

ON VANISHING THEOREMS FOR NON-COMPACT ANALYTIC SPACES

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ABSTRACT. We present several classical vanishing theorems for complex analytic spaces, together with related results that are useful in the minimal model theory for projective morphisms between complex analytic spaces. In particular, we use Nakano's vanishing theorem on weakly 1-complete complex manifolds as a main ingredient.

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1. VANISHING THEOREMS FOR NON-COMPACT ANALYTIC SPACES

We present several classical vanishing theorems for complex analytic spaces, together with related results that are useful in the minimal model theory for projective morphisms between complex analytic spaces. The main purpose of this paper is to provide detailed proofs of results whose proofs are either not available in standard references or difficult to locate in the literature. On the other hand, we omit some arguments that can be readily found in standard references. More concretely, this paper is intended as a guide to the tools required to understand the proof of the relative Kawamata–Viehweg vanishing theorem for complex analytic spaces in [Nay] (see [Nay, Theorem 3.7]).

We begin with Nakano's vanishing theorem on weakly 1-complete complex manifolds, which plays a central role in this paper.

Theorem 1.1 (Nakano's vanishing theorem). *Let X be a weakly 1-complete complex manifold, that is, there exists a smooth plurisubharmonic function Ψ on X such that*

$$X_c := \{x \in X \mid \Psi(x) < c\}$$

is relatively compact for every $c \in \mathbb{R}$. Let \mathcal{E} be a Nakano positive vector bundle on X . Then

$$H^i(X, \omega_X \otimes \mathcal{E}) = 0$$

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for every $i > 0$. In particular, for every $c \in \mathbb{R}$, we have

$$H^i(X_c, \omega_X \otimes \mathcal{E}) = 0$$

for every $i > 0$.

As a special case of Theorem 1.1, we obtain the following.

Corollary 1.2 (Kodaira vanishing theorem for weakly 1-complete complex manifolds). *Let X be a weakly 1-complete complex manifold and let \mathcal{L} be a positive line bundle on X . Then*

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0$$

for every $i > 0$. In particular, for every $c \in \mathbb{R}$, we have

$$H^i(X_c, \omega_X \otimes \mathcal{L}) = 0$$

for every $i > 0$.

If X is compact in Corollary 1.2, then the statement reduces to Kodaira's original vanishing theorem. For various geometric applications, the following corollary of Theorem 1.1 is useful.

Corollary 1.3. *Let $\pi: X \rightarrow Y$ be a proper morphism of complex analytic spaces such that X is smooth. Let \mathcal{E} be a Nakano positive vector bundle on X . Then*

$$R^i \pi_* (\omega_X \otimes \mathcal{E}) = 0$$

for every $i > 0$.

For the minimal model theory for projective morphisms between complex analytic spaces established in [Fn3], we need the following result.

Corollary 1.4 (Relative Kodaira vanishing theorem). *Let $\pi: X \rightarrow Y$ be a projective morphism of complex analytic spaces. Let \mathcal{L} be a π -ample line bundle on X . Then*

$$R^i \pi_* (\omega_X \otimes \mathcal{L}) = 0$$

for every $i > 0$.

Once Corollary 1.4 is established, it is straightforward to prove the relative Kawamata–Viehweg vanishing theorem for projective morphisms between complex varieties. This theorem plays a crucial role in the minimal model theory developed in [Fn3].

Theorem 1.5 (Kawamata–Viehweg vanishing theorem for projective morphisms of complex varieties). *Let X be a smooth complex variety and let $\pi: X \rightarrow Y$ be a projective morphism of complex varieties. Assume that D is an \mathbb{R} -divisor on X such that D is π -nef and π -big, and that $\text{Supp}\{D\}$ is a simple normal crossing divisor on X . Then*

$$R^i \pi_* \mathcal{O}_X(K_X + [D]) = 0$$

for every $i > 0$.

Remark 1.6. In earlier literature related to the minimal model program, vanishing theorems are often formulated for *normal crossing divisors*. From a modern perspective, however, *simple normal crossing divisors* provide a more appropriate framework, and there is little benefit in extending vanishing theorems to the broader class of normal crossing divisors.

In [Nay], Nakayama formulates the vanishing theorem (see, for example, [Nay, Theorem 3.7]) for *proper* morphisms rather than *projective* morphisms. From the viewpoint of the

minimal model program, however, it is more natural to restrict attention to projective morphisms. This also avoids unnecessary technical complications.

Furthermore, many classical references, including [Nay], treat only \mathbb{Q} -divisors. However, extending results such as Theorem 1.5 to \mathbb{R} -divisors is not difficult. Ultimately, it suffices to observe that by a slight perturbation of the coefficients of a π -ample \mathbb{R} -divisor, one can approximate it by a π -ample \mathbb{Q} -divisor.

One of the main purposes of this paper is to present a vanishing theorem for weakly 1-complete complex analytic spaces (see Theorem 1.7), originally due to Fujiki and Hironaka, in a more accessible form. Although the results in Theorems 1.7, 1.9, and 1.10, as well as Corollary 1.12, are not needed for the proof of Theorem 1.5, and hence are not required for the minimal model theory discussed in [Fn3], they are of independent interest. We also note that these results play an important role in Nakayama's original treatment of vanishing theorems in the complex analytic setting (see [Nay, §3]). For this reason, we include them here.

Theorem 1.7 (Fujiki's vanishing theorem, see [Fk, Theorem N']). *Let X be a weakly 1-complete complex analytic space and let \mathcal{S} be a coherent sheaf on X . Let \mathcal{L} be a positive line bundle on X . Then, for every $c \in \mathbb{R}$, there exists a positive integer m_0 such that*

$$H^i(X_c, \mathcal{S} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{M}) = 0$$

for all $i \geq 1$, $m \geq m_0$, and for every semipositive line bundle \mathcal{M} on X .

We give an important remark on Theorem 1.7.

Remark 1.8. Hironaka contributed to both the formulation and the proof of Theorem 1.7. In [Fk], Fujiki wrote:

The author learned the formulation of this theorem and the idea of its proof from Prof. Hironaka.

For Hironaka's contribution, see also [Na1, §3, Comments].

As an immediate application of Theorem 1.7, we obtain the following.

Theorem 1.9 ([K, Theorem 2.1]). *Let X be a weakly 1-complete complex analytic space and let \mathcal{S} be a coherent sheaf on X . Let \mathcal{L} be a positive line bundle on X . Then, for every $c \in \mathbb{R}$, there exists a positive integer k_0 such that, for every $k \geq k_0$, the sheaf $\mathcal{S} \otimes \mathcal{L}^{\otimes k}$ is generated by finitely many global sections over some open neighborhood of \overline{X}_c .*

Hence, we obtain the following embedding theorem, which generalizes Kodaira's embedding theorem. When X is compact, Theorem 1.10 coincides with Grauert's generalization of Kodaira's embedding theorem (see [G] and [No, Theorem 8.5.8]).

Theorem 1.10 (Embedding theorem, Hironaka). *Let X be a weakly 1-complete complex analytic space and let \mathcal{L} be a positive line bundle on X . Fix $c \in \mathbb{R}$ and consider X_c . Then there exists a positive integer m_0 such that, for every $m \geq m_0$, there exist finitely many sections $\varphi_0, \dots, \varphi_N \in H^0(X_c, \mathcal{L}^{\otimes m})$ that define an embedding*

$$\Phi: X_c \hookrightarrow \mathbb{P}^N$$

as a locally closed analytic subspace such that

$$\mathcal{L}^{\otimes m} \simeq \Phi^* \mathcal{O}_{\mathbb{P}^N}(1).$$

Note that in Theorem 1.10 we cannot replace X_c with X (see [O]). For further details on the global embedding problem of weakly 1-complete complex analytic spaces, see [T] and [W].

Remark 1.11. In [Na2, §21], Nakano wrote that Theorem 1.10 was first obtained by Hironaka. For Hironaka's contributions to the theory of weakly 1-complete complex analytic spaces, see also Remark 1.8.

As an easy consequence of Theorem 1.10, we obtain a metric characterization of ample line bundles on compact complex analytic spaces. This is a generalization of Kodaira's result.

Corollary 1.12. *Let X be a compact complex analytic space and let \mathcal{L} be a line bundle on X . Then \mathcal{L} is positive if and only if it is ample.*

We close this section with Fujita's vanishing theorem. We will see that it is also a consequence of Theorem 1.7.

Theorem 1.13 (Fujita's vanishing theorem). *Let X be a projective scheme over \mathbb{C} and let \mathcal{L} be an ample line bundle on X . Let \mathcal{F} be a coherent sheaf on X . Then there exists a positive integer $m(\mathcal{F}, \mathcal{L})$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{M}) = 0$$

for all $i > 0$, $m \geq m(\mathcal{F}, \mathcal{L})$, and for every nef line bundle \mathcal{M} on X .

For details on Fujita's vanishing theorem, see, for example, [Fn2, Section 3.8].

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2. BASIC DEFINITIONS

We are mainly interested in singular complex analytic spaces. In this section, we recall some basic definitions needed to understand the results in Section 1. For details, see [D2, Chapter IX. §2].

Definition 2.1 (Strictly plurisubharmonic and plurisubharmonic functions). Let V be an analytic subspace of an open subset $U \subset \mathbb{C}^n$. A smooth function φ on V is, by definition, the restriction to V of a smooth function $\tilde{\varphi}$ defined on an open neighborhood U' of V in U .

An \mathbb{R} -valued smooth function φ on V is said to be *strictly plurisubharmonic* (resp. *plurisubharmonic*) if there exists such a smooth function $\tilde{\varphi}$ that is strictly plurisubharmonic (resp. plurisubharmonic).

Recall that a smooth function $\tilde{\varphi}$ is said to be strictly plurisubharmonic (resp. plurisubharmonic) if its complex Hessian is positive definite (resp. semipositive).

Remark 2.2. In this paper, (strictly) plurisubharmonic functions are always assumed to be smooth.

We globalize Definition 2.1 as follows.

Definition 2.3 (Strictly plurisubharmonic and plurisubharmonic functions on complex analytic spaces). Let X be a complex analytic space. A function v on X is said to

be *smooth* (resp. strictly plurisubharmonic, plurisubharmonic) if there exists an open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ together with isomorphisms

$$\varphi_\lambda: U_\lambda \xrightarrow{\sim} V_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{N_\lambda}$$

for each $\lambda \in \Lambda$, where V_λ is a closed analytic subspace of an open set $\Omega_\lambda \subset \mathbb{C}^{N_\lambda}$, such that for each λ there exists a smooth (resp. strictly plurisubharmonic, plurisubharmonic) function \tilde{v}_λ on Ω_λ satisfying

$$v|_{U_\lambda} = \tilde{v}_\lambda|_{V_\lambda} \circ \varphi_\lambda.$$

The following lemma shows that Definition 2.3 is independent of the choice of open covering.

Lemma 2.4 (cf. [D2, Chapter IX. (2.6) Lemma]). *Let*

$$\varphi_\lambda: U_\lambda \xrightarrow{\sim} V_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{N_\lambda} \quad \text{and} \quad \varphi_{\lambda'}: U_{\lambda'} \xrightarrow{\sim} V_{\lambda'} \subset \Omega_{\lambda'} \subset \mathbb{C}^{N_{\lambda'}}$$

be two charts of X . Let \tilde{v}_λ be a smooth (resp. strictly plurisubharmonic, plurisubharmonic) function on Ω_λ such that

$$v|_{U_\lambda} = \tilde{v}_\lambda|_{V_\lambda} \circ \varphi_\lambda.$$

Then, for every $x \in U_\lambda \cap U_{\lambda'}$, there exists a smooth (resp. strictly plurisubharmonic, plurisubharmonic) function $\tilde{v}_{\lambda'}$ defined on a neighborhood $W_{\lambda'} \subset \Omega_{\lambda'}$ of $\varphi_{\lambda'}(x)$ such that

$$\tilde{v}_{\lambda'}|_{V_{\lambda'} \cap W_{\lambda'}} \circ \varphi_{\lambda'} = v|_{U_{\lambda'} \cap \varphi_{\lambda'}^{-1}(W_{\lambda'})}.$$

Proof of Lemma 2.4. We consider the transition maps

$$\varphi_{\lambda'} \circ \varphi_\lambda^{-1}: \varphi_\lambda(U_\lambda \cap U_{\lambda'}) \xrightarrow{\sim} \varphi_{\lambda'}(U_\lambda \cap U_{\lambda'})$$

and

$$\varphi_\lambda \circ \varphi_{\lambda'}^{-1}: \varphi_{\lambda'}(U_\lambda \cap U_{\lambda'}) \xrightarrow{\sim} \varphi_\lambda(U_\lambda \cap U_{\lambda'}).$$

These are restrictions of holomorphic maps

$$\psi: W_\lambda \rightarrow \Omega_{\lambda'}, \quad \psi': W_{\lambda'} \rightarrow \Omega_\lambda,$$

defined on neighborhoods $W_\lambda \ni \varphi_\lambda(x)$ and $W_{\lambda'} \ni \varphi_{\lambda'}(x)$, respectively. By shrinking $W_{\lambda'}$ if necessary, we may assume that $\psi'(W_{\lambda'}) \subset W_\lambda$.

We first treat the case where \tilde{v}_λ is strictly plurisubharmonic. Consider the function

$$\Phi(z, z') := \tilde{v}_\lambda(z) + |z' - \psi(z)|^2$$

on $W_\lambda \times \Omega_{\lambda'}$. Since $(z, z') \mapsto (z, z' - \psi(z))$ is a holomorphic automorphism of $W_\lambda \times \mathbb{C}^{N_{\lambda'}}$, it follows that Φ is strictly plurisubharmonic.

Now embed $W_{\lambda'}$ into $W_\lambda \times \Omega_{\lambda'}$ via

$$z' \mapsto (\psi'(z'), z').$$

Since the restriction of a strictly plurisubharmonic function to a complex submanifold is again strictly plurisubharmonic, the function

$$\tilde{v}_{\lambda'}(z') := \Phi(\psi'(z'), z') = \tilde{v}_\lambda(\psi'(z')) + |z' - \psi \circ \psi'(z')|^2$$

is strictly plurisubharmonic on $W_{\lambda'}$.

Moreover, since

$$\psi \circ \varphi_\lambda = \varphi_{\lambda'} \quad \text{and} \quad \psi' \circ \varphi_{\lambda'} = \varphi_\lambda$$

on $U_\lambda \cap U_{\lambda'}$, we obtain

$$\tilde{v}_{\lambda'} \circ \varphi_{\lambda'} = \tilde{v}_\lambda \circ \varphi_\lambda = v$$

on $U_{\lambda'} \cap \varphi_{\lambda'}^{-1}(W_{\lambda'})$.

This proves the claim when \tilde{v}_λ is strictly plurisubharmonic. The cases where \tilde{v}_λ is smooth or plurisubharmonic are treated in the same way. \square

Let us recall the definition of weakly 1-complete complex analytic spaces.

Definition 2.5 (Weakly 1-complete complex analytic spaces). Let X be a complex analytic space. If there exists an \mathbb{R} -valued smooth plurisubharmonic function Ψ on X such that $X_c := \{x \in X \mid \Psi(x) < c\}$ is relatively compact in X for every $c \in \mathbb{R}$, then X is said to be *weakly 1-complete*. We call Ψ an *exhaustion function* of X .

Of course, if X is smooth in Definition 2.5, then X is called a *weakly 1-complete complex manifold*. We note that every compact analytic space is weakly 1-complete, since constant functions provide smooth plurisubharmonic functions. We also note that every closed analytic subspace of a weakly 1-complete complex analytic space is weakly 1-complete.

Remark 2.6 (Pseudoconvex complex manifolds). In some literature (see, for example, [D1, (8.6) Definition] and [D2, Chapter VIII. (5.1) Definition]), a weakly 1-complete complex manifold is called a *weakly pseudoconvex complex manifold*.

We will freely use the following remark in Section 4 without explicit mention.

Remark 2.7. Let X be a weakly 1-complete complex analytic space with an exhaustion function Ψ . As in Definition 2.5, we put

$$X_c := \{x \in X \mid \Psi(x) < c\}.$$

We set $\Psi_c := (c - \Psi)^{-1}$. Then X_c is a weakly 1-complete complex analytic space and Ψ_c is an exhaustion function on X_c . In this case, for every $c' < c$, we have

$$X_{c'} = \{x \in X \mid \Psi(x) < c'\} = \{x \in X_c \mid \Psi_c(x) < (c - c')^{-1}\}.$$

Let $f: Y \rightarrow X$ be a proper morphism of complex analytic spaces. Then $f^*\Psi$ is a smooth plurisubharmonic function on Y . It is obvious that

$$Y_c := \{y \in Y \mid f^*\Psi(y) < c\} = f^{-1}(X_c)$$

holds for every $c \in \mathbb{R}$. In particular, Y_c is relatively compact in Y since f is proper. This means that Y is a weakly 1-complete complex analytic space with exhaustion function $f^*\Psi$.

Let us recall the definition of positive and semipositive line bundles on complex analytic spaces. This notion is indispensable for understanding Fujiki's vanishing theorem (see Theorem 1.7).

Definition 2.8 (Positive and semipositive line bundles on complex analytic spaces). Let X be a complex analytic space and let \mathcal{L} be a line bundle on X . Assume that \mathcal{L} is defined by a system of transition functions $\{f_{\alpha\beta}\}$ with respect to some open covering $\mathcal{U} = \{U_\alpha\}$ of X . In this situation, a metric on \mathcal{L} is given by a system of $\mathbb{R}_{>0}$ -valued functions $h = \{h_\alpha\}$, where each h_α is a smooth function defined on U_α , such that $h_\alpha = \frac{1}{|f_{\alpha\beta}|^2} h_\beta$ holds on $U_\alpha \cap U_\beta$. If $-\log h_\alpha$ is strictly plurisubharmonic (resp. plurisubharmonic) on U_α for every α , then \mathcal{L} is said to be *positive* (resp. *semipositive*).

When X is a complex manifold, the notions of positive and semipositive line bundles in Definition 2.8 coincides with the usual ones. In this short paper, we do not define Nakano positive vector bundles explicitly, since the definition is somewhat complicated. The following lemma, which easily follows from the definition of Nakano positive vector bundles, is sufficient for most of our applications.

Lemma 2.9. *Let X be a complex manifold, let \mathcal{E} be a vector bundle on X , and let \mathcal{L} be a positive line bundle on X . Let U be a relatively compact open subset of X . Then there exists a positive integer m_0 such that $\mathcal{E} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{M}$ is Nakano positive on U for every $m \geq m_0$ and for every semipositive line bundle \mathcal{M} on X .*

Proof of Lemma 2.9. Once we understand the definition of Nakano positive vector bundles (see, for example, [D1, (6.9) Definition] and [D2, Chapter VII. (6.3) Definition]), there is no difficulty in verifying this statement. \square

3. LEMMAS ON POSITIVITY OF LINE BUNDLES

For the reader's convenience, we recall the definitions of relatively ample line bundles and projective morphisms in the complex analytic setting.

Definition 3.1 (Relatively ample line bundles and projective morphisms, cf. [BS, Chapter IV. §1. Preliminaries (d)]). Let $\pi: X \rightarrow Y$ be a proper morphism of complex analytic spaces, and let \mathcal{L} be a line bundle on X . We say that \mathcal{L} is π -very ample (or relatively very ample over Y) if it is π -free, i.e. the natural morphism

$$\pi^* \pi_* \mathcal{L} \longrightarrow \mathcal{L}$$

is surjective, and the induced morphism

$$X \longrightarrow \mathbb{P}_Y(\pi_* \mathcal{L})$$

over Y is a closed embedding.

A line bundle \mathcal{L} on X is called π -ample (or ample over Y) if for every point $y \in Y$, there exist an open neighborhood U of y and a positive integer m such that $\mathcal{L}^{\otimes m}|_{\pi^{-1}(U)}$ is relatively very ample over U .

We say that π is *projective* if there exists a π -ample line bundle on X .

Remark 3.2. It is well known that \mathcal{L} is π -ample if and only if $\mathcal{L}|_{\pi^{-1}(y)}$ is ample in the usual sense for every $y \in Y$. We also note that $\mathcal{L}|_{\pi^{-1}(y)}$ is ample if and only if it is positive by Corollary 1.12.

To prove Corollary 1.4 and Theorem 1.7, we prepare the following fundamental lemma.

Lemma 3.3 (cf. [Fk, Lemma 2]). *Let $\pi: Y \rightarrow X$ be a projective morphism between complex analytic spaces, let \mathcal{L} be a positive line bundle on X , and let \mathcal{N} be a π -ample line bundle on Y . Let U be a relatively compact open subset of X . Then there exists a positive integer m_0 such that $\mathcal{N} \otimes \pi^* \mathcal{L}^{\otimes m}$ is positive on $\pi^{-1}(U)$ for every $m \geq m_0$.*

For the sake of completeness, we give a detailed proof of Lemma 3.3.

Proof of Lemma 3.3. Let $h_{\mathcal{L}}$ be a positive metric on \mathcal{L} in the sense of Definition 2.8. We divide the proof into several steps.

Step 1. Choose a positive integer k such that $\mathcal{N}^{\otimes k}$ is very ample over an open neighborhood of \bar{U} . Note that $\mathcal{N} \otimes \pi^* \mathcal{L}^{\otimes m}$ admits a positive metric if and only if $(\mathcal{N} \otimes \pi^* \mathcal{L}^{\otimes m})^{\otimes k}$ does. Hence, replacing \mathcal{N} with $\mathcal{N}^{\otimes k}$ and shrinking X around \bar{U} if necessary, we may assume that \mathcal{N} is very ample over X .

Step 2. Choose a finite open cover $\{U_{\lambda}\}_{\lambda \in \Lambda}$ of a neighborhood of \bar{U} such that:

- (a) each U_{λ} is a relatively compact Stein open subset of X ,
- (b) each U_{λ} is realized as a closed analytic subspace of an open subset $V_{\lambda} \subset \mathbb{C}^n$,
- (c) \mathcal{L} can be trivialized on U_{λ} for every $\lambda \in \Lambda$,

- (d) $h_{\mathcal{L}}$ is given by a system of smooth $\mathbb{R}_{>0}$ -valued functions $\{h_{\mathcal{L},\lambda}\}_{\lambda \in \Lambda}$ such that $-\log h_{\mathcal{L},\lambda}$ is the restriction of a strictly plurisubharmonic function on V_λ for every $\lambda \in \Lambda$,
- (e) there exists an embedding

$$\begin{array}{ccccccc}
 \pi^{-1}(U_\lambda) & \hookrightarrow & U_\lambda \times \mathbb{P}^N & \hookrightarrow & V_\lambda \times \mathbb{P}^N & \xrightarrow{p_2} & \mathbb{P}^N \\
 & \searrow & \nearrow p_1 & & \nearrow p_1 & & \\
 & & U_\lambda & \hookrightarrow & V_\lambda & &
 \end{array}$$

such that

$$\mathcal{N}|_{\pi^{-1}(U_\lambda)} \simeq p_2^* \mathcal{O}_{\mathbb{P}^N}(1)|_{\pi^{-1}(U_\lambda)},$$

where p_1 and p_2 denote the projections.

Step 3. Let h_{FS} be the standard smooth Hermitian metric on $\mathcal{O}_{\mathbb{P}^N}(1)$ whose curvature form is the Fubini–Study form. Set

$$h_\lambda := p_2^* h_{\text{FS}} \otimes \pi^* h_{\mathcal{L}}.$$

Then h_λ defines a positive metric on $(\mathcal{N} \otimes \pi^* \mathcal{L})|_{\pi^{-1}(U_\lambda)}$.

Step 4. Let $\{\theta_\lambda\}_{\lambda \in \Lambda}$ be a partition of unity subordinate to $\{U_\lambda\}_{\lambda \in \Lambda}$ such that $0 \leq \theta_\lambda \leq 1$, $\text{supp } \theta_\lambda \subset U_\lambda$, and $\sum_{\lambda \in \Lambda} \theta_\lambda \equiv 1$ on a neighborhood of \bar{U} . Consider the metric

$$\sum_{\lambda \in \Lambda} \pi^* \theta_\lambda \cdot h_\lambda$$

on $\mathcal{N} \otimes \pi^* \mathcal{L}$ over a neighborhood of \bar{U} . We take a point $P \in Y$ such that

$$\pi(P) \in \text{supp} \left(\prod_{\lambda \in \Lambda} (1 - \theta_\lambda) \right) \cap \bar{U}.$$

We can choose a sufficiently small open neighborhood U° of $\pi(P)$ in X such that

$$\pi(P) \in U^\circ \subset V^\circ \subset \mathbb{C}^n,$$

that is, U° is realized as a closed analytic subspace of an open subset $V^\circ \subset \mathbb{C}^n$, and each $\theta_\lambda|_{U^\circ}$ is given by the restriction of a smooth function on V° (see Lemma 2.4).

Moreover, we may assume that there exists a small open subset W of \mathbb{P}^N such that

$$P \in U^\circ \times W \subset V^\circ \times W \subset V^\circ \times \mathbb{P}^N,$$

and that $-\log h_\lambda$ on $\pi^{-1}(U^\circ) \cap (U^\circ \times W)$ is given by the restriction of a strictly plurisubharmonic function on $V^\circ \times W$ for every $\lambda \in \Lambda$ (see Lemma 2.4). We may also assume that $-\log h_{\mathcal{L}}$ is given by the restriction of a smooth strictly plurisubharmonic function on V° .

We consider the complex Hessian of

$$-\log \left(\sum_{\lambda \in \Lambda} \pi^* \theta_\lambda \cdot h_\lambda \right)$$

on $W \times V^\circ$ and that of

$$-\log h_{\mathcal{L}}$$

on V° . By Lemma 3.4 below, there exists an integer $m(P)$ such that

$$\left(\sum_{\lambda \in \Lambda} \pi^* \theta_\lambda \cdot h_\lambda \right) \otimes (\pi^* h_{\mathcal{L}})^{m-1}$$

defines a positive metric on $\mathcal{N} \otimes \pi^* \mathcal{L}^{\otimes m}$ in some open neighborhood of P for every $m \geq m(P)$.

Since $\text{supp} \left(\prod_{\lambda \in \Lambda} (1 - \theta_\lambda) \right) \cap \bar{U}$ is compact and π is proper, we can choose a sufficiently large integer m_0 such that

$$\left(\sum_{\lambda \in \Lambda} \pi^* \theta_\lambda \cdot h_\lambda \right) \otimes (\pi^* h_{\mathcal{L}})^{m-1}$$

defines a positive metric of $\mathcal{N} \otimes \pi^* \mathcal{L}^{\otimes m}$ on $\pi^{-1}(U)$ for every $m \geq m_0$.

This completes the proof of Lemma 3.3. \square

We have already used the following elementary lemma from linear algebra in the proof of Lemma 3.3.

Lemma 3.4. *Let*

$$H = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

be a Hermitian matrix written in block form, where A is positive definite. Let B' be a positive-definite Hermitian matrix of the same size as B . Then there exists m_0 such that

$$\begin{pmatrix} A & C \\ C^* & B + mB' \end{pmatrix}$$

is positive definite for every $m \geq m_0$.

We include a proof of Lemma 3.4 for the sake of completeness.

Proof of Lemma 3.4. Let

$$H_m = \begin{pmatrix} A & C \\ C^* & B + mB' \end{pmatrix}$$

and let I denote the identity matrix of appropriate size. Since A is positive definite, it is invertible. Thus, we can write

$$H_m = \begin{pmatrix} I & 0 \\ C^* A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (B + mB') - C^* A^{-1} C \end{pmatrix} \begin{pmatrix} I & A^{-1} C \\ 0 & I \end{pmatrix}.$$

Therefore, H_m is positive definite if and only if the block diagonal matrix

$$\begin{pmatrix} A & 0 \\ 0 & S_m \end{pmatrix}, \quad S_m := (B + mB') - C^* A^{-1} C,$$

is positive definite. Since A is positive definite, this is equivalent to S_m being positive definite.

Since B' is positive definite by assumption, there exists a constant $c > 0$ such that $B' \geq cI$. Hence,

$$S_m = B - C^* A^{-1} C + mB' \geq B - C^* A^{-1} C + mcI,$$

which is positive definite for all sufficiently large m . Therefore, there exists m_0 such that S_m is positive definite for every $m \geq m_0$. Consequently, H_m is positive definite for every $m \geq m_0$. \square

Lemma 3.3 admits several important corollaries. Among them, Corollary 3.5 appears to be necessary for the proof of [Nay, Theorem 3.4], although this is not stated explicitly there.

Corollary 3.5. *Let $\pi: Y \rightarrow X$ be a finite morphism between complex analytic spaces, let \mathcal{L} be a positive line bundle on X , and let U be a relatively compact open subset of X . Then $\pi^*\mathcal{L}$ is positive on $\pi^{-1}(U)$.*

Proof of Corollary 3.5. We note that \mathcal{O}_Y is a π -ample line bundle on Y since π is finite. Thus, by Lemma 3.3, $\pi^*\mathcal{L}$ is positive on $\pi^{-1}(U)$. \square

We give an important remark related to Corollary 3.5.

Remark 3.6. Let $\pi: Y \rightarrow X$ be a finite morphism of complex analytic spaces and let φ be a smooth strictly plurisubharmonic function on X . Then $\pi^*\varphi$ is smooth and plurisubharmonic but is not always strictly plurisubharmonic.

Corollary 3.7 is a basic statement that should be well known. However, since it does not seem to be stated explicitly in the literature, we include it here for completeness.

Corollary 3.7. *Let $\pi: X \rightarrow Y$ be a projective morphism of complex analytic spaces and let \mathcal{L} be a π -ample line bundle on X . Then, for any $P \in Y$, there exists a relatively compact Stein open neighborhood U_P of P such that \mathcal{L} is positive on $\pi^{-1}(U_P)$.*

Proof of Corollary 3.7. Without loss of generality, we may assume that Y is realized as a closed analytic subspace of an open subset $\Omega \subset \mathbb{C}^n$ by shrinking Y around P suitably. Then \mathcal{O}_Y is a positive line bundle on Y . Let U_P be a relatively compact open neighborhood of P in Y . Then \mathcal{L} is positive on $\pi^{-1}(U_P)$ by Lemma 3.3. \square

We close this section with an important remark on the definition of ampleness in the complex analytic setting.

Remark 3.8. Let $f: X \rightarrow S$ be a proper morphism between complex analytic spaces. Nakayama adopts his own definition of an f -ample line bundle (see [Nay, Definition 1.1]). However, by Corollary 1.12 and Corollary 3.7, one sees that Nakayama's definition coincides with the usual one (see Definition 3.1).

4. PROOFS

In this section, we prove the theorems and corollaries stated in Section 1. We begin with the proof of Nakano's vanishing theorem: Theorem 1.1.

Proof of Theorem 1.1. This result is well known as Nakano's vanishing theorem. For details, see, for example, [D1, (9.1) Nakano vanishing theorem] and [D2, Chapter VIII. (5.5) Theorem]. Let Ψ be an exhaustion function of X and set $X_c = \{x \in X \mid \Psi(x) < c\}$ and $\Psi_c = (c - \Psi)^{-1}$. Then we easily see that X_c is a weakly 1-complete complex manifold with exhaustion function Ψ_c (see Remark 2.7). Hence we obtain the desired vanishing theorem on X_c . \square

Remark 4.1. Kodaira established the foundations of the theory of compact complex manifolds. Nakano extended Kodaira's results to non-compact complex manifolds (see, for example, [Na2]). On the other hand, Demailly generalized Hörmander's techniques (see, for example, [D1] and [D2]). Hence, Nakano's approach is more differential-geometric in nature than Demailly's. The author feels that the treatment of Theorem 1.1 in [D1] and [D2] is simpler than that in [Na2].

Corollaries 1.2 and 1.3 follow easily from Theorem 1.1.

Proof of Corollary 1.2. This is a special case of Theorem 1.1. We note that a line bundle \mathcal{L} is positive if and only if it is Nakano positive. \square

Proof of Corollary 1.3. Let $P \in Y$ be any point. Then we can choose an arbitrarily small Stein open neighborhood U of P in Y and a strictly plurisubharmonic exhaustion function Ψ on U . Then $\pi^{-1}(U)$ is a weakly 1-complete complex manifold with exhaustion function $\pi^*\Psi$. By Theorem 1.1, we have

$$H^i(\pi^{-1}(U), \omega_X \otimes \mathcal{E}) = 0 \quad \text{for every } i > 0.$$

This implies that

$$R^i\pi_*(\omega_X \otimes \mathcal{E}) = 0 \quad \text{for every } i > 0.$$

This is what we wanted to show. \square

Let us prove Corollary 1.4.

Proof of Corollary 1.4. We use the same notation as in the proof of Corollary 1.3. On U , $e^{-\Psi}$ defines a positive metric on the trivial line bundle \mathcal{O}_U . Take $c \in \mathbb{R}$ such that $P \in U_c$. By Lemma 3.3, the line bundle \mathcal{L} is positive on $\pi^{-1}(U_c)$. Hence we have

$$H^i(\pi^{-1}(U_c), \omega_X \otimes \mathcal{L}) = 0$$

for every $i > 0$ by Corollary 1.2. This implies

$$R^i\pi_*(\omega_X \otimes \mathcal{L}) = 0$$

for every $i > 0$. \square

We can prove Theorem 1.5 as an application of Corollary 1.4. The reader can find an alternative approach to Theorem 1.5 in [Fn1].

Proof of Theorem 1.5. Let $P \in Y$ be an arbitrary point. It suffices to prove the desired vanishing in a neighborhood of P . Therefore, we may replace Y with a sufficiently small open neighborhood of P without further mention. By a standard reduction argument (see, for example, Step 2 in the proof of [Fn2, Theorem 3.2.1]), we may assume that D is π -ample. By a small perturbation of the coefficients of D , we may further assume that D is a \mathbb{Q} -divisor. Let H be a π -ample Cartier divisor on X . Let Δ be a given reduced simple normal crossing divisor on X , and let m_Δ be a sufficiently large and divisible integer. Then, after possibly shrinking Y around P , we can find a reduced divisor Δ' on X such that

$$\mathcal{O}_X(m_\Delta H) \simeq \mathcal{O}_X(\Delta + \Delta'),$$

and $\Delta + \Delta'$ is a simple normal crossing divisor on X , where Δ' is obtained by taking a general member of $|m_\Delta H - \Delta|$. Hence we can use Kawamata's covering argument. By applying Kawamata's covering trick, we can reduce the problem to the case where D is a π -ample Cartier divisor on X . Therefore, the desired vanishing follows from Corollary 1.4. This completes the proof of Theorem 1.5. \square

From now on, we prove Theorem 1.7. We begin with the following fundamental lemma.

Lemma 4.2. *Let X be a complex variety and let \mathcal{S} be a coherent sheaf on X . Let \mathcal{L} be a positive line bundle on X . Set*

$$T := T_1 \cup T_2,$$

where T_1 is the singular locus of X and

$$T_2 := \{x \in X \mid \mathcal{S} \text{ is not locally free at } x\}.$$

Let U be a relatively compact open subset of X . Then, after shrinking X around \bar{U} suitably, we can construct a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ & \searrow f & \downarrow h \\ & & X \end{array}$$

satisfying the following properties:

- (i) $h: Z \rightarrow X$ is the normalization,
- (ii) g is a finite composite of blow-ups,
- (iii) Y is smooth,
- (iv) f is an isomorphism over $X \setminus T$,
- (v) $f^*\mathcal{S}/T(f^*\mathcal{S})$ is locally free, where $T(f^*\mathcal{S})$ is the torsion part of $f^*\mathcal{S}$, and
- (vi) there exists an effective Cartier divisor E on Y such that $\mathcal{O}_Y(-E)$ is f -ample, $\mathcal{O}_Y(-E) \otimes f^*\mathcal{L}^{\otimes a}$ is positive for some integer $a > 0$, and $g(E) = h^{-1}(T)$ holds set-theoretically.

Lemma 4.2 will play a crucial role in the proof of Theorem 1.7.

Proof of Lemma 4.2. Let $h: Z \rightarrow X$ be the normalization. By [R, 3.5. Theorem], there exists a proper bimeromorphic morphism $\rho: Z' \rightarrow X$ such that ρ is an isomorphism over $X \setminus T$ and

$$\rho^*\mathcal{S}/T(\rho^*\mathcal{S})$$

is locally free, where $T(\rho^*\mathcal{S})$ is the torsion part of $\rho^*\mathcal{S}$.

Let $g_1: Z_1 \rightarrow Z$ be the blow-up along the ideal sheaf \mathcal{J} , where \mathcal{J} is the defining ideal sheaf of $h^{-1}(T)$ on Z . Let Γ be the graph of the rational map $Z' \dashrightarrow Z_1$.

By applying the flattening theorem (see [Hi, Corollary 1 and Definition 4.4.3]) to $\Gamma \rightarrow Z_1$ and using the desingularization theorem (see, for example, [BM, Theorem 13.3]), after shrinking X around \bar{U} suitably, we obtain a finite sequence of blow-ups

$$Z \leftarrow Z_1 \leftarrow Z_2 \leftarrow \cdots \leftarrow Z_m =: Y$$

such that Y is smooth, $f: Y \rightarrow X$ factors through Z' , and $g: Y \rightarrow Z$ is an isomorphism over $Z \setminus h^{-1}(T)$.

Since $g: Y \rightarrow Z$ is a finite composite of blow-ups, we can construct an effective Cartier divisor E on Y such that $\mathcal{O}_Y(-E)$ is g -ample and $g(E) = h^{-1}(T)$ set-theoretically, after shrinking X suitably. Since h is finite, $\mathcal{O}_Y(-E)$ is f -ample.

Therefore, by Lemma 3.3, we can choose a positive integer a such that

$$\mathcal{O}_Y(-E) \otimes f^*\mathcal{L}^{\otimes a}$$

is positive after shrinking X suitably again.

By construction, $f: Y \rightarrow X$ factors through Z' . Hence, we can check that

$$f^*\mathcal{S}/T(f^*\mathcal{S})$$

is locally free.

Thus, the morphisms $Y \rightarrow Z \rightarrow X$ satisfy all the desired properties. \square

For the proof of Theorem 1.7, we prepare the following two easy lemmas.

Lemma 4.3. *Let*

$$(4.1) \quad 0 \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_2 \rightarrow \mathcal{S}_3 \rightarrow 0$$

be a short exact sequence of coherent sheaves on X . If Theorem 1.7 holds for \mathcal{S}_1 and \mathcal{S}_3 (resp. \mathcal{S}_1 and \mathcal{S}_2), then it also holds for \mathcal{S}_2 (resp. \mathcal{S}_3).

Proof of Lemma 4.3. This follows immediately. It is sufficient to consider the long exact sequence in cohomology induced by (4.1). \square

Lemma 4.4. *Let*

$$(4.2) \quad 0 \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_2 \rightarrow \mathcal{S}_3 \rightarrow \mathcal{S}_4 \rightarrow 0$$

be an exact sequence of coherent sheaves on X . If Theorem 1.7 holds for \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_4 , then it also holds for \mathcal{S}_3 .

Proof of Lemma 4.4. We split (4.2) into the following two short exact sequences:

$$(4.3) \quad 0 \rightarrow \mathcal{S}_1 \rightarrow \mathcal{S}_2 \rightarrow \mathcal{S}' \rightarrow 0$$

and

$$(4.4) \quad 0 \rightarrow \mathcal{S}' \rightarrow \mathcal{S}_3 \rightarrow \mathcal{S}_4 \rightarrow 0.$$

By assumption and (4.3), Theorem 1.7 holds for \mathcal{S}' (see Lemma 4.3). Then, by Lemma 4.3 and (4.4), we conclude that Theorem 1.7 holds for \mathcal{S}_3 . \square

Lemma 4.5. *Let X be a weakly 1-complete complex manifold and let \mathcal{A} be a positive line bundle on X . Then, for every $c \in \mathbb{R}$, there exists a positive integer m such that*

$$H^0(X_c, \mathcal{A}^{\otimes m}) \neq 0.$$

Proof of Lemma 4.5. Fix a point $P \in X_c$. Let $f: Y \rightarrow X$ be the blow-up at P , and let $E \simeq \mathbb{P}^{n-1}$ be the exceptional divisor, where $n = \dim X$. Then we have

$$\omega_Y \simeq f^* \omega_X \otimes \mathcal{O}_Y((n-1)E).$$

Note that Y is again a weakly 1-complete complex manifold. We can take a positive integer m such that

$$\omega_Y^{-1} \otimes f^* \mathcal{A}^{\otimes m} \otimes \mathcal{O}_Y(-E) \simeq f^*(\mathcal{A}^{\otimes m} \otimes \omega_X^{-1}) \otimes \mathcal{O}_Y(-nE)$$

is positive on Y_c (see Lemmas 2.9 and 3.3). Hence, by Corollary 1.2, we obtain

$$H^i(Y_c, f^* \mathcal{A}^{\otimes m} \otimes \mathcal{O}_Y(-E)) = 0 \quad \text{for every } i > 0.$$

We note that $f_* \mathcal{O}_Y(-E) \simeq \mathfrak{m}_P$ holds, where \mathfrak{m}_P is the ideal sheaf corresponding to P . Therefore, we have

$$H^1(X_c, \mathcal{A}^{\otimes m} \otimes \mathfrak{m}_P) = 0$$

since it is a subspace of $H^1(Y_c, f^* \mathcal{A}^{\otimes m} \otimes \mathcal{O}_Y(-E)) = 0$. Hence, the natural restriction map

$$H^0(X_c, \mathcal{A}^{\otimes m}) \rightarrow \mathcal{A}^{\otimes m} \otimes \mathcal{O}_X/\mathfrak{m}_P \simeq \mathbb{C}$$

is surjective. In particular,

$$H^0(X_c, \mathcal{A}^{\otimes m}) \neq 0.$$

This completes the proof. \square

Let us prove Theorem 1.7.

Proof of Theorem 1.7. We use induction on $n := \dim \text{Supp } \mathcal{S}$. If $n = 0$, then it is obvious. From now on, we assume that Theorem 1.7 holds in the lower dimensional case.

Step 1. In this step, we will reduce the proof to the case where X is a variety and \mathcal{S} is a torsion-free coherent sheaf on X . This reduction argument is standard (see, for example, [Fn2, Theorem 3.8.1]). We include it for the sake of completeness.

Let \mathcal{N}_X be the nilradical of \mathcal{O}_X . By shrinking X around \overline{X}_c suitably, we have $\mathcal{N}_X^k = 0$ for some positive integer k . We consider the following short exact sequence:

$$0 \rightarrow \mathcal{N}_X^{i+1}\mathcal{S} \rightarrow \mathcal{N}_X^i\mathcal{S} \rightarrow \mathcal{N}_X^i\mathcal{S}/\mathcal{N}_X^{i+1}\mathcal{S} \rightarrow 0.$$

Then, by Lemma 4.3, we can reduce the proof to the case where X is reduced since $\mathcal{N}_X^{k-1}\mathcal{S}$ and $\mathcal{N}_X^i\mathcal{S}/\mathcal{N}_X^{i+1}\mathcal{S}$ are supported on X_{red} for every i . Hence we can assume that X is reduced. Let $X = X^1 \cup \dots \cup X^l$ be its decomposition into irreducible components after shrinking X around \overline{X}_c suitably. Let \mathcal{I} be the defining ideal sheaf of X^1 on X . We consider the short exact sequence

$$0 \rightarrow \mathcal{I}\mathcal{S} \rightarrow \mathcal{S} \rightarrow \mathcal{S}/\mathcal{I}\mathcal{S} \rightarrow 0.$$

The outer terms of the above short exact sequence are supported on $X^2 \cup \dots \cup X^l$ and X^1 , respectively. By induction on the number of irreducible components and Lemma 4.3, we can reduce the proof to the case where X is irreducible. Thus, from now on, we can assume that X is a variety. Finally, we consider the short exact sequence:

$$0 \rightarrow T(\mathcal{S}) \rightarrow \mathcal{S} \rightarrow \mathcal{S}/T(\mathcal{S}) \rightarrow 0,$$

where $T(\mathcal{S})$ is the torsion part of \mathcal{S} . Since $\dim \text{Supp } T(\mathcal{S}) < n = \dim X$, we can reduce the proof to the case where \mathcal{S} is torsion-free by induction on n and Lemma 4.3. We finish the proof of Step 1.

Hence, from now on, we assume that X is a variety and \mathcal{S} is a torsion-free coherent sheaf on X . We take some $d > c$. We put $U := X_d$ and use Lemma 4.2. We replace X with $X_{d+\varepsilon}$ for $0 < \varepsilon \ll 1$. Then we get a proper bimeromorphic morphism $f: Y \rightarrow X$ from a complex manifold Y . Note that Y is weakly 1-complete by construction.

Step 2. In this step, we will prove that Theorem 1.7 holds for $f_*(f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE))$ for some positive integer k .

Let Ψ be a smooth exhaustion function of X . Then Y is a weakly 1-complete complex manifold with an exhaustion function $f^*\Psi$. The line bundle $\mathcal{O}_Y(-E) \otimes f^*\mathcal{L}^{\otimes a}$ is positive (see Lemma 4.2). Since $f^*\mathcal{S}/T(f^*\mathcal{S})$ is locally free, we can find a positive integer k such that

$$(4.5) \quad H^i(Y_c, f^*\mathcal{S}/T(f^*\mathcal{S}) \otimes \mathcal{O}_Y(-kE) \otimes f^*\mathcal{L}^{\otimes ka} \otimes \mathcal{M}') = 0$$

for every $i > 0$ and for every semipositive line bundle \mathcal{M}' on Y by Lemma 2.9 and Theorem 1.1. Since $T(f^*\mathcal{S})$ is torsion, we may assume that

$$(4.6) \quad H^i(Y_c, T(f^*\mathcal{S}) \otimes \mathcal{O}_Y(-kE) \otimes f^*\mathcal{L}^{\otimes ka} \otimes \mathcal{M}') = 0$$

for every $i > 0$ and for every semipositive line bundle \mathcal{M}' on Y . Since $\mathcal{O}_Y(-E)$ is f -ample, we may further assume that

$$(4.7) \quad R^i f_*(f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE)) = 0$$

on X_c for every $i > 0$ (see, for example, [BS, Chapter IV. Theorem 2.1]). Hence, by (4.5), (4.6), and (4.7), we have

$$\begin{aligned} & H^i(X_c, f_*(f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE)) \otimes \mathcal{L}^{\otimes ka} \otimes \mathcal{M}) \\ & \simeq H^i(Y_c, f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE) \otimes f^*\mathcal{L}^{\otimes ka} \otimes f^*\mathcal{M}) = 0 \end{aligned}$$

for every $i > 0$ and for every semipositive line bundle \mathcal{M} on X . This means that Theorem 1.7 holds for $f_*(f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE))$.

Step 3. In this step, we will prove that Theorem 1.7 holds true for $f_*f^*\mathcal{S}$.

We use the positive integer k obtained in Step 2. We consider the exact sequence

$$(4.8) \quad 0 \rightarrow \mathcal{T}_1 \rightarrow f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE) \rightarrow f^*\mathcal{S} \rightarrow \mathcal{T}_2 \rightarrow 0$$

induced by the natural map $f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE) \rightarrow f^*\mathcal{S}$, where \mathcal{T}_1 and \mathcal{T}_2 are torsion coherent sheaves on Y . We split (4.8) into the following two short exact sequences:

$$(4.9) \quad 0 \rightarrow \mathcal{T}_1 \rightarrow f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE) \rightarrow \mathcal{S}^\dagger \rightarrow 0$$

and

$$(4.10) \quad 0 \rightarrow \mathcal{S}^\dagger \rightarrow f^*\mathcal{S} \rightarrow \mathcal{T}_2 \rightarrow 0.$$

By (4.9), we obtain

$$(4.11) \quad 0 \rightarrow \mathcal{T}_3 \rightarrow f_*(f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE)) \rightarrow f_*\mathcal{S}^\dagger \rightarrow \mathcal{T}_4 \rightarrow 0$$

for some torsion coherent sheaves \mathcal{T}_3 and \mathcal{T}_4 on X . By (4.10), we have

$$(4.12) \quad 0 \rightarrow f_*\mathcal{S}^\dagger \rightarrow f_*f^*\mathcal{S} \rightarrow \mathcal{T}_5 \rightarrow 0$$

for some torsion coherent sheaf \mathcal{T}_5 on X . Since \mathcal{T}_3 and \mathcal{T}_4 are torsion coherent sheaves on X , by Lemma 4.4, (4.11), and the result in Step 2, we see that Theorem 1.7 holds for $f_*\mathcal{S}^\dagger$. Hence Theorem 1.7 holds true for $f_*f^*\mathcal{S}$ by Lemma 4.3 and (4.12) since \mathcal{T}_5 is a torsion coherent sheaf on X . This is what we wanted.

Step 4. In this step, we will prove that Theorem 1.7 holds true for \mathcal{S} . More precisely, we will construct the following exact sequence:

$$(4.13) \quad 0 \rightarrow \mathcal{K} \rightarrow f_*f^*\mathcal{S} \rightarrow \mathcal{S} \otimes \mathcal{L}^{\otimes l} \rightarrow \mathcal{C} \rightarrow 0,$$

where l is some positive integer, \mathcal{K} and \mathcal{C} are torsion coherent sheaves.

Once we obtain (4.13), by Lemma 4.4, we see that Theorem 1.7 holds for $\mathcal{S} \otimes \mathcal{L}^{\otimes l}$ since \mathcal{K} and \mathcal{C} are torsion sheaves on X and Theorem 1.7 holds for $f_*f^*\mathcal{S}$ by Step 3. Thus, Theorem 1.7 holds for \mathcal{S} . Hence it is sufficient to construct (4.13). We consider the following short exact sequence:

$$(4.14) \quad 0 \rightarrow \mathcal{S} \rightarrow f_*f^*\mathcal{S} \rightarrow \mathcal{T}' \rightarrow 0$$

induced by the natural map $\mathcal{S} \rightarrow f_*f^*\mathcal{S}$. By construction, \mathcal{T}' is a torsion coherent sheaf and $\text{Supp } \mathcal{T}'$ is contained in T in Lemma 4.2. Let \mathcal{A} be the sheaf of annihilators of \mathcal{T}' .

Let $f: Y \xrightarrow{g} Z \xrightarrow{h} X$ be as in Lemma 4.2. We put

$$\mathbf{cond}_X := \mathcal{H}om_{\mathcal{O}_X}(h_*\mathcal{O}_Z, \mathcal{O}_X) \subset \mathcal{O}_X$$

and call it the *conductor ideal sheaf* on X . It is also an ideal sheaf on Z . We write it as \mathbf{cond}_Z when we view the conductor ideal sheaf as an ideal sheaf on Z . Since $\mathcal{O}_Y(-E) \otimes f^*\mathcal{L}^{\otimes a}$ is positive on Y , we can find a non-zero global section φ of $\mathcal{O}_Y(-b_1E) \otimes f^*\mathcal{L}^{\otimes ab_1}$ on Y_d for some positive integer b_1 by Lemma 4.5. We can see φ as a global section of $g_*\mathcal{O}_Y(-b_1E) \otimes h^*\mathcal{L}^{\otimes ab_1}$ on Z_d . We write it as φ_Z when we view φ as a global section of $g_*\mathcal{O}_Y(-b_1E) \otimes h^*\mathcal{L}^{\otimes ab_1}$. Since Z is normal, $g_*\mathcal{O}_Y(-b_1E)$ is an ideal sheaf on Z . We note that $g(E) = h^{-1}(T)$ by construction. Hence, there exists a positive integer b_2 such that $\varphi_Z^{\otimes b_2} \in H^0(Z_{c+\varepsilon}, h^*\mathcal{L}^{\otimes ab_1b_2} \otimes \mathbf{cond}_Z)$ for $0 < \varepsilon \ll 1$ by the Nullstellensatz (see, for example, [No, Theorem 6.4.20]). Therefore, we can see $\varphi_Z^{\otimes b_2}$ as a global section φ_X of $\mathcal{L}^{\otimes ab_1b_2} \otimes \mathbf{cond}_X$ on $X_{c+\varepsilon}$. Since $\varphi \in H^0(Y_d, \mathcal{O}_Y(-b_1E) \otimes f^*\mathcal{L}^{\otimes ab_1})$, φ_X vanishes along

$T = f(E)$ over some open neighborhood of $\overline{X_c}$. By the Nullstellensatz (see, for example, [No, Theorem 6.4.20]) again, we can find a sufficiently large positive integer b_3 such that

$$\psi := \varphi_X^{\otimes b_3} \in H^0(X_c, \mathcal{L}^{\otimes ab_1b_2b_3} \otimes \mathcal{A}).$$

By taking $\otimes \psi$, we get the following commutative diagram:

$$(4.15) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & f_*f^*\mathcal{S} & \longrightarrow & \mathcal{T}' \longrightarrow 0 \\ & & \downarrow \otimes \psi & & \downarrow \otimes \psi & & \downarrow \otimes 0 \\ 0 & \longrightarrow & \mathcal{S} \otimes \mathcal{L}^{\otimes l} & \longrightarrow & f_*f^*\mathcal{S} \otimes \mathcal{L}^{\otimes l} & \longrightarrow & \mathcal{T}' \otimes \mathcal{L}^{\otimes l} \longrightarrow 0 \end{array}$$

from (4.14), where $l = ab_1b_2b_3$. Thus we get a desired exact sequence:

$$0 \rightarrow \mathcal{K} \rightarrow f_*f^*\mathcal{S} \xrightarrow{\otimes \psi} \mathcal{S} \otimes \mathcal{L}^{\otimes l} \rightarrow \mathcal{C} \rightarrow 0$$

from (4.15). This is what we wanted.

We finish the proof of Theorem 1.7. \square

Proof of Theorem 1.9. This is an immediate application of Theorem 1.7. For details, see the proof of [K, Theorem 2.1]. We also note that the argument in [K] is well known to algebraic geometers. \square

Proof of Theorem 1.10. The following argument is more or less well known to algebraic geometers (see [Ha, Chapter II. Section 7]). Hence we omit some details.

We put $\mathcal{S} = \mathcal{O}_X$ and apply Theorem 1.9. Then there exists a positive integer k_0 such that $\mathcal{L}^{\otimes k}$ is generated by finitely many global sections on some open neighborhood of $\overline{X_c}$. Take $P, Q \in \overline{X_c}$. Let \mathfrak{m}_P (resp. \mathfrak{m}_Q) be the ideal sheaf corresponding to P (resp. Q). Take some $d > c$. By Theorem 1.7, we may assume that the evaluation map

$$(4.16) \quad H^0(X_d, \mathcal{L}^{\otimes k}) \rightarrow \mathcal{L}^{\otimes k} \otimes \mathcal{O}_X/\mathfrak{m}_P \oplus \mathcal{L}^{\otimes k} \otimes \mathcal{O}_X/\mathfrak{m}_Q$$

is surjective for every $k \geq k_0$. By Theorem 1.7 again, we may assume that

$$(4.17) \quad H^0(X_d, \mathfrak{m}_P \otimes \mathcal{L}^{\otimes k}) \rightarrow \mathfrak{m}_P/\mathfrak{m}_P^2 \otimes \mathcal{L}^{\otimes k}$$

is surjective for every $k \geq k_0$. By (4.16), (4.17), and the compactness of $\overline{X_c}$, we can find a positive integer m_0 with the desired properties (see also [Ha, Chapter II. Proposition 7.3]). \square

Corollary 1.12 follows immediately from Theorem 1.10.

Proof of Corollary 1.12. If \mathcal{L} is ample, then it is positive. Conversely, if \mathcal{L} is positive, then it is ample by the embedding theorem (see Theorem 1.10). \square

We prove Theorem 1.13 as an application of Theorem 1.7.

Proof of Theorem 1.13. We can regard X as a complex analytic space via Serre's GAGA. By Corollary 1.12, \mathcal{L} is positive. Since X is compact, Theorem 1.7 implies that there exists m_0 such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{M}') = 0$$

for $i > 0$, $m \geq m_0$, and for every semipositive line bundle \mathcal{M}' on X . Let \mathcal{M} be a nef line bundle. Then $\mathcal{L} \otimes \mathcal{M}$ is ample, hence positive. In particular, it is semipositive. Therefore,

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{L} \otimes \mathcal{M}) = 0$$

for $i > 0$ and $m \geq m_0$. Thus it suffices to take $m(\mathcal{F}, \mathcal{L}) := m_0 + 1$. \square

REFERENCES

- [BS] C. Bănică, O. Stănășilă, *Algebraic methods in the global theory of complex spaces*, Translated from the Romanian. Editura Academiei, Bucharest; John Wiley & Sons, London–New York–Sydney, 1976.
- [BM] E. Bierstone, P. D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, *Invent. Math.* **128** (1997), no. 2, 207–302.
- [D1] J.-P. Demailly, L^2 estimates for the $\bar{\partial}$ -operator on complex manifolds, Notes de cours, Ecole d’été de Mathématiques (Analyse Complexe), Institut Fourier, Grenoble, Juin 1996, available at https://www-fourier.univ-grenoble-alpes.fr/~demailly/manuscripts/estimations_l2.pdf
- [D2] J.-P. Demailly, *Complex analytic and differential geometry*, available at <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>
- [Fk] A. Fujiki, On the blowing down of analytic spaces, *Publ. Res. Inst. Math. Sci.* **10** (1974/75), 473–507.
- [Fn1] O. Fujino, A transcendental approach to Kollár’s injectivity theorem II, *J. Reine Angew. Math.* **681** (2013), 149–174.
- [Fn2] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.
- [Fn3] O. Fujino, Minimal model program for projective morphisms between complex analytic spaces, preprint (2022). arXiv:2201.11315 [math.AG]
- [G] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen (German), *Math. Ann.* **146** (1962), 331–368.
- [Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**. Springer-Verlag, New York-Heidelberg, 1977.
- [Hi] H. Hironaka, Flattening theorem in complex-analytic geometry, *Amer. J. Math.* **97** (1975), 503–547.
- [K] H. Kazama, Existence and approximation theorems on a weakly 1-complete analytic space, *Manuscripta Math.* **19** (1976), no. 1, 57–74.
- [Na1] S. Nakano, Vanishing theorems for weakly 1-complete manifolds, *Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki*, pp. 169–179. Kinokuniya, Tokyo, 1973.
- [Na2] S. Nakano, *Function theory of several complex variables—Differential geometric approach* (in Japanese), Asakura Shoten, 1981.
- [Nay] N. Nakayama, The lower semicontinuity of the plurigenera of complex varieties, *Algebraic geometry, Sendai*, 1985, 551–590, *Adv. Stud. Pure Math.*, **10**, North-Holland, Amsterdam, 1987.
- [No] J. Noguchi, *Analytic function theory of several variables. Elements of Oka’s coherence*, Springer, Singapore, 2016.
- [O] T. Ohsawa, A counter example of ampleness of positive line bundles, *Proc. Japan Acad. Ser. A Math. Sci.* **55** (1979), no. 5, 193–194.
- [R] H. Rossi, Picard variety of an isolated singular point, *Rice Univ. Stud.* **54** (1968), no. 4, 63–73.
- [T] S. Takayama, Adjoint linear series on weakly 1-complete Kähler manifolds. I. Global projective embedding, *Math. Ann.* **311** (1998), no. 3, 501–531.
- [W] Y. Yuta, Global embeddings of weakly pseudoconvex complex spaces and refined approximation theorems, preprint (2025). arXiv:2512.03572 [math.CV]

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