta')$ be the associated toric variety. We note that $X\simeq X'$ as toric varieties. We consider the natural inclusion $\varphi: N\to N'$. Then φ induces a finite surjective toric morphism $F: X\to X'$. We call it the *l-times multiplication map* of X.

Remark 2.2.2. The l-times multiplication map $F: X\to X'$ should be called the *l-th power map* of X. However, we follow [F1] in this paper.

2.2.3 (Convention). Let $\mathcal{A}$ be an object on X. Then we write $\mathcal{A}'$ to indicate the corresponding object on X'. Let Φ be a star closed subset of Δ and let Y be the toric polyhedron associated to Φ. Then $F: X\to X'$ induces a finite surjective morphism $F: Y\to Y'$.

2.2.4 (Split injections on the big torus). By fixing a base of M, we have $k[M]\simeq k[x_1,x_1^{-1},\cdots,x_n,x_n^{-1}]$. We can write $x^m = x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ for $m=(m_1,\cdots,m_n)\in\mathbb{Z}^n = M$. Let T be the big torus of X. Then we have the isomorphism of $\mathcal{O}_T = k[M]$-modules $k[M]\otimes_{\mathbb{Z}}\wedge^a M\to\Omega^a_T$ for any $a\geq 0$ induced by

$$x^m\otimes m_{\alpha_1}\wedge\cdots\wedge m_{\alpha_a}\mapsto x^m\frac{dx^{m_{\alpha_1}}}{x^{m_{\alpha_1}}}\wedge\cdots\wedge\frac{dx^{m_{\alpha_a}}}{x^{m_{\alpha_a}}},$$

where $m, m_{\alpha_1},\cdots,m_{\alpha_a}\in\mathbb{Z}^n = M$. Therefore, $F_*\Omega^a_T$ corresponds to a $k[M']$-module $k[M]\otimes_{\mathbb{Z}}\wedge^a M$. We consider the $k[M']$-module homomorphisms $k[M']\otimes_{\mathbb{Z}}\wedge^a M'\to k[M]\otimes_{\mathbb{Z}}\wedge^a M$ given by $x^{m_\beta}\otimes m_{\alpha_1}\wedge\cdots\wedge m_{\alpha_a}\mapsto x^{lm_\beta}\otimes m_{\alpha_1}\wedge\cdots\wedge m_{\alpha_a}$, and $k[M]\otimes_{\mathbb{Z}}\wedge^a M\to k[M']\otimes_{\mathbb{Z}}\wedge^a M'$ induced by $x^{m_\gamma}\otimes m_{\alpha_1}\wedge\cdots\wedge m_{\alpha_a}\mapsto x^{m_\beta}\otimes m_{\alpha_1}\wedge\cdots\wedge m_{\alpha_a}$ if $m_\gamma = lm_\beta$ and $x^{m_\gamma}\otimes m_{\alpha_1}\wedge\cdots\wedge m_{\alpha_a}\mapsto 0$ otherwise. Thus, these $k[M']$-module homomorphisms give split injections $\Omega^a_{T'}\to F_*\Omega^a_T$ for any $a\geq 0$.

§ 2.3. Proof of the vanishing theorem and E_1-degeneration

Let us start the proof of the vanishing theorem and E_1-degeneration. The next proposition plays a key role in the proof.

Proposition 2.3.1. *Let X be a toric variety and let $Y \subset X$ be a toric polyhedron. Let $F : X \to X'$ be the l-times multiplication map and let $F : Y \to Y'$ be the induced map. Then there exists a split injection $\widetilde{\Omega}^a_{Y'} \to F_*\widetilde{\Omega}^a_Y$ for any $a \geq 0$.*

Proof. We write $X = X(\Delta)$ and $Y = Y(\Phi)$. Then $Y(\Phi)$ has the open covering $\{Y(\Phi) \cap U(\pi); \pi \in \Phi\}$, where $U(\pi) = \mathrm{Spec} k[M \cap \pi^\vee]$. We put $Z = Z(\Psi) = Y(\Phi) \cap U(\pi)$. Then, by the description of $\widetilde{\Omega}^a_Z$ in [I, Section 2] or 2.1.7, we have natural embeddings $\mathcal{O}_Z \subset k[M]$ and $\widetilde{\Omega}^a_Z \subset k[M] \otimes_{\mathbb{Z}} \wedge^a M$ for any $a > 0$ as k-vector spaces. Note that $\mathcal{O}_Z$ is spanned by $\{x^m; m \in M \cap (\pi^\vee \cap (\cup_{\sigma\in\Psi}\sigma^\perp))\}$ as a k-vector space. In 2.2.4, we constructed split injections $k[M'] \otimes_{\mathbb{Z}} \wedge^a M' \to k[M] \otimes_{\mathbb{Z}} \wedge^a M$ for any $a \geq 0$. This split injections induce split injections

$$\begin{array}{ccc} \widetilde{\Omega}^a_{Z'} & \to & F_*\widetilde{\Omega}^a_Z \\ \cap & & \cap \\ k[M'] \otimes_{\mathbb{Z}} \wedge^a M' & \to & k[M] \otimes_{\mathbb{Z}} \wedge^a M \end{array}$$

for all $a \geq 0$ as k-vector spaces. However, it is not difficult to see that $\widetilde{\Omega}^a_{Z'} \to F_*\widetilde{\Omega}^a_Z$ and its split $F_*\widetilde{\Omega}^a_Z \to \widetilde{\Omega}^a_{Z'}$ are $\mathcal{O}_{Z'}$-homomorphisms for any $a \geq 0$. The above constructed split injections for $Y(\Phi) \cap U(\pi)$ can be patched together. Thus, we obtain split injections $\widetilde{\Omega}^a_{Y'} \to F_*\widetilde{\Omega}^a_Y$ for any $a \geq 0$. □

The following theorem is one of the main theorems of this paper. It is a generalization of Danilov's vanishing theorem for toric varieties (see [D, 7.5.2. Theorem]).

Theorem 2.3.2 (cf. Theorem 1.1)**.** *Let $Y = Y(\Phi)$ be a projective toric polyhedron defined over a field k of any characteristic. Then*

$$H^i(Y, \widetilde{\Omega}^a_Y \otimes L) = 0 \quad \textit{for} \quad i \neq 0$$

holds for every ample line bundle L on Y.

Proof. We assume that $l = p > 0$, where p is the characteristic of k. In this case, $F^*L' \simeq L^{\otimes p}$. Thus, we obtain

$$\begin{aligned} H^i(Y, \widetilde{\Omega}^a_Y \otimes L) &\simeq H^i(Y', \widetilde{\Omega}^a_{Y'} \otimes L') \\ &\subset H^i(Y', F_*\widetilde{\Omega}^a_Y \otimes L') \\ &\simeq H^i(Y, \widetilde{\Omega}^a_Y \otimes L^{\otimes p}), \end{aligned}$$

where we used the split injection in Proposition 2.3.1 and the projection formula. By iterating the above arguments, we obtain $H^i(Y, \widetilde{\Omega}^a_Y \otimes L) \subset H^i(Y, \widetilde{\Omega}^a_Y \otimes L^{\otimes p^r})$ for any positive integer r. By Serre's vanishing theorem, we obtain $H^i(Y, \widetilde{\Omega}^a_Y \otimes L) = H^i(Y, \widetilde{\Omega}^a_Y \otimes L^{\otimes p^r}) = 0$ for $i > 0$. When the characteristic of k is zero, we can assume that everything is defined over R, where $R(\supset \mathbb{Z})$ is a finitely generated ring. By the above result, the vanishing theorem holds over R/P, where P is any general maximal ideal of R, since R/P is a finite field and the ampleness is an open condition. Therefore, we have the desired vanishing theorem over the generic point of Spec$R$. Of course, it holds over k. $\square$

If Y is a toric variety, then Theorem 2.3.2 is nothing but Danilov's vanishing theorem. For the other vanishing theorems on toric varieties, see [F1] and the results in Sections 3 and 4. The next corollary is a special case of Theorem 2.3.2.

Corollary 2.3.3. *Let $Y = Y(\Phi)$ be a projective toric polyhedron and L an ample line bundle on Y. Then we obtain $H^i(Y, L) = 0$ for any $i > 0$.*

Proof. It is sufficient to remember that $\widetilde{\Omega}^0_Y \simeq \mathcal{O}_Y$. $\square$

Remark 2.3.4. Let X be a projective toric variety. Then, it is obvious that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. However, $H^i(Y, \mathcal{O}_Y)$ is not necessarily zero for some $i > 0$ when Y is a projective toric polyhedron. See Example 2.1.3. More explicitly, let X be an n-dimensional non-singular complete toric variety. We put $Y = X \setminus T$, where T is the big torus. Then $H^{n-1}(Y, \mathcal{O}_Y)$ is dual to $H^0(Y, \mathcal{O}_Y)$ since $K_Y \sim 0$. Therefore, $H^{n-1}(Y, \mathcal{O}_Y) \neq \{0\}$.

The following theorem is a supplement to Theorem 2.3.2.

Theorem 2.3.5. *Let Y be a toric polyhedron on a toric variety X. Let L be a line bundle on Y. Assume that $L = \mathcal{L}|_Y$ for some line bundle $\mathcal{L}$ on X. If $H^i(Y, \widetilde{\Omega}^a_Y \otimes L^{\otimes l}) = 0$ for some positive integer l, then $H^i(Y, \widetilde{\Omega}^a_Y \otimes L) = 0$.*

Proof. Let $F : X \to X'$ be the l-times multiplication map. Then $F^*\mathcal{L}' \simeq \mathcal{L}^{\otimes l}$. Therefore, $F^*L' \simeq L^{\otimes l}$. By the same argument as in the proof of Theorem 2.3.2, we obtain the desired statement. $\square$

By the construction of the split injections in Proposition 2.3.1 and the definition of the exterior derivative, we have the following proposition.

Proposition 2.3.6. *We assume that $l = p > 0$, where p is the characteristic of k. Then there exist morphisms of complexes*

$$\phi : \bigoplus_{a \geq 0} \widetilde{\Omega}^a_{Y'}[-a] \to F_* \widetilde{\Omega}^\bullet_Y$$

and

$$\psi : F_*\widetilde{\Omega}^\bullet_Y \to \bigoplus_{a\geq 0} \widetilde{\Omega}^a_{Y'}[-a]$$

such that $\psi \circ \phi$ *is a quasi-isomorphism. Note that the complex* $\bigoplus_{a\geq 0} \widetilde{\Omega}^a_{Y'}[-a]$ *has zero differentials.*

Proof. We consider the following diagram.

$$\begin{tikzcd}
\widetilde{\Omega}^a_{Y'} \arrow[r, "\phi_a"] \arrow[d, "0"'] & F_*\widetilde{\Omega}^a_Y \arrow[r, "\psi_a"] \arrow[d, "d"] & \widetilde{\Omega}^a_{Y'} \arrow[d, "0"] \\
\widetilde{\Omega}^{a+1}_{Y'} \arrow[r, "\phi_{a+1}"'] & F_*\widetilde{\Omega}^{a+1}_Y \arrow[r, "\psi_{a+1}"'] & \widetilde{\Omega}^{a+1}_{Y'}
\end{tikzcd}$$

Here, ϕ_i and ψ_i are $\mathcal{O}_{Y'}$-homomorphisms constructed in Proposition 2.3.1 for any i. Since we assume that $l = p$, the above diagram is commutative. Therefore, we obtain the desired morphisms of complexes ϕ and ψ. □

As an application of Proposition 2.3.6, we can prove the E_1-degeneration of Hodge to de Rham type spectral sequence for toric polyhedra.

Theorem 2.3.7 (cf. Theorem 1.2)**.** *Let* $Y = Y(\Phi)$ *be a complete toric polyhedron. Then the spectral sequence*

$$E_1^{a,b} = H^b(Y, \widetilde{\Omega}^a_Y) \Rightarrow \mathbb{H}^{a+b}(Y, \widetilde{\Omega}^\bullet_Y)$$

degenerates at the E_1*-term.*

Proof. The following proof is well known. See, for example, the proof of Theorem 4 in [BTLM]. We assume that $l = p > 0$, where p is the characteristic of k. Then, by Proposition 2.3.6,

$$\begin{aligned}\sum_{a+b=n} \dim_k E_\infty^{a,b} &= \dim_k \mathbb{H}^n(Y, \widetilde{\Omega}^\bullet_Y) = \dim_k \mathbb{H}^n(Y, F_*\widetilde{\Omega}^\bullet_Y)\\ &\geq \sum_{a+b=n} \dim_k H^b(Y', \widetilde{\Omega}^a_{Y'}) = \sum_{a+b=n} \dim_k E_1^{a,b}.\end{aligned}$$

In general, $\sum_{a+b=n} \dim_k E_\infty^{a,b} \leq \sum_{a+b=n} \dim_k E_1^{a,b}$. Therefore, $E_\infty^{a,b} \simeq E_1^{a,b}$ holds and the spectral sequence degenerates at E_1. When the characteristic of k is zero, we can assume that everything is defined over $\mathbb{Q}$. Moreover, we can construct a toric polyhedron $\mathcal{Y}$ defined over $\mathbb{Z}$ such that $Y = \mathcal{Y} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Q}$. By applying the above E_1-degeneration on a general fiber of $f : \mathcal{Y} \to \mathrm{Spec}\mathbb{Z}$ and the base change theorem, we obtain that $\sum_{a+b=n}\dim_{\mathbb{Q}} E_1^{a,b} = \dim_{\mathbb{Q}} \mathbb{H}^n(Y, \widetilde{\Omega}^\bullet_Y)$. In particular, $\sum_{a+b=n}\dim_k E_1^{a,b} = \dim_k \mathbb{H}^n(Y, \widetilde{\Omega}^\bullet_Y)$ and we have the desired E_1-degeneration over k. □

We close this section with the following two remarks on Ishida's results.

Remark 2.3.8. If $k = \mathbb{C}$ and Δ consists of a finite number of cones, then Ishida's de Rham complex $\widetilde{\Omega}^\bullet_Y$ is canonically isomorphic to the Du Bois complex $\underline{\underline{\Omega}}^\bullet_Y$ (see Theorem 4.1 in [I]). Therefore, the E_1-degeneration in Theorem 2.3.7 was known when $k = \mathbb{C}$. We note that $\mathbb{H}^{a+b}(Y, \widetilde{\Omega}^\bullet_Y)$ is isomorphic to $H^{a+b}(Y, \mathbb{C})$ in this case.

Remark 2.3.9. Let $Y = Y(\Phi)$ be a toric polyhedron. In [I, p.130], Ishida introduced a complex $C^\bullet(\Phi^{(2)}, \mathcal{O} \otimes \varLambda^a)$. For the definition and the basic properties, see [I, Sections 2 and 3]. Note that $\widetilde{\Omega}^a_Y \to C^0(\Phi^{(2)}, \mathcal{O}\otimes\varLambda^a) \to \cdots \to C^j(\Phi^{(2)}, \mathcal{O}\otimes\varLambda^a) \to \cdots$ is a resolution of $\widetilde{\Omega}^a_Y$, that is, $\widetilde{\Omega}^a_Y \simeq \mathcal{H}^0(C^\bullet(\Phi^{(2)}, \mathcal{O}\otimes\varLambda^a))$ and $\mathcal{H}^i(C^\bullet(\Phi^{(2)}, \mathcal{O}\otimes\varLambda^a)) = 0$ for $i \neq 0$ (cf. [I, Proposition 2.4]). Assume that Y is complete. Let L be a nef line bundle on Y. Then it is not difficult to see that $\widetilde{\Omega}^a_Y \otimes L \to C^\bullet(\Phi^{(2)}, \mathcal{O} \otimes \varLambda^a) \otimes L$ is a Γ-acyclic resolution of $\widetilde{\Omega}^a_Y \otimes L$. Therefore, if L is ample, then Theorem 2.3.2 implies that $H^0(Y, \widetilde{\Omega}^a_Y \otimes L) = H^0(D^\bullet)$ and $H^i(Y, \widetilde{\Omega}^a_Y \otimes L) = H^i(D^\bullet) = 0$ for $i \neq 0$, where $D^\bullet$ is a complex of k-vector spaces $\Gamma(Y, C^\bullet(\Phi^{(2)}, \mathcal{O} \otimes \varLambda^a) \otimes L)$.

§ 3. Suppplements

In this section, we make some remarks on my paper [F1]. Let $X = X(\Delta)$ be a toric variety defined over a field k of any characteristic. Note that Δ is not assumed to be finite in this section. First, we define $\widetilde{\Omega}^a_X(\log(A+B))(-A)$, which is a slight generalization of $\widetilde{\Omega}^a_X(\log B)$ in [F1, Definition 1.2].

Definition 3.1. Let X be a toric variety and let A and B be reduced torus invariant Weil divisors on X without common irreducible components. We put $W = X \setminus \mathrm{Sing}(X)$, where $\mathrm{Sing}(X)$ is the singular locus of X. Then we define $\widetilde{\Omega}^a_X(\log(A + B))(-A) = \iota_*(\Omega^a_W(\log(A + B)) \otimes \mathcal{O}_W(-A))$ for any $a \geq 0$, where $\iota : W \hookrightarrow X$ is the natural open immersion.

By the same argument as in [F1, Section 2] (see also Subsection 2.2), the split injection $\Omega^a_{T'} \to F_*\Omega^a_T$ induces the following split injection.

Proposition 3.2. *Let $F : X \to X'$ be the l-times multiplication map. Then the split injection $\Omega^a_{T'} \to F_*\Omega^a_T$ naturally induces the following split injection $\widetilde{\Omega}^a_{X'}(\log(A' + B'))(-A') \to F_*\widetilde{\Omega}^a_X(\log(A + B))(-A)$ for any $a \geq 0$.*

The next proposition is obvious by the definition of the exterior derivative and the construction of the split injections in Proposition 3.2 (cf. Proposition 2.3.6).

Proposition 3.3. *We assume that $l = p > 0$, where p is the characteristic of k. Then there exist morphisms of complexes*

$$\phi : \bigoplus_{a\geq 0} \widetilde{\Omega}^a_{X'}(\log(A'+B'))(-A')[-a] \to F_*\widetilde{\Omega}^\bullet_X(\log(A+B))(-A)$$

and

$$\psi : F_*\widetilde{\Omega}^\bullet_X(\log(A+B))(-A) \to \bigoplus_{a\geq 0} \widetilde{\Omega}^a_{X'}(\log(A'+B'))(-A')[-a]$$

such that the composition $\psi \circ \phi$ is a quasi-isomorphism. We note that the complex $\bigoplus_{a\geq 0} \widetilde{\Omega}^a_{X'}(\log(A'+B'))(-A')[-a]$ has zero differentials.

The following E_1-degeneration is a direct consequence of Proposition 3.3. See the proof of Theorem 2.3.7.

Theorem 3.4 (cf. Theorem 1.4). *Let X be a complete toric variety and let A and B be reduced torus invariant Weil divisors on X without common irreducible components. Then the spectral sequence*

$$E_1^{a,b} = H^b(X, \widetilde{\Omega}^a_X(\log(A+B))(-A)) \Rightarrow \mathbb{H}^{a+b}(X, \widetilde{\Omega}^\bullet_X(\log(A+B))(-A))$$

degenerates at the E_1-term.

Remark 3.5. If $k = \mathbb{C}$ and X is non-singular and complete, then it is well known that $\mathbb{H}^{a+b}(X, \Omega^\bullet_X) = H^{a+b}(X, \mathbb{C})$, $\mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log B)) = H^{a+b}(X \setminus B, \mathbb{C})$, and $\mathbb{H}^{a+b}(X, \Omega^\bullet_X(\log A) \otimes \mathcal{O}_X(-A)) = H^{a+b}_c(X \setminus A, \mathbb{C})$, where $H^{a+b}_c(X \setminus A, \mathbb{C})$ is the cohomology group with compact support.

Finally, we state a generalization of [F1, Theorem 1.1]. The proof is obvious. See also Theorem 4.3 below.

Theorem 3.6 (cf. Theorem 1.3). *Let X be a toric variety and let A and B be reduced torus invariant Weil divisors on X without common irreducible components. Let L be a line bundle on X. If $H^i(X, \widetilde{\Omega}^a_X(\log(A+B))(-A) \otimes L^{\otimes l}) = 0$ for some positive integer l, then $H^i(X, \widetilde{\Omega}^a_X(\log(A+B))(-A) \otimes L) = 0$.*

Some other vanishing theorems in [F1] can be generalized by using Theorem 3.6. We leave the details for the reader's exercise.

§ 4. Kollár type vanishing theorems and extension theorem

In this section, we treat a variant of the method in [F1]. Here, every toric variety is defined over a field k of any characteristic and a fan is not necessarily finite. Let

$f : Z \to X$ be a toric morphism of finite type. Then we have the following commutative diagram of l-times multiplication maps.

$$\begin{array}{ccc} Z & \xrightarrow{F^Z} & Z' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow[F^X]{} & X' \end{array}$$

This means that $F^X : X \to X'$ and $F^Z : Z \to Z'$ are the l-times multiplication maps explained in 2.2 and that $F^X \circ f = f' \circ F^Z$. Let $\mathcal{F}$ be a coherent sheaf on Z such that there exists a split injection $\alpha : \mathcal{F}' \to F^Z_* \mathcal{F}$. Then we have an obvious lemma.

Lemma 4.1. *We have a split injection*

$$\beta = R^j f'_* \alpha : R^j f'_* \mathcal{F}' \to F^X_* R^j f_* \mathcal{F}$$

for any j.

Proof. Since F^X and F^Z are finite, we have the following isomorphisms

$$F^X_* R^j f_* \mathcal{F} \simeq R^j (F^X \circ f)_* \mathcal{F} \simeq R^j (f' \circ F^Z)_* \mathcal{F} \simeq R^j f'_* (F^Z_* \mathcal{F})$$

by Leray's spectral sequence. Therefore, we obtain a split injection

$$\beta = R^j f'_* \alpha : R^j f'_* \mathcal{F}' \to F^X_* R^j f_* \mathcal{F}$$

for any j. □

Let L be a line bundle on X. Then we obtain the following useful proposition.

Proposition 4.2. *If $H^i(X, R^j f_* \mathcal{F} \otimes L^{\otimes l}) = 0$ for some positive integer l, then $H^i(X, R^j f_* \mathcal{F} \otimes L) = 0$.*

Proof. Let F^X be the l-times multiplication map. As usual, we have

$$\begin{aligned} H^i(X, R^j f_* \mathcal{F} \otimes L) &\simeq H^i(X', R^j f'_* \mathcal{F}' \otimes L') \\ &\subset H^i(X', F^X_* R^j f_* \mathcal{F} \otimes L') \\ &\simeq H^i(X, R^j f_* \mathcal{F} \otimes L^{\otimes l}) \end{aligned}$$

because $(F^X)^* L' \simeq L^{\otimes l}$. So, we obtain the desired statement. □

Therefore, we get a very powerful vanishing theorem.

Theorem 4.3. *Let $f : Z \to X$ be a proper toric morphism and let A and B be reduced torus invariant Weil divisors on Z without common irreducible components. Assume that $\pi : X \to S$ is a projective toric morphism and L is a π-ample line bundle on X. Then $R^i\pi_*(L \otimes R^j f_* \widetilde{\Omega}^a_Z(\log(A+B))(-A)) = 0$ for any $i > 0$, $j \geq 0$, and $a \geq 0$.*

Proof. The problem is local. So, we can assume that S is affine. We put $\mathcal{F} = \widetilde{\Omega}^a_Z(\log(A+B))(-A)$. Then, this is a direct consequence of Proposition 3.2 and Proposition 4.2 by Serre's vanishing theorem. □

We obtain Kollár type vanishing theorem for toric varieties as a special case of Theorem 4.3.

Corollary 4.4 (Kollár type vanishing theorem)**.** *Let $f : Z \to X$ be a proper toric morphism and let B be a reduced torus invariant Weil divisor on Z. Assume that X is projective and L is an ample line bundle on X. Then $H^i(X, R^j f_*\mathcal{O}_Z(K_Z+B) \otimes L) = 0$ for any $i > 0$ and $j \geq 0$.*

Proof. It is sufficient to put $a = \dim Z$ in Theorem 4.3. □

The next theorem is one of the main results of this paper. See also Theorem 5.3.

Theorem 4.5 (cf. [M, Theorem 5.1])**.** *Let $\pi : X \to S$ be a proper toric morphism and $Y = Y(\Phi)$ a toric polyhedron on $X = X(\Delta)$. Let L be a π-ample line bundle on X. Let $\mathcal{I}_Y$ be the defining ideal sheaf of Y on X. Then $R^i\pi_*(\mathcal{I}_Y \otimes L) = 0$ for any $i > 0$. Since $R^i\pi_*L = 0$ for any $i > 0$, we have that $R^i(\pi|_Y)_*(L|_Y) = 0$ for any $i > 0$ and that the restriction map $\pi_*L \to (\pi|_Y)_*(L|_Y)$ is surjective.*

Proof. If $Y = X$, then there is nothing to prove. So, we can assume that $Y \subsetneq X$. Let $f : V \to X$ be a toric resolution such that $K_V + E = f^*(K_X + D)$ and that $\mathrm{Supp}(f^{-1}(Y))$ is a simple normal crossing divisor on V. We decompose $E = E_1 + E_2$, where $E_1 = \mathrm{Supp}(f^{-1}(Y))$ and $E_2 = E - E_1$.

Claim. *We have an isomorphism $\mathcal{I}_Y \simeq f_*\mathcal{O}_V(-E_1)$.*

Proof of Claim. By the definition of E_1, $f : V \to X$ induces a morphism $f : E_1 \to Y$. We consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_Y \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "\simeq"] & \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\mathcal{O}_V(-E_1) \arrow[r] & \mathcal{O}_X \arrow[r] & f_*\mathcal{O}_{E_1} \arrow[r] & \cdots
\end{tikzcd}$$

Since $\mathcal{O}_Y \to f_*\mathcal{O}_{E_1}$ is injective, we have $\mathcal{I}_Y \simeq f_*\mathcal{O}_V(-E_1)$. □

By the vanishing theorem (cf. Theorem 4.3 and Corollary 4.4), we obtain that

$$R^i\pi_*(f_*\mathcal{O}_V(-E_1) \otimes L) \simeq R^i\pi_*(\mathcal{I}_Y \otimes L) = 0$$

for any $i > 0$ because $-E_1 \sim K_V + E_2$. The other statements are obvious by exact sequences. □

§ 5. Toric polyhedra as quasi-log varieties

In this section, all (toric) varieties are assumed to be of finite type over the complex number field $\mathbb{C}$ to use the results in [F2]. We will explain the background and motivation of the results obtained in the previous sections. Note that this section is independent of the other sections. We quickly review the notation of the log minimal model program.

Notation. Let V be a normal variety and let B be an effective $\mathbb{Q}$-divisor on V such that $K_V + B$ is $\mathbb{Q}$-Cartier. Then we can define the *discrepancy* $a(E, V, B) \in \mathbb{Q}$ for any prime divisor E over V. If $a(E, V, B) \geq -1$ for any E, then (V, B) is called *log canonical.* Let (V, B) be a log canonical pair. If E is a prime divisor over V such that $a(E, V, B) = -1$, then $c_V(E)$ is called *log canonical center* of (V, B), where $c_V(E)$ is the closure of the image of E on V.

Let $X = X(\Delta)$ be a toric variety and let D be the complement of the big torus. Then the next proposition is well known. So, we omit the proof.

Proposition 5.1. *The pair (X, D) is log canonical and $K_X + D \sim 0$. Let W be a closed subvariety of X. Then, W is a log canonical center of (X, D) if and only if $W = V(\sigma)$ for some $\sigma \in \Delta \setminus \{0\}$.*

By Proposition 5.1 and adjunction in [A, Theorem 4.4] and [F3, Theorem 3.12], we have the following useful theorem.

Theorem 5.2. *Let $Y = Y(\Phi)$ be a toric polyhedron on X. Then, the log canonical pair (X, D) induces a natural quasi-log structure on $(Y, 0)$. Note that $(Y, 0)$ has only qlc singularities.*

Here, we do not explain the definition of quasi-log varieties. It is because it is very difficult to grasp. See the introduction of [F3] and 5.6 below. The essential point of the theory of quasi-log varieties is contained in the proof of Theorem 5.3 below. The following theorem: Theorem 5.3 is my motivation for Theorem 4.5. It depends on the deep results obtained in [F2].

Theorem 5.3 (cf. [M, Theorem 5.1]). *Let $\pi : X \to S$ be a proper toric morphism and $Y = Y(\Phi)$ a toric polyhedron on $X = X(\Delta)$. Let M be a Cartier divisor on X such that M is π-nef and π-big and $M|_{V(\sigma)}$ is π-big for any $\sigma \in \Delta \setminus \Phi$. Let $\mathcal{I}_Y$ be the defining ideal sheaf of Y on X. Then $R^i\pi_*(\mathcal{I}_Y \otimes \mathcal{O}_X(M)) = 0$ for any $i > 0$. Since $R^i\pi_*\mathcal{O}_X(M) = 0$ for any $i > 0$, we have that $R^i\pi_*\mathcal{O}_Y(M) = 0$ for any $i > 0$ and that the restriction map $\pi_*\mathcal{O}_X(M) \to \pi_*\mathcal{O}_Y(M)$ is surjective.*

Sketch of the proof. If $Y = X$, then there is nothing to prove. So, we can assume that $Y \subsetneq X$. Let $f : V \to X$ be a toric resolution such that $K_V + E = f^*(K_X + D)$ and that $\mathrm{Supp}(f^{-1}(Y))$ is a simple normal crossing divisor on V. We decompose $E = E_1 + E_2$, where $E_1 = \mathrm{Supp}(f^{-1}(Y))$ and $E_2 = E - E_1$. We consider the short exact sequence

$$0 \to \mathcal{O}_V(-E_1) \to \mathcal{O}_V \to \mathcal{O}_{E_1} \to 0.$$

Then we obtain the exact sequence

$$0 \to f_*\mathcal{O}_V(-E_1) \to \mathcal{O}_X \to f_*\mathcal{O}_{E_1} \to R^1f_*\mathcal{O}_V(-E_1) \to \cdots .$$

Since $-E_1 \sim K_V + E_2$, $R^1f_*\mathcal{O}_V(-E_1) \simeq R^1f_*\mathcal{O}_V(K_V + E_2)$ and every non-zero local section of $R^1f_*\mathcal{O}_V(-E_1)$ contains in its support the f-image of some strata of (V, E_2) (see, for example, [A, Theorem 7.4] or [F3, Theorem 3.13]). Note that W is a stratum of (V, E_2) if and only if W is V or a log canonical center of (V, E_2). On the other hand, the support of $f_*\mathcal{O}_{E_1}$ is contained in Y. Therefore, the connecting homomorphism $f_*\mathcal{O}_{E_1} \to R^1f_*\mathcal{O}_V(-E_1)$ is a 0-map. Thus, we obtain

$$0 \to f_*\mathcal{O}_V(-E_1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

and $\mathcal{I}_Y \simeq f_*\mathcal{O}_V(-E_1)$. We consider $f^*M \sim f^*M - E_1 - (K_V + E_2)$. By the vanishing theorem (see [F2] and [F3, Theorem 3.13]), we obtain $R^i\pi_*(f_*\mathcal{O}_V(f^*M - E_1)) \simeq R^i\pi_*(\mathcal{I}_Y \otimes \mathcal{O}_X(M)) = 0$ for any $i > 0$. The other statements are obvious by exact sequences. □

Remark 5.4. In Theorem 5.3, by the Lefschetz principle, we can replace the base field $\mathbb{C}$ with a field k of characteristic zero. I believe that Theorem 5.3 holds true for toric varieties defined over a field k of any characteristic. However, I did not check it.

Remark 5.5. In the proof of Theorem 5.3, we did not use the fact that $\pi : X \to S$ is *toric*. We just needed the properties in Proposition 5.1.

5.6 (Comments on Theorem 5.2). We freely use the notation in the proof of Theorem 5.3. We assume that $Y \subsetneq X$. Then we have the following properties.

1. $g^*0 \sim K_{E_1} + E_2|_{E_1}$, where $g = f|_{E_1} : E_1 \to Y$.
2. $E_2|_{E_1}$ is reduced and $g_*\mathcal{O}_{E_1} \simeq \mathcal{O}_Y$.
3. The collection of subvarieties $\{V(\sigma)\}_{\sigma\in\Phi}$ coincides with the image of torus invariant irreducible subvarieties of V which are contained in E_1.

Therefore, Y is a *quasi-log variety* with the *quasi-log canonical class* 0 and the subvarieties $V(\sigma)$ for $\sigma \in \Phi$ are the *qlc centers* of Y. We sometimes call $g : (E_1, E_2|_{E_1}) \to Y$ a *quasi-log resolution*. For the details, see [F3].

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