Vanishing theorems

Osamu Fujino

Dedicated to Professor Shigefumi Mori

Abstract.

We prove some injectivity, torsion-free, and vanishing theorems for simple normal crossing pairs. Our results heavily depend on the theory of mixed Hodge structures on cohomology groups with compact support. We also treat several basic properties of semi divisorial log terminal pairs.

Contents

§1. Introduction

In this paper, we prove some vanishing theorems for simple normal crossing pairs, which will play important roles in the study of higher dimensional algebraic varieties. We note that the notion of *simple normal crossing pairs* includes here the case when the ambient variety itself has several irreducible components with simple normal crossings. Theorem 1.1 is a generalization of the works of several authors: Kawamata, Viehweg, Kollár, Esnault–Viehweg, Ambro, Fujino, and others

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(cf. [Ko1], [KMM], [EV], [Ko2], [A], [F2], [F4], [F7], [F8], [F11], and so on).

Theorem 1.1 (see Theorem 3.7)**.** *Let (Y, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on Y. Let $f : Y \to X$ be a proper morphism to an algebraic variety X and let L be a Cartier divisor on Y such that $L - (K_Y + \Delta)$ is f-semi-ample.*

(i) *every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ) for every q.*

(ii) *let $\pi : X \to V$ be a projective morphism to an algebraic variety V such that*

$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^* H$$

for some π-ample $\mathbb{R}$-divisor H on X. Then $R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic, that is,

$$R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$$

for every $p > 0$ and $q \geq 0$.

When X is a divisor on a smooth variety M, Theorem 1.1 is contained in [A] and plays crucial roles in the theory of quasi-log varieties. For the details, see [F8, Chapter 3] and [F9]. When X is quasi-projective, it is proved in [FF, Section 6]. Here, we need no extra assumptions on X. Therefore, Theorem 1.1 is new. The theory of resolution of singularities for *reducible* varieties has recently been developing (cf. [BM] and [BP]). It refines several vanishing theorems in [F8]. It is one of the main themes of this paper. We will give a proof of Theorem 1.1 in Section 3. Note that we do not treat *normal crossing varieties*. We only discuss *simple* normal crossing varieties because the theory of resolution of singularities for reducible varieties works well only for *simple* normal crossing varieties. We note that the fundamental theorems for the log minimal model program for log canonical pairs can be proved without using the theory of quasi-log varieties (cf. [F10] and [F11]). The case when Y is smooth in Theorem 1.1 is sufficient for [F10] and [F11]. For that case, see [F7] and [F11, Sections 5 and 6]. Our proof of Theorem 1.1 heavily depends on the theory of mixed Hodge structures on cohomology groups with compact support.

1.2 (Hodge theoretic viewpoint)**.** Let X be a projective simple normal crossing variety with $\dim X = n$. We are mainly interested in $H^\bullet(X, \omega_X)$ or $H^\bullet(X, \omega_X \otimes L)$ for some line bundle L on X. By the theory of mixed Hodge structures,

$$\mathrm{Gr}_F^n H^\bullet(X, \mathbb{C}) \simeq H^{\bullet - n}(X, \nu_* \omega_{X^\nu}),$$

where $\nu : X^\nu \to X$ is the normalization, and

$$\mathrm{Gr}_F^0 H^\bullet(X, \mathbb{C}) \simeq H^\bullet(X, \mathcal{O}_X).$$

Note that F is the Hodge filtration on the natural mixed Hodge structure on $H^\bullet(X, \mathbb{Q})$. Let D be a simple normal crossing divisor on X. Then we obtain

$$\mathrm{Gr}_F^n H^\bullet(X \setminus D, \mathbb{C}) \simeq H^{\bullet-n}(X, \nu_*\omega_{X^\nu} \otimes \mathcal{O}_X(D))$$

and

$$\mathrm{Gr}_F^0 H_c^\bullet(X \setminus D, \mathbb{C}) \simeq H^\bullet(X, \mathcal{O}_X(-D)).$$

Note that

$$\mathrm{Gr}_F^0 H^\bullet(X \setminus D, \mathbb{C}) \simeq H^\bullet(X, \mathcal{O}_X)$$

and that

$$H^{\bullet-n}(X, \nu_*\omega_{X^\nu} \otimes \mathcal{O}_X(D)) \not\simeq H^{\bullet-n}(X, \omega_X \otimes \mathcal{O}_X(D)).$$

We also note that $H_c^\bullet(X \setminus D, \mathbb{Q})$ need not be the dual vector space of $H^{2n-\bullet}(X \setminus D, \mathbb{Q})$ when X is not smooth. In this setting, we are interested in $H^\bullet(X, \omega_X(D))$ or $H^\bullet(X, \omega_X(D) \otimes L)$. Therefore, we consider the natural mixed Hodge structure on $H_c^\bullet(X \setminus D, \mathbb{C})$ and take the dual vector space of

$$\mathrm{Gr}_F^0 H_c^\bullet(X \setminus D, \mathbb{C}) \simeq H^\bullet(X, \mathcal{O}_X(-D))$$

by Serre duality. Then we obtain $H^{n-\bullet}(X, \omega_X(D))$. We note that if L is semi-ample then we can reduce the problem to the case when L is trivial by the usual covering trick. The above observation is crucial for our treatment of the vanishing theorems and the semipositivity theorems in [F8] and [FF]. In this paper, we do not discuss the Hodge theoretic part of vanishing and semipositivity theorems. We prove Theorem 1.1 by assuming the Hodge theoretic injectivity theorem: Theorem 3.1. For the details of the Hodge theoretic part, see [F8], [FF], and [F15].

The author learned the following example from Kento Fujita.

Example 1.3. Let $X_1 = \mathbb{P}^2$ and let C_1 be a line on X_1. Let $X_2 = \mathbb{P}^2$ and let C_2 be a smooth conic on X_2. We fix an isomorphism $\tau : C_1 \to C_2$. By gluing X_1 and X_2 along $\tau : C_1 \to C_2$, we obtain a simple normal crossing surface X such that $-K_X$ is ample (cf. [Ft]). We can check that X can not be embedded into any smooth varieties as a simple normal crossing divisor.

Example 1.3 shows that Theorem 1.1 is not covered by the results in [A], [F4], and [F8].

Remark 1.4 (cf. [F8, Proposition 3.65])**.** We can construct a proper simple normal crossing variety X with the following property. Let $f : Y \to X$ be a proper morphism from a simple normal crossing variety Y such that f induces an isomorphism $f|_V : V \simeq U$ where V (resp. U) is a dense Zariski open subset of Y (resp. X) which contains the generic point of any stratum of Y (resp. X). Then Y is non-projective. Therefore, we can not directly use Chow's lemma to reduce our main theorem (cf. Theorem 1.1) to the quasi-projective case (cf. [FF, Section 6]).

There exists another standard approach to various Kodaira type vanishing theorems. It is an analytic method (see, for example, [F5] and [F6]). At the present time, the relationship between our Hodge theoretic approach and the analytic method is not clear.

We summarize the contents of this paper. In Section 2, we collect some basic definitions and results for the study of simple normal crossing varieties and divisors on them. Section 3 is the main part of this paper. It is devoted to the study of injectivity, torsion-free, and vanishing theorems for simple normal crossing pairs. We note that we do not prove the Hodge theoretic injectivity theorem: Theorem 3.1. We just quote it from [F8] (see also [F15]). Section 4 is an easy application of the vanishing theorem in Section 3. We prove the basic properties of semi divisorial log terminal pairs in the sense of Kollár. In Section 5, we explain our new semipositivity theorem, which is a generalization of the Fujita–Kawamata semipositivity theorem, without proof. It depends on the theory of variations of mixed Hodge structures on cohomology groups with compact support and is related to the results obtained in Section 3. Anyway, the vanishing theorem and the semipositivity theorem discussed in this paper follow from the theory of mixed Hodge structures on cohomology groups with compact support.

For various applications of Theorem 1.1 and related topics, see [F8], [FF], [F13], [F14], [F15], and so on.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. But we note that, by using the Lefschetz principle, all the results in this paper hold over an algebraically closed field k of characteristic zero.

§2. Preliminaries

First, we quickly recall basic definitions of divisors. We note that we have to deal with reducible algebraic schemes in this paper. For details, see, for example, [H, Section 2] and [L, Section 7.1].

2.1. Let X be a noetherian scheme with structure sheaf $\mathcal{O}_X$ and let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$. Let $\mathcal{K}_X^*$ denote the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$, and $\mathcal{O}_X^*$ the sheaf of invertible elements in $\mathcal{O}_X$. We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}_X^* \subset \mathcal{K}_X^*$.

2.2 (Cartier, $\mathbb{Q}$-Cartier, and $\mathbb{R}$-Cartier divisors)**.** A *Cartier divisor* D on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, D is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. A $\mathbb{Q}$-*Cartier divisor* (resp. $\mathbb{R}$-*Cartier divisor*) is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{R}$).

2.3 (Linear, $\mathbb{Q}$-linear, and $\mathbb{R}$-linear equivalence)**.** Let D_1 and D_2 be two $\mathbb{R}$-Cartier divisors on X. Then D_1 is *linearly* (resp. $\mathbb{Q}$-*linearly*, or $\mathbb{R}$-*linearly*) *equivalent* to D_2, denoted by $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$, or $D_1 \sim_{\mathbb{R}} D_2$) if

$$D_1 = D_2 + \sum_{i=1}^{k} r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Let $f : X \to Y$ be a morphism. If there is an $\mathbb{R}$-Cartier divisor B on Y such that $D_1 \sim_{\mathbb{R}} D_2 + f^*B$, then D_1 is said to be *relatively* $\mathbb{R}$-*linearly equivalent* to D_2. It is denoted by $D_1 \sim_{\mathbb{R},f} D_2$.

2.4 (Supports)**.** Let D be a Cartier divisor on X. The *support* of D, denoted by Supp$D$, is the subset of X consisting of points x such that a local equation for D is not in $\mathcal{O}_{X,x}^*$. The support of D is a closed subset of X.

2.5 (Weil divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors)**.** Let X be an equidimensional reduced separated algebraic scheme. We note that X is not necessarily regular in codimension one. A (*Weil*) *divisor* D on X is a finite formal sum

$$\sum_{i=1}^{n} d_i D_i$$

where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is an integer for every i such that $D_i \neq D_j$ for $i \neq j$.

If $d_i \in \mathbb{Q}$ (resp. $d_i \in \mathbb{R}$) for every i, then D is called a $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*). We define the *round-up* $\lceil D \rceil = \sum_{i=1}^{r} \lceil d_i \rceil D_i$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{i=1}^{r} \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x+1$ (resp. $x-1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We define $D^{<1} = \sum_{d_i<1} d_i D_i$ and so on. We call D a *boundary* $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ for every i.

Next, we recall the definition of *simple normal crossing pairs*.

Definition 2.6 (Simple normal crossing pairs)**.** We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular system of parameters $(x_1, \cdots, x_p, y_1, \cdots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety.* If X is a simple normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. Therefore, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$ (cf. [L, Section 7.1 Corollary 1.19]). It is a Cartier divisor on X and is well-defined up to linear equivalence.

We note that a simple normal crossing pair is called a *semi-snc pair* in [Ko3, Definition 1.10].

Definition 2.7 (Strata and permissibility)**.** Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \cdots, i_k\} \subset I$. A Cartier divisor D on X is *permissible* if D contains no strata of X in its support. A finite $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisors is called a *permissible* $\mathbb{Q}$-*Cartier divisor* (resp. $\mathbb{R}$-*Cartier divisor*) on X.

2.8. Let X be a simple normal crossing variety. Let $\mathrm{PerDiv}(X)$ be the abelian group generated by permissible Cartier divisors on X and let $\mathrm{Weil}(X)$ be the abelian group generated by Weil divisors on X. Then we can define natural injective homomorphisms of abelian groups

$$\psi : \mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} \to \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}$$

for $\mathbb{K} = \mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Let $\nu : \widetilde{X} \to X$ be the normalization. Then we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Div}(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} & \xrightarrow[\widetilde{\psi}]{\sim} & \mathrm{Weil}(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} \\
{\scriptstyle \nu^*}\uparrow & & \downarrow{\scriptstyle \nu_*} \\
\mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} & \xrightarrow[\psi]{} & \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}
\end{array}$$

Note that $\mathrm{Div}(\widetilde{X})$ is the abelian group generated by Cartier divisors on $\widetilde{X}$ and that $\widetilde{\psi}$ is an isomorphism since $\widetilde{X}$ is smooth.

By ψ, every permissible Cartier (resp. $\mathbb{Q}$-Cartier or $\mathbb{R}$-Cartier) divisor can be considered as a Weil divisor (resp. $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor). Therefore, various operations, for example, $\lfloor D \rfloor$, $D^{<1}$, and so on, make sense for a permissible $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X.

We note the following easy example.

Example 2.9. Let X be a simple normal crossing variety in $\mathbb{C}^3 = \mathrm{Spec}\mathbb{C}[x, y, z]$ defined by $xy = 0$. We set $D_1 = (x + z = 0) \cap X$ and $D_2 = (x - z = 0) \cap X$. Then $D = \frac{1}{2}D_1 + \frac{1}{2}D_2$ is a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. In this case, $\lfloor D \rfloor = (x = z = 0)$ on X. Therefore, $\lfloor D \rfloor$ is not a Cartier divisor on X.

Definition 2.10 (Simple normal crossing divisors)**.** Let X be a simple normal crossing variety and let D be a Cartier divisor on X. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X.

Remark 2.11. Let X be a simple normal crossing variety and let D be a $\mathbb{K}$-divisor on X where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. If $\mathrm{Supp}D$ is a simple normal crossing divisor on X and D is $\mathbb{K}$-Cartier, then $\lfloor D \rfloor$ and $\lceil D \rceil$ (resp. $\{D\}$, $D^{<1}$, and so on) are Cartier (resp. $\mathbb{K}$-Cartier) divisors on X (cf. [BP, Section 8]).

The following lemma is easy but important.

Lemma 2.12. *Let X be a simple normal crossing variety and let B be a permissible $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\lfloor B \rfloor = 0$. Let A be a Cartier divisor on X. Assume that $A \sim_{\mathbb{R}} B$. Then there exists a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\lfloor C \rfloor = 0$, and* $\mathrm{Supp} C = \mathrm{Supp} B$.

Proof. We can write $B = A + \sum_{i=1}^{k} r_i (f_i)$, where $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{R}$ for every i. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ (see 2.1). Let $P \in X$ be a scheme theoretic point corresponding to some stratum of X. We consider the following affine map

$$\mathbb{K}^k \to H^0(X_P, \mathcal{K}_{X_P}^*/\mathcal{O}_{X_P}^*) \otimes_{\mathbb{Z}} \mathbb{K}$$

given by $(a_1, \cdots, a_k) \mapsto A + \sum_{i=1}^{k} a_i (f_i)$, where $X_P = \mathrm{Spec} \mathcal{O}_{X,P}$ and $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Then we can check that

$$\mathcal{P} = \{(a_1, \cdots, a_k) \in \mathbb{R}^k \,|\, A + \sum_i a_i (f_i) \text{ is permissible}\} \subset \mathbb{R}^k$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Therefore, we see that

$$\mathcal{S} = \{(a_1, \cdots, a_k) \in \mathcal{P} \,|\, \mathrm{Supp}(A + \sum_i a_i (f_i)) \subset \mathrm{Supp} B\} \subset \mathcal{P}$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Since $(r_1, \cdots, r_k) \in \mathcal{S}$, we know that $\mathcal{S} \neq \emptyset$. We take a point $(s_1, \cdots, s_k) \in \mathcal{S} \cap \mathbb{Q}^k$ which is general in $\mathcal{S}$ and sufficiently close to $(r_1, \cdots, r_k)$ and set $C = A + \sum_{i=1}^{k} s_i (f_i)$. By construction, C is a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $C \sim_{\mathbb{Q}} A$, $\lfloor C \rfloor = 0$, and $\mathrm{Supp} C = \mathrm{Supp} B$. Q.E.D.

We need the following important definition in Section 3.

Definition 2.13 (Strata and permissibility for pairs)**.** Let (X, D) be a simple normal crossing pair such that D is a boundary $\mathbb{R}$-divisor on X. Let $\nu : X^\nu \to X$ be the normalization. We define Θ by the formula

$$K_{X^\nu} + \Theta = \nu^*(K_X + D).$$

Then a *stratum* of (X, D) is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Θ). We note that (X^ν, Θ) is log canonical. When $D = 0$, this definition is compatible with Definition 2.7. A Cartier divisor B on X is *permissible with respect to* (X, D) if B contains no strata of (X, D) in its support. A finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of permissible Cartier divisors with respect to (X, D) is called a *permissible* $\mathbb{R}$*-divisor* (resp. $\mathbb{Q}$*-divisor*)*with respect to* (X, D).

For the reader's convenience, we recall Grothendieck's Quot scheme. For the details, see, for example, [N, Theorem 5.14] and [AK, Section 2]. We will use it in the proof of the main theorem: Theorem 3.7.

Theorem 2.14 (Grothendieck)**.** *Let S be a noetherian scheme, let $\pi : X \to S$ be a projective morphism, and let L be a relatively very ample line bundle on X. Then for any coherent $\mathcal{O}_X$-module E and any polynomial $\Phi \in \mathbb{Q}[\lambda]$, the functor $\mathfrak{Quot}^{\Phi,L}_{E/X/S}$ is representable by a projective S-scheme $\mathrm{Quot}^{\Phi,L}_{E/X/S}$.*

§3. Vanishing theorems

Let us start with the following injectivity theorem (cf. [F4, Proposition 3.2] and [F8, Proposition 2.23]). The proof of Theorem 3.1 in [F8] is purely Hodge theoretic. It uses the theory of mixed Hodge structures on cohomology groups with compact support (cf. 1.2). For the details, see [F8, Chapter 2] and [F15].

Theorem 3.1 (Hodge theoretic injectivity theorem)**.** *Let $(X, S+B)$ be a simple normal crossing pair such that X is proper, $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\lfloor B \rfloor = 0$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in* $\mathrm{Supp} B$. *Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Remark 3.2. In [F15], we prove a slight generalization of Theorem 3.1. However, Theorem 3.1 is sufficient for the proof of Theorem 3.4 below.

The next lemma is an easy generalization of the vanishing theorem of Reid–Fukuda type for simple normal crossing pairs, which is a very special case of Theorem 3.7 (i). However, we need Lemma 3.3 for our proof of Theorem 3.7.

Lemma 3.3 (Relative vanishing lemma)**.** *Let $f : Y \to X$ be a proper morphism from a simple normal crossing pair (Y, Δ) to an algebraic variety X such that Δ is a boundary $\mathbb{R}$-divisor on Y. We assume that f is an isomorphism at the generic point of any stratum of the pair (Y, Δ). Let L be a Cartier divisor on Y such that $L \sim_{\mathbb{R}, f} K_Y + \Delta$. Then $R^q f_* \mathcal{O}_Y(L) = 0$ for every $q > 0$.*

Proof. By shrinking X, we may assume that $L \sim_{\mathbb{R}} K_Y + \Delta$. By applying Lemma 2.12 to $\{\Delta\}$, we may further assume that Δ is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_Y + \Delta$.

Step 1. We assume that Y is irreducible. In this case, $L-(K_Y+\Delta)$ is nef and log big over X with respect to the pair (Y, Δ), that is, $L-(K_Y+\Delta)$ is nef and big over X and $(L-(K_Y+\Delta))|_W$ is big over $f(W)$ for every stratum W of the pair (Y, Δ). Therefore, $R^q f_*\mathcal{O}_Y(L)=0$ for every $q>0$ by the vanishing theorem of Reid–Fukuda type (see, for example, [F8, Lemma 4.10]).

Step 2. Let Y_1 be an irreducible component of Y and let Y_2 be the union of the other irreducible components of Y. Then we have a short exact sequence

$$0 \to \mathcal{O}_{Y_1}(-Y_2|_{Y_1}) \to \mathcal{O}_Y \to \mathcal{O}_{Y_2} \to 0.$$

We set $L' = L|_{Y_1} - Y_2|_{Y_1}$. Then we have a short exact sequence

$$0 \to \mathcal{O}_{Y_1}(L') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y_2}(L|_{Y_2}) \to 0$$

and $L' \sim_{\mathbb{Q}} K_{Y_1} + \Delta|_{Y_1}$. On the other hand, we can check that

$$L|_{Y_2} \sim_{\mathbb{Q}} K_{Y_2} + Y_1|_{Y_2} + \Delta|_{Y_2}.$$

We have already known that $R^q f_*\mathcal{O}_{Y_1}(L')=0$ for every $q>0$ by Step 1. By induction on the number of the irreducible components of Y, we have $R^q f_*\mathcal{O}_{Y_2}(L|_{Y_2})=0$ for every $q>0$. Therefore, $R^q f_*\mathcal{O}_Y(L)=0$ for every $q>0$ by the exact sequence:

$$\cdots \to R^q f_*\mathcal{O}_{Y_1}(L') \to R^q f_*\mathcal{O}_Y(L) \to R^q f_*\mathcal{O}_{Y_2}(L|_{Y_2}) \to \cdots.$$

So, we finish the proof of Lemma 3.3. Q.E.D.

Although Lemma 3.3 is a very easy generalization of the relative Kawamata–Viehweg vanishing theorem, it is sufficiently powerful for the study of reducible varieties once we combine it with the recent results in [BM] and [BP]. In Section 4, we will see an application of Lemma 3.3 for the study of semi divisorial log terminal pairs.

It is the time to state the main injectivity theorem for simple normal crossing pairs. It is a direct application of Theorem 3.1. Our formulation of Theorem 3.4 is indispensable for the proof of our main theorem: Theorem 3.7.

Theorem 3.4 (Injectivity theorem for simple normal crossing pairs)**.** *Let (X, Δ) be a simple normal crossing pair such that X is proper and* *that Δ is a boundary $\mathbb{R}$-divisor on X. Let L be a Cartier divisor on X and let D be an effective Cartier divisor that is permissible with respect to (X, Δ). Assume the following conditions.*

(i) $L \sim_{\mathbb{R}} K_X + \Delta + H$,

(ii) *H is a semi-ample $\mathbb{R}$-divisor, and*

(iii) *$tH \sim_{\mathbb{R}} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to (X, Δ).*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Remark 3.5. For the definition and the basic properties of semi-ample $\mathbb{R}$-divisors, see [F11, Definition 4.11, Lemma 4.13, and Lemma 4.14].

Proof of Theorem 3.4. We set $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$ throughout this proof. We obtain a projective birational morphism $f : Y \to X$ from a simple normal crossing variety Y such that f is an isomorphism over $X \setminus \operatorname{Supp}(D + D' + B)$, and that the union of the support of $f^*(S + B + D + D')$ and the exceptional locus of f has a simple normal crossing support on Y (cf. [BP, Theorem 1.4]). Let B' be the strict transform of B on Y. We may assume that $\operatorname{Supp}B'$ is disjoint from any strata of Y that are not irreducible components of Y by taking blowing-ups. We write

$$K_Y + S' + B' = f^*(K_X + S + B) + E,$$

where S' is the strict transform of S and E is f-exceptional. By the construction of $f : Y \to X$, S' is Cartier and B' is $\mathbb{R}$-Cartier. Therefore, E is also $\mathbb{R}$-Cartier. It is easy to see that $E_+ = \lceil E \rceil \geq 0$. We set $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is $\mathbb{R}$-Cartier because $\operatorname{Supp}E$ is simple normal crossing on Y (cf. Remark 2.11). Since f^*H is an $\mathbb{R}_{>0}$-linear combination of semi-ample Cartier divisors, we can write $f^*H \sim_{\mathbb{R}} \sum_i a_i H_i$, where $0 < a_i < 1$ and H_i is a general Cartier divisor on Y for every i. We set

$$B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D + D') + (1 - \varepsilon) \sum_i a_i H_i$$

for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{R}} K_Y + S' + B''$. By construction, $\lfloor B'' \rfloor = 0$, the support of $S' + B''$ is simple normal crossing on Y, and

$\mathrm{Supp}B'' \supset \mathrm{Supp}f^*D$. So, Theorem 3.1 implies that the homomorphisms

$$H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D))$$

are injective for all q. By Lemma 3.3, $R^qf_*\mathcal{O}_Y(L') = 0$ for every $q > 0$ and it is easy to see that $f_*\mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

are injective for all q. Q.E.D.

Lemma 3.6. *Let $f : Z \to X$ be a proper morphism from a simple normal crossing pair (Z, B) to an algebraic variety X. Let $\overline{X}$ be a projective variety such that $\overline{X}$ contains X as a Zariski dense open subset. Then there exist a proper simple normal crossing pair $(\overline{Z}, \overline{B})$ that is a compactification of (Z, B) and a proper morphism $\overline{f} : \overline{Z} \to \overline{X}$ that compactifies $f : Z \to X$. Moreover, $\overline{Z} \setminus Z$ is a divisor on $\overline{Z}$, $\mathrm{Supp}\overline{B} \cup \mathrm{Supp}(\overline{Z} \setminus Z)$ is a simple normal crossing divisor on $\overline{Z}$, and $\overline{Z} \setminus Z$ has no common irreducible components with $\overline{B}$. We note that we can make $\overline{B}$ a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on $\overline{Z}$ when so is B on Z, where $\mathbb{K}$ is $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. When f is projective, we can make $\overline{Z}$ projective.*

Proof. Let $\overline{B} \subset \overline{Z}$ be any compactification of $B \subset Z$. By blowing up $\overline{Z}$ inside $\overline{Z} \setminus Z$, we may assume that $f : Z \to X$ extends to $\overline{f} : \overline{Z} \to \overline{X}$, $\overline{Z}$ is a simple normal crossing variety, and $\overline{Z} \setminus Z$ is of pure codimension one (see [BM, Theorem 1.5]). By [BP, Theorem 1.4], we can construct a desired compactification. Note that we can make $\overline{B}$ a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor by the argument in [BP, Section 8]. Q.E.D.

Theorem 3.7 below is our main theorem of this paper, which is a generalization of Kollár's torsion-free and vanishing theorem (see [Ko1, Theorem 2.1]). The reader find various applications of Theorem 3.7 in [F8], [FF], and [F13]. We note that Theorem 3.7 for *embedded normal crossing pairs* was first formulated by Florin Ambro for his theory of *quasi-log varieties* (cf. [A]). For the details of the theory of quasi-log varieties, see [F8, Chapter 3] and [F9].

Theorem 3.7. *Let (Y, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on Y. Let $f : Y \to X$ be a proper morphism to an algebraic variety X and let L be a Cartier divisor on Y such that $L - (K_Y + \Delta)$ is f-semi-ample.*

(i) *every associated prime of $R^qf_*\mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ) for every q.*

(ii) *let $\pi : X \to V$ be a projective morphism to an algebraic variety V such that*

$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^*H$$

for some π-ample $\mathbb{R}$-divisor H on X. Then $R^q f_*\mathcal{O}_Y(L)$ is π_*-acyclic, that is,

$$R^p\pi_* R^q f_*\mathcal{O}_Y(L) = 0$$

for every $p > 0$ and $q \geq 0$.

Proof. We set $S = \lfloor \Delta \rfloor$, $B = \{\Delta\}$, and $H' \sim_{\mathbb{R}} L - (K_Y + \Delta)$ throughout this proof. Let us start with the proof of (i).

Step 1. First, we assume that X is projective. We may assume that H' is semi-ample by replacing L (resp. H') with $L + f^*A'$ (resp. $H' + f^*A'$), where A' is a very ample Cartier divisor on X. Suppose that $R^q f_*\mathcal{O}_Y(L)$ has a local section whose support does not contain the f-images of any strata of $(Y, S+B)$. More precisely, let U be a non-empty Zariski open set and let $s \in \Gamma(U, R^q f_*\mathcal{O}_Y(L))$ be a non-zero section of $R^q f_*\mathcal{O}_Y(L)$ on U whose support $V \subset U$ does not contain the f-images of any strata of $(Y, S+B)$. Let $\overline{V}$ be the closure of V in X. We note that $\overline{V} \setminus V$ may contain the f-image of some stratum of $(Y, S+B)$. Let Y_2 be the union of the irreducible components of Y that are mapped into $\overline{V} \setminus V$ and let Y_1 be the union of the other irreducible components of Y. We set

$$K_{Y_1} + S_1 + B_1 = (K_Y + S + B)|_{Y_1}$$

such that S_1 is reduced and that $\lfloor B_1 \rfloor = 0$. By replacing Y, S, B, L, and H' with Y_1, S_1, B_1, $L|_{Y_1}$, and $H'|_{Y_1}$, we may assume that no irreducible components of Y are mapped into $\overline{V} \setminus V$. Let C be a stratum of $(Y, S+B)$ that is mapped into $\overline{V} \setminus V$. Let $\sigma : Y' \to Y$ be the blowing-up along C. We set $E = \sigma^{-1}(C)$. We can write

$$K_{Y'} + S' + B' = \sigma^*(K_Y + S + B)$$

such that S' is reduced and $\lfloor B' \rfloor = 0$. Thus,

$$\sigma^*H' \sim_{\mathbb{R}} \sigma^*L - (K_{Y'} + S' + B')$$

and

$$\sigma^*H' \sim_{\mathbb{R}} \sigma^*L - E - (K_{Y'} + (S' - E) + B').$$

We note that $S' - E$ is effective. We replace Y, H', L, S, and B with Y', σ^*H', $\sigma^*L - E$, $S' - E$, and B'. By repeating this process finitely many times, we may assume that $\overline{V}$ does not contain the f-images of any strata of $(Y, S+B)$. Then we can find a very ample Cartier divisor A on X with the following properties.

(a) f^*A is permissible with respect to $(Y, S+B)$, and
(b) $R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We may assume that $H' - f^*A$ is semi-ample by replacing L (resp. H') with $L + f^*A$ (resp. $H' + f^*A$). If necessary, we replace L (resp. H') with $L + f^*A''$ (resp. $H' + f^*A''$), where A'' is a very ample Cartier divisor. Then, we have

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \simeq H^q(Y, \mathcal{O}_Y(L))$$

and

$$H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y, \mathcal{O}_Y(L + f^*A)).$$

We obtain that

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \to H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A))$$

is not injective by (b) if A'' is sufficiently ample. So,

$$H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + f^*A))$$

is not injective. It contradicts Theorem 3.4. Therefore, the support of every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains the f-image of some stratum of (Y, Δ), equivalently, the support of every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ is equal to the union of the f-images of some strata of (Y, Δ). This means that every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ). We finish the proof when X is projective.

Step 2. Next, we assume that X is not projective. Note that the problem is local. So, we shrink X and may assume that X is affine. We can write $H' \sim_{\mathbb{R}} H'_1 + H'_2$, where H'_1 (resp. H'_2) is a semi-ample $\mathbb{Q}$-divisor (resp. a semi-ample $\mathbb{R}$-divisor) on Y. We can write $H'_2 \sim_{\mathbb{R}} \sum_i a_i A_i$, where $0 < a_i < 1$ and A_i is a general effective Cartier divisor on Y for every i. Replacing B (resp. H') with $B + \sum_i a_i A_i$ (resp. H'_1), we may assume that H' is a semi-ample $\mathbb{Q}$-divisor. Without loss of generality, we may further assume that $(Y, B + S + H')$ is a simple normal crossing pair. We compactify X and apply Lemma 3.6. Then we obtain a compactification $\overline{f} : \overline{Y} \to \overline{X}$ of $f : Y \to X$. Let $\overline{H'}$ be the closure of H' on $\overline{Y}$. If $\overline{H'}$ is not a semi-ample $\mathbb{Q}$-divisor, then we take blowing-ups of $\overline{Y}$ inside $\overline{Y} \setminus Y$ and obtain a semi-ample $\mathbb{Q}$-divisor $\widetilde{H'}$ on $\overline{Y}$ such that $\widetilde{H'}|_Y = H'$. Let $\overline{B}$ (resp. $\overline{S}$) be the closure of B (resp. S)

on $\overline{Y}$. We may assume that $\overline{S}$ is Cartier and $\overline{B}$ is $\mathbb{R}$-Cartier (cf. Lemma 3.6). We construct a coherent sheaf $\mathcal{F}$ on $\overline{Y}$ which is an extension of $\mathcal{O}_Y(L)$. We consider Grothendieck's Quot scheme $\mathrm{Quot}^{1,\mathcal{O}_{\overline{Y}}}_{\mathcal{F}/\overline{Y}/\overline{Y}}$ (see Theorem 2.14). Note that the restriction of $\mathrm{Quot}^{1,\mathcal{O}_{\overline{Y}}}_{\mathcal{F}/\overline{Y}/\overline{Y}}$ to Y is nothing but Y because $\mathcal{F}|_Y = \mathcal{O}_Y(L)$ is a line bundle on Y. Therefore, the structure morphism from $\mathrm{Quot}^{1,\mathcal{O}_{\overline{Y}}}_{\mathcal{F}/\overline{Y}/\overline{Y}}$ to $\overline{Y}$ has a section s over Y. By taking the closure of $s(Y)$ in $\mathrm{Quot}^{1,\mathcal{O}_{\overline{Y}}}_{\mathcal{F}/\overline{Y}/\overline{Y}}$, we have a compactification $Y^\dagger$ of Y and a line bundle $\mathcal{L}$ on $Y^\dagger$ with $\mathcal{L}|_Y = \mathcal{O}_Y(L)$. If necessary, we take more blowing-ups of $Y^\dagger$ outside Y (cf. [BP, Theorem 1.4]). Then we obtain a new compactification $\overline{Y}$ and a Cartier divisor $\overline{L}$ on $\overline{Y}$ with $\overline{L}|_Y = L$ (cf. Lemma 3.6). In this situation, $\widetilde{H'} \sim_{\mathbb{R}} \overline{L} - (K_{\overline{Y}} + \overline{S} + \overline{B})$ does not necessarily hold. We can write

$$H' + \sum_i b_i(f_i) = L - (K_Y + S + B),$$

where b_i is a real number and $f_i \in \Gamma(Y, \mathcal{K}^*_Y)$ for every i. We set

$$E = \widetilde{H'} + \sum_i b_i(f_i) - (\overline{L} - (K_{\overline{Y}} + \overline{S} + \overline{B})).$$

We note that we can see $f_i \in \Gamma(\overline{Y}, \mathcal{K}^*_{\overline{Y}})$ for every i (cf. [L, Section 7.1 Proposition 1.15]). We replace $\overline{L}$ (resp. $\overline{B}$) with $\overline{L} + \lceil E \rceil$ (resp. $\overline{B} + \{-E\}$). Then we obtain the desired property of $R^q\overline{f}_*\mathcal{O}_{\overline{Y}}(\overline{L})$ since $\overline{X}$ is projective. We note that $\lceil E \rceil$ is Cartier because $\mathrm{Supp}E$ is in $\overline{Y} \setminus Y$ and E is $\mathbb{R}$-Cartier (cf. Remark 2.11). So, we finish the whole proof of (i).

From now on, we prove (ii).

Step 1. We assume that $\dim V = 0$. In this case, we can write $H \sim_{\mathbb{R}} H_1 + H_2$, where H_1 (resp. H_2) is an ample $\mathbb{Q}$-divisor (resp. an ample $\mathbb{R}$-divisor) on X. So, we can write $H_2 \sim_{\mathbb{R}} \sum_i a_i A_i$, where $0 < a_i < 1$ and A_i is a general very ample Cartier divisor on X for every i. Replacing B (resp. H) with $B + \sum_i a_i f^*A_i$ (resp. H_1), we may assume that H is an ample $\mathbb{Q}$-divisor. We take a general member $A \in |mH|$, where m is a sufficiently large and divisible integer, such that $A' = f^*A$ and $R^qf_*\mathcal{O}_Y(L + A')$ is π_*-acyclic, that is, Γ-acyclic, for all q. By (i), we have the following short exact sequences,

$$0 \to R^qf_*\mathcal{O}_Y(L) \to R^qf_*\mathcal{O}_Y(L + A') \to R^qf_*\mathcal{O}_{A'}(L + A') \to 0.$$

for every q. Note that $R^qf_*\mathcal{O}_{A'}(L + A')$ is π_*-acyclic by induction on $\dim X$ and $R^qf_*\mathcal{O}_Y(L + A')$ is also π_*-acyclic by the above assumption.

Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{array}{ccc} E_2^{pq} = H^p(X, R^q f_*\mathcal{O}_Y(L)) & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L)) \\ \varphi^{pq}\downarrow & & \downarrow\varphi^{p+q} \\ \overline{E}_2^{pq} = H^p(X, R^q f_*\mathcal{O}_Y(L+A')) & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L+A')) \end{array}$$

We note that φ^{1+q} is injective by Theorem 3.4. We have that

$$E_2^{1q} \xrightarrow{\alpha} H^{1+q}(Y, \mathcal{O}_Y(L))$$

is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$. We also have that $\overline{E}_2^{1q} = 0$ by the above assumption. Therefore, we obtain $E_2^{1q} = 0$ since the injection

$$E_2^{1q} \xrightarrow{\alpha} H^{1+q}(Y, \mathcal{O}_Y(L)) \xrightarrow{\varphi^{1+q}} H^{1+q}(Y, \mathcal{O}_Y(L+A'))$$

factors through $\overline{E}_2^{1q} = 0$. This implies that $H^p(X, R^q f_*\mathcal{O}_Y(L)) = 0$ for every $p > 0$.

Step 2. We assume that V is projective. By replacing H (resp. L) with $H + \pi^*G$ (resp. $L + (\pi \circ f)^*G$), where G is a very ample Cartier divisor on V, we may assume that H is an ample $\mathbb{R}$-divisor. If G is a sufficiently ample Cartier divisor on V, then we have

$$H^k(V, R^p\pi_* R^q f_*\mathcal{O}_Y(L) \otimes G) = 0$$

for every $k \geq 1$,

$$\begin{aligned} H^0(V, R^p\pi_* R^q f_*\mathcal{O}_Y(L) \otimes \mathcal{O}_V(G)) &\simeq H^p(X, R^q f_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(\pi^*G)) \\ &\simeq H^p(X, R^q f_*\mathcal{O}_Y(L + f^*\pi^*G)) \end{aligned}$$

for every p and q, and $R^p\pi_* R^q f_*\mathcal{O}_Y(L) \otimes \mathcal{O}_V(G)$ is generated by its global sections for every p and q. Since

$$L + f^*\pi^*G - (K_Y + \Delta) \sim_{\mathbb{R}} f^*(H + \pi^*G),$$

and $H + \pi^*G$ is ample, we can apply Step 1 and obtain

$$H^p(X, R^q f_*\mathcal{O}_Y(L + f^*\pi^*G)) = 0$$

for every $p > 0$. Thus, $R^p\pi_* R^q f_*\mathcal{O}_Y(L) = 0$ for every $p > 0$ by the above arguments.

Step 3. When V is not projective, we shrink V and may assume that V is affine. By the same argument as in Step 1, we may assume that H is $\mathbb{Q}$-Cartier. Let $\overline{\pi} : \overline{X} \to \overline{V}$ be a compactification of $\pi : X \to V$ such that $\overline{X}$ and $\overline{V}$ are projective. We may assume that there exists a $\overline{\pi}$-ample $\mathbb{Q}$-divisor $\overline{H}$ on $\overline{X}$ such that $\overline{H}|_X = H$. By Lemma 3.6, we can compactify $f : (Y, S + B) \to X$ and obtain $\overline{f} : (\overline{Y}, \overline{S} + \overline{B}) \to \overline{X}$. We note that $\overline{f}^* \overline{H} \sim_{\mathbb{R}} \overline{L} - (K_{\overline{Y}} + \overline{S} + \overline{B})$ does not necessarily hold, where $\overline{L}$ is a Cartier divisor on $\overline{Y}$ constructed as in Step 2 in the proof of (i). By the same argument as in Step 2 in the proof of (i), we obtain that $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.

We finish the whole proof of (ii). Q.E.D.

§4. Semi divisorial log terminal pairs

Let us start with the definition of *semi divisorial log terminal pairs* in the sense of Kollár. For details of singularities which appear in the minimal model program, see [F3] and [Ko3].

Definition 4.1 (Semi divisorial log terminal pairs)**.** Let X be an equidimensional reduced separated S_2 algebraic scheme which is simple normal crossing in codimension one. Let $\Delta = \sum_i a_i \Delta_i$ be an $\mathbb{R}$-Weil divisor on X such that $0 < a_i \leq 1$ for every i and that Δ_i is not contained in the singular locus of X, where Δ_i is a prime divisor on X for every i and $\Delta_i \neq \Delta_j$ for $i \neq j$. Assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The pair (X, Δ) is *semi divisorial log terminal* (*sdlt*, for short) if $a(E, X, \Delta) > -1$ for every exceptional divisor E over X such that (X, Δ) is not a simple normal crossing pair at the generic point of $c_X(E)$, where $c_X(E)$ is the center of E on X.

We note that if (X, Δ) is sdlt and X is irreducible then (X, Δ) is a divisorial log terminal pair (dlt, for short). The following theorem is a direct generalization of [F8, Theorem 4.14] (cf. [F12, Proposition 2.4]). It is an easy application of Lemma 3.3.

Theorem 4.2 (cf. [F12, Theorem 5.2])**.** *Let (X, D) be a semi divisorial log terminal pair. Let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition. We set*

$$Y = \bigcup_{i \in J} X_i \subset X$$

for $J \subset I$. Then Y is Cohen–Macaulay, seminormal, and has only Du Bois singularities. In particular, each irreducible component of X is normal and X itself is Cohen–Macaulay.

We note that an irreducible component of a seminormal scheme need not be seminormal (see [Ko3, Example 10.12]). We also note that an irreducible component of a Cohen–Macaulay scheme need not be Cohen–Macaulay. The author learned the following example from Shunsuke Takagi.

Example 4.3. We set

$$R = \mathbb{C}[x, y, z, w]/(yz - xw, xz^2 - y^2 w).$$

Then $X = \mathrm{Spec} R$ is a reduced reducible two-dimensional Cohen–Macaulay scheme. An irreducible component

$$Y = \mathrm{Spec} R/(y^3 - x^2 z, z^3 - yw^2)$$

of X is not Cohen–Macaulay. It is because

$$R/(y^3 - x^2 z, z^3 - yw^2) \simeq \mathbb{C}[s^4, s^3 t, st^3, t^4].$$

The Cohen–Macaulayness of X is very important for various duality theorems. We use it in the proof of Theorem 5.1 in [FF].

Let us start the proof of Theorem 4.2.

Proof of Theorem 4.2. By [BP, Theorem 1.4], there is a morphism $f : Z \to X$ given by a composite of blowing-ups with smooth centers such that $(Z, f_*^{-1} D + \mathrm{Exc}(f))$ is a simple normal crossing pair and that f is an isomorphism over U, where U is the largest Zariski open subset of X such that $(U, D|_U)$ is a simple normal crossing pair. Then we can write

$$K_Z + D' = f^*(K_X + D) + E,$$

where D' and E are effective and have no common irreducible components. By construction, E is f-exceptional and $\mathrm{Supp}(E + D')$ is a simple normal crossing divisor on Z. Since X is S_2 and simple normal crossing in codimension one, X is seminormal. Then we obtain $f_* \mathcal{O}_Z \simeq \mathcal{O}_X$. Let $Z = \bigcup_{i \in I} Z_i$ be the irreducible decomposition. We consider the short exact sequence

$$0 \to \mathcal{O}_V(-W|_V) \to \mathcal{O}_Z \to \mathcal{O}_W \to 0,$$

where $W = \bigcup_{i \in J} Z_i$ and $V = \bigcup_{i \in I \setminus J} Z_i$. Therefore,

$$0 \to \mathcal{O}_V(\lceil E \rceil - W|_V) \to \mathcal{O}_Z(\lceil E \rceil) \to \mathcal{O}_W(\lceil E \rceil) \to 0$$

is exact. We note that $\lceil E \rceil$ is Cartier (cf. Remark 2.11). By Lemma 3.3, $R^i f_* \mathcal{O}_Z(\lceil E \rceil) = 0$ for every $i > 0$. We note that

$$\lceil E \rceil \sim_{\mathbb{R}, f} K_Z + D' + \{-E\}.$$

Since

$$(\lceil E\rceil - W)|_V \sim_{\mathbb{R},f} K_V + (D' + \{-E\})|_V,$$

$R^i f_*\mathcal{O}_V(\lceil E\rceil - W|_V) = 0$ for every $i > 0$ by Lemma 3.3 again. Therefore, we obtain that

$$0 \to f_*\mathcal{O}_V(\lceil E\rceil - W|_V) \to f_*\mathcal{O}_Z(\lceil E\rceil) \simeq \mathcal{O}_X \to f_*\mathcal{O}_W(\lceil E\rceil) \to 0$$

is exact and that $R^i f_*\mathcal{O}_W(\lceil E\rceil) = 0$ for every $i > 0$. Since $\lceil E\rceil|_W$ is effective and $\mathcal{O}_X \to f_*\mathcal{O}_W(\lceil E\rceil) \to 0$ factors through $\mathcal{O}_Y$, we have $\mathcal{O}_Y \simeq f_*\mathcal{O}_W \simeq f_*\mathcal{O}_W(\lceil E\rceil)$. Therefore, Y is seminormal because so is W. In the derived category of coherent sheaves on Y, the composition

$$\mathcal{O}_Y \to Rf_*\mathcal{O}_W \to Rf_*\mathcal{O}_W(\lceil E\rceil) \simeq \mathcal{O}_Y \tag{1}$$

is a quasi-isomorphism. Therefore, Y has only Du Bois singularities because W is a simple normal crossing variety. On the other hand, $R^i f_*\omega_W = 0$ for every $i > 0$ by Lemma 3.3. By applying Grothendieck duality to (1):

$$\mathcal{O}_Y \to Rf_*\mathcal{O}_W \to \mathcal{O}_Y,$$

we obtain

$$\omega^\bullet_Y \xrightarrow{a} Rf_*\omega^\bullet_W \xrightarrow{b} \omega^\bullet_Y, \tag{2}$$

where $\omega^\bullet_Y$ (resp. $\omega^\bullet_W$) is the dualizing complex of Y (resp. W). Note that $b \circ a$ is a quasi-isomorphism. Thus we have

$$h^i(\omega^\bullet_Y) \subset R^i f_*\omega^\bullet_W = R^{i+d} f_*\omega_W$$

where $d = \dim Y = \dim W$. This implies that $h^i(\omega^\bullet_Y) = 0$ for every $i > -\dim Y$. Thus, Y is Cohen–Macaulay and $\omega^\bullet_Y \simeq \omega_Y[d]$. Q.E.D.

As a byproduct of the proof of Theorem 4.2, we obtain the following useful vanishing theorem. Roughly speaking, Proposition 4.4 says that Y has only *semi-rational* singularities.

Proposition 4.4. *In the notation of the proof of* Theorem 4.2*,* $f_*\mathcal{O}_W \simeq \mathcal{O}_Y$ *and* $R^i f_*\mathcal{O}_W = 0$ *for every* $i > 0$*.*

Proof. By (2) in the proof of Theorem 4.2, we obtain

$$\omega_Y \xrightarrow{\alpha} f_*\omega_W \xrightarrow{\beta} \omega_Y$$

where $\beta \circ \alpha$ is an isomorphism. Since ω_W is locally free and f is an isomorphism over U, $f_*\omega_W$ is a pure sheaf of dimension d. Thus $f_*\omega_W \simeq \omega_Y$ because they are isomorphic over U. Then we obtain $Rf_*\omega^\bullet_W \simeq \omega^\bullet_Y$

in the derived category of coherent sheaves on Y. By Grothendieck duality, $Rf_*\mathcal{O}_W \simeq R\mathcal{H}om(Rf_*\omega^\bullet_W, \omega^\bullet_Y) \simeq \mathcal{O}_Y$ in the derived category of coherent sheaves on Y. Therefore, $f_*\mathcal{O}_W \simeq \mathcal{O}_Y$ and $R^if_*\mathcal{O}_W = 0$ for every $i > 0$. Q.E.D.

As an easy application of Theorem 4.2, we have an adjunction formula for sdlt pairs.

Corollary 4.5 (Adjunction for sdlt pairs). *In the notation of* Theorem 4.2*, we define* D_Y *by*

$$(K_X + D)|_Y = K_Y + D_Y.$$

Then the pair (Y, D_Y) *is semi divisorial log terminal.*

Proof. By Theorem 4.2, Y is Cohen–Macaulay. In particular, Y satisfies Serre's S_2 condition. Then it is easy to see that the pair (Y, D_Y) is semi divisorial log terminal. Q.E.D.

We close this section with an important remark.

Remark 4.6. Let (X, D) be a semi divisorial log terminal pair in the sense of Kollár (see Definition 4.1). Then it is a semi divisorial log terminal pair in the sense of [F1, Definition 1.1]. A key point is that any irreducible component of X is normal (see Theorem 4.2). When the author defined semi divisorial log terminal pairs in [F1, Definition 1.1], the theory of resolution of singularities for *reducible* varieties (cf. [BM] and [BP]) was not available.

§5. Semipositivity theorem

In [F8, Chapter 2], we discuss mixed Hodge structures on cohomology groups with compact support for the proof of Theorem 3.1 (see also [F15]). In [FF], we investigate variations of mixed Hodge structures on cohomology groups with compact support. By the Hodge theoretic semipositivity theorem obtained in [FF, Section 5], we can prove the following theorem as an application of Theorem 3.7.

Theorem 5.1 (Semipositivity theorem). *Let* (X, D) *be a simple normal crossing pair such that* D *is reduced and let* $f : X \to Y$ *be a projective surjective morphism onto a smooth complete algebraic variety* Y*. Assume that every stratum of* (X, D) *is dominant onto* Y*. Let* Σ *be a simple normal crossing divisor on* Y *such that every stratum of* (X, D) *is smooth over* $Y_0 = Y \setminus \Sigma$*. Then* $R^if_*\omega_{X/Y}(D)$ *is locally free for every* i*. We set* $X_0 = f^{-1}(Y_0)$*,* $D_0 = D|_{X_0}$*, and* $d = \dim X - \dim Y$*. We further* *assume that all the local monodromies on* $R^{d-i}(f|_{X_0\setminus D_0})_!\mathbb{Q}_{X_0\setminus D_0}$ *around* Σ *are unipotent. Then we obtain that* $R^if_*\omega_{X/Y}(D)$ *is a semipositive* (*in the sense of Fujita–Kawamata*) *locally free sheaf on* Y*, that is, a* nef *locally free sheaf on* Y*.*

We note that Theorem 5.1 is a generalization of the Fujita–Kawamata semipositivity theorem (cf. [Ka]). We also note that Theorem 5.1 contains the main theorem of [F2]. In [F2], we use variations of mixed Hodge structures on cohomology groups of smooth quasi-projective varieties. However, our formulation in [FF] based on mixed Hodge structures on cohomology groups with compact support is more suitable for reducible varieties than the formulation in [F2] (cf. 1.2). Theorem 3.7 and Theorem 5.1 show that the theory of mixed Hodge structures on cohomology groups with compact support is useful for the study of higher dimensional algebraic varieties. For details, see [F8, Chapter 2], [FF], and [F15].

Finally, we note that in [F14] we prove the projectivity of the moduli spaces of stable varieties as an application of Theorem 5.1 by Kollár's projectivity criterion.

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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujino@math.kyoto-u.ac.jp