ON VANISHING THEOREMS FOR NON-COMPACT ANALYTIC SPACES

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Abstract. In this paper, we explain some vanishing theorems for non-compact complex analytic spaces. We see that Nakano's vanishing theorem on weakly 1-complete complex manifolds is very useful.

1. Vanishing theorems for non-compact analytic spaces

Let us start with Nakano's vanishing theorem on weakly 1-complete complex manifolds.

Theorem 1.1 (Nakano's vanishing theorem). *Let X be a weakly 1-complete complex manifold and let $\mathcal{E}$ be a Nakano positive vector bundle on X. Then, we have*

$$H^i(X, \omega_X \otimes \mathcal{E}) = 0$$

for every $i > 0$. In particular, for every $c \in \mathbb{R}$, we see that

$$H^i(X_c, \omega_X \otimes \mathcal{E}) = 0$$

holds for every $i > 0$.

As a special case of Theorem 1.1, we have:

Corollary 1.2 (Kodaira vanishing theorem for weakly 1-complete complex manifolds). *Let X be a weakly 1-complete complex manifold and let $\mathcal{L}$ be a positive line bundle on X. Then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ holds for every $i > 0$. In particular, we have $H^i(X_c, \omega_X \otimes \mathcal{L}) = 0$ for every $i > 0$.*

If X is compact in Corollary 1.2, then the statement is nothing but Kodaira's original vanishing theorem. For various geometric applications, the following corollary of Theorem 1.1 may be useful.

Corollary 1.3. *Let $\pi \colon X \to Y$ be a proper morphism of complex analytic spaces such that X is smooth. Let $\mathcal{E}$ be a Nakano positive vector bundle on X. Then $R^i\pi_*(\omega_X \otimes \mathcal{E}) = 0$ holds for every $i > 0$.*

For the minimal model theory for projective morphisms between complex analytic spaces (see [Fn2]), we need:

Corollary 1.4 (Relative Kodaira vanishing theorem). *Let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces. Let $\mathcal{L}$ be a π-ample line bundle on X. Then $R^i\pi_*(\omega_X \otimes \mathcal{L}) = 0$ holds for every $i > 0$.*

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Once we establish Corollary 1.4, there is no difficulty to prove the relative Kawamata–Viehweg vanishing theorem for projective morphisms between complex varieties, which will play a crucial role in the theory of minimal models (see [Fn2]).

Theorem 1.5 (Kawamata–Viehweg vanishing theorem for projective morphisms of complex varieties)**.** *Let X be a smooth complex variety and let $\pi\colon X \to Y$ be a projective morphism of complex varieties. Assume that D is an $\mathbb{R}$-divisor on X such that D is π-nef and π-big and that* $\operatorname{Supp}\{D\}$ *is a simple normal crossing divisor on X. Then $R^i\pi_*\mathcal{O}_X(K_X+\lceil D\rceil)=0$ for every $i>0$.*

One of the main purposes of this paper is to make the following vanishing theorem for weakly 1-complete complex analytic spaces, which is mainly due to Fujiki and Hironaka, more accessible.

Theorem 1.6 (Fujiki's vanishing theorem, see [Fk, Theorem N′])**.** *Let X be a weakly 1-complete complex analytic space and let $\mathcal{S}$ be a coherent sheaf on X. Let $\mathcal{L}$ be a positive line bundle on X. Then, for every $c\in\mathbb{R}$, there exists a positive integer m_0 such that*

$$H^i(X_c, \mathcal{S}\otimes\mathcal{L}^{\otimes m}\otimes\mathcal{M})=0$$

holds for $i\geq 1$, $m\geq m_0$, and for every semipositive line bundle $\mathcal{M}$ on X.

We give an important remark on Theorem 1.6. .

Remark 1.7. Hironaka contributed to the formulation and the proof of Theorem 1.6. In [Fk], Fujiki wrote:

> The author learned the formulation of this theorem and the idea of its proof from Prof. Hironaka.

For Hironaka's contribution, see also [Na1, §3. Comments].

As an obvious application of Theorem 1.6, we have:

Theorem 1.8 ([K, Theorem 2.1])**.** *Let X be a weakly 1-complete complex analytic space and let $\mathcal{S}$ be a coherent sheaf on X. Let $\mathcal{L}$ be a positive line bundle on X. Then, for every $c\in\mathbb{R}$, there exists a positive integer k_0 such that there exist finitely many global sections of $\mathcal{S}\otimes\mathcal{L}^{\otimes k}$ over some open neighborhood of $\overline{X_c}$ which generate $\mathcal{S}\otimes\mathcal{L}^{\otimes k}$ there for every $k\geq k_0$.*

Hence, we have the following embedding theorem, which is a generalization of Kodaira's embedding theorem. When X is compact, Theorem 1.9 is Grauert's generalization of Kodaira's embedding theorem (see [G] and [No, Theorem 8.5.8]).

Theorem 1.9 (Embedding theorem, Hironaka)**.** *Let X be a weakly 1-complete complex analytic space and let $\mathcal{L}$ be a positive line bundle on X. We take some $c\in\mathbb{R}$ and consider X_c. Then there exists a positive integer m_0 such that for every $m\geq m_0$ we can find finite elements $\varphi_0,\cdots,\varphi_N$ of $H^0(X_c,\mathcal{L}^{\otimes m})$ which embed X_c as a locally closed analytic subspace of $\mathbb{P}^N$ with $\mathcal{L}^{\otimes m}\simeq\Phi^*\mathcal{O}_{\mathbb{P}^N}(1)$, where $\Phi\colon X_c\hookrightarrow\mathbb{P}^N$ is the induced embedding.*

Remark 1.10. In [Na2, §21], Nakano wrote that Theorem 1.9 was first obtained by Hironaka. For Hironaka's contribution to the theory of weakly 1-complete complex analytic spaces, see also Remark 1.7.

As an easy consequence of Theorem 1.9, we have a metric characterization of ample line bundles on compact complex analytic spaces.

Corollary 1.11. *Let X be a compact complex analytic space and let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is positive if and only if $\mathcal{L}$ is ample.*

We close this section with Fujita's vanishing theorem. We will see that it is also a corollary of Theorem 1.6.

Theorem 1.12 (Fujita's vanishing theorem)**.** *Let X be a projective scheme over $\mathbb{C}$ and let $\mathcal{L}$ be an ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf on X. Then there exists a positive integer $m(\mathcal{F}, \mathcal{L})$ such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{M}) = 0$ for $i > 0$, $m \geq m(\mathcal{F}, \mathcal{L})$, and for every nef line bundle $\mathcal{M}$ on X.*

For the details of Fujita's vanishing theorem, see, for example, [Fn1, Section 3.8].

2. Basic definitions

We are mainly interested in singular complex analytic spaces. Here, we will only explain some basic definitions necessary for understanding theorems in Section 1. For the details, see [D2, Chapter IX. §2.].

Definition 2.1 (Strictly plurisubharmonic and plurisubharmonic functions)**.** Let V be an analytic subspace of an open subset U in $\mathbb{C}^n$. A smooth function φ on V is by definition the restriction on V of some smooth function $\widetilde{\varphi}$ defined on an open neighborhood U' of V in U. An $\mathbb{R}$-valued smooth function φ on V is said to be *strictly plurisubharmonic* (resp. *plurisubharmonic*) on V if there is a smooth function $\widetilde{\varphi}$ as above such that it is strictly plurisubharmonic (resp. plurisubharmonic) in the usual sense.

By Definition 2.1, we can define smooth (strictly) plurisubharmonic functions on complex analytic spaces. For the details, see [D2, Chapter IX. (2.5) Definition and (2.6) Lemma].

Definition 2.2 (Weakly 1-complete complex analytic spaces)**.** Let X be a complex analytic space. If there exists an $\mathbb{R}$-valued smooth function Ψ on X which is plurisubharmonic such that $X_c := \{x \in X \mid \Psi(x) < c\}$ is relatively compact in X for every $c \in \mathbb{R}$, then X is said to be *weakly* 1*-complete*. We call Ψ an *exhaustion function* of X.

Of course, if X is smooth in Definition 2.2, then X is called a *weakly* 1*-complete complex manifold*. We note that every compact analytic space is weakly 1-complete by definition. We also note that every closed analytic subspace of a weakly 1-complete complex analytic space is weakly 1-complete.

Remark 2.3 (Pseudoconvex complex manifolds)**.** In some literature (see, for example, [D1, (8.6) Definition] and [D2, Chapter VIII. (5.1) Definition]), a weakly 1-complete complex manifold is called a *weakly pseudoconvex complex manifold*. We note that a compact complex manifold is automatically a weakly 1-complete complex manifold.

We will freely use the following remark in Section 3 without mentioning it explicitly.

Remark 2.4. Let X be a weakly 1-complete complex analytic space with an exhaustion function Ψ. As in Definition 2.2, we put

$$X_c := \{x \in X \mid \Psi(x) < c\}.$$

We set $\Psi_c := (c - \Psi)^{-1}$. Then X_c is a weakly 1-complete complex analytic space and Ψ_c is an exhaustion function on X_c. In this case, for every $c' < c$, we have

$$X_{c'} = \{x \in X \mid \Psi(x) < c'\} = \{x \in X_c \mid \Psi_c(x) < (c - c')^{-1}\}.$$

Let $f\colon Y \to X$ be a proper morphism of complex analytic spaces. Then $f^*\Psi$ is a smooth plurisubharmonic function on Y. It is obvious that

$$Y_c := \{y \in Y \mid f^*\Psi(y) < c\} = f^{-1}(X_c)$$

holds for every $c \in \mathbb{R}$. In particular, Y_c is compact since f is proper. This means that Y is a weakly 1-complete complex analytic space with an exhaustion function $f^*\Psi$.

Let us recall the definition of positive and semipositive line bundles on singular complex analytic spaces. It is indispensable in order to understand Fujiki's vanishing theorem (see Theorem 1.6).

Definition 2.5 (Positive and semipositive line bundles)**.** Let X be a complex analytic space and let $\mathcal{L}$ be a line bundle on X. Assume that $\mathcal{L}$ is defined by the system of transition functions $\{f_{\alpha\beta}\}$ with respect to some open covering $\mathcal{U} = \{U_\alpha\}$ of X. In this situation, a metric on $\mathcal{L}$ is given by the system of $\mathbb{R}_{>0}$-valued functions $h = \{h_\alpha\}$, where each h_α is a smooth function defined on U_α, such that $h_\alpha = \frac{1}{|f_{\alpha\beta}|^2} h_\beta$ holds on $U_\alpha \cap U_\beta$. If $-\log h_\alpha$ is strictly plurisubharmonic (resp. plurisubharmonic) on U_α for every α, then $\mathcal{L}$ is said to be *positive* (resp. *semipositive*).

In this short paper, we do not define Nakano positive vector bundles explicitly since the definition is somewhat complicated. The following lemma, which easily follows from the definition of Nakano positive vector bundles, seems to be sufficient for our almost all applications.

Lemma 2.6. *Let X be a complex manifold, let $\mathcal{E}$ be a vector bundle on X, and let $\mathcal{L}$ be a a positive line bundle on X. Let U be a relatively compact open subset of X. Then there exists a positive integer m_0 such that $\mathcal{E} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{M}$ is Nakano positive on U for every $m \geq m_0$ and for every semipositive line bundle $\mathcal{M}$ on X.*

Proof. Once we understand the definition of Nakano positive vector bundles (see, for example, [D1, (6.9) Definition] and [D2, Chapter VII. (6.3) Definition]), we have no difficulty in checking this statement. □

3. Proof of theorems and corollaries

In this section, we will prove theorems and corollaries in Section 1. Let us look at the proof of Nakano's vanishing theorem: Theorem 1.1.

Proof of Theorem 1.1. This statement is well known as Nakano's vanishing theorem. For the details, see, for example, [D1, (9.1) Nakano vanishing theorem] and [D2, Chapter VIII. (5.5) Theorem]. We take an exhaustion function Ψ of X such that $X_c = \{x \in X \mid \Psi(x) < c\}$ and put $\Psi_c = (c - \Psi)^{-1}$. Then we can easily see that X_c is a weakly 1-complete complex manifold with an exhaustion function Ψ_c (see Remark 2.4). Hence we get the desired vanishing theorem on X_c. □

Remark 3.1. Kodaira established the foundation of the theory of compact complex manifolds. Nakano generalized Kodaira's results for non-compact complex manifolds (see, for example, [Na2]). On the other hand, Demailly generalized Hörmander's techniques (see, for example, [D1] and [D2]). Hence, Nakano's approach is more differential geometric than Demailly's. The author feels that the treatment of Theorem 1.1 in [D1] and [D2] is simpler than the one in [Na2].

Corollaries 1.2 and 1.3 are almost obvious by Theorem 1.1.

Proof of Corollary 1.2. This is a special case of Theorem 1.1. We note that a line bundle $\mathcal{L}$ is positive if and only if it is Nakano positive. $\square$

Proof of Corollary 1.3. We take any point $P \in Y$. Then we can find an arbitrary small Stein open neighborhood U of P in Y and a strictly plurisubharmonic exhaustion function Ψ on U. Then $\pi^{-1}(U)$ is a weakly 1-complete complex manifold with an exhaustion function $\pi^*\Psi$. By Theorem 1.1, $H^i(\pi^{-1}(U), \omega_X \otimes \mathcal{E}) = 0$ for every $i > 0$. This implies that $R^i\pi_* (\omega_X \otimes \mathcal{E}) = 0$ holds for every $i > 0$. This is what we wanted. $\square$

For the proof of Corollary 1.4 and Theorem 1.6, we prepare a very important lemma.

Lemma 3.2. *Let $\pi\colon Y \to X$ be a projective morphism of complex analytic spaces, let $\mathcal{L}$ be a positive line bundle on X, and let $\mathcal{N}$ be a π-ample line bundle on Y. Let U be any relatively compact open subset of X. Then there exists a positive integer m_0 such that $\mathcal{N} \otimes \pi^*\mathcal{L}^{\otimes m}$ is positive on $\pi^{-1}(U)$ for every $m \geq m_0$.*

Proof. We take a positive integer k such that $\mathcal{N}^{\otimes k}$ is very ample over some open neighborhood of $\overline{U}$. By replacing $\mathcal{N}$ with $\mathcal{N}^{\otimes k}$ and shrinking X around $\overline{U}$ suitably, we may assume that $\mathcal{N}$ is very ample over X. Then, by the proof of [Fk, Lemma 2], we get a desired positive integer m_0. For the details, see [Fk, Lemma 2]. $\square$

Let us prove Corollary 1.4.

Proof of Corollary 1.4. We use the same notation as in the proof of Corollary 1.3. On U, $e^{-\Psi}$ is a positive metric on the trivial line bundle $\mathcal{O}_U$. We take some $c \in \mathbb{R}$ such that $P \in U_c$. By Lemma 3.2, the line bundle $\mathcal{L}$ is positive on $\pi^{-1}(U_c)$. Hence we have $H^i(\pi^{-1}(U_c), \omega_X \otimes \mathcal{L}) = 0$ for every $i > 0$ by Corollary 1.2. This implies $R^i\pi_* (\omega_X \otimes \mathcal{L}) = 0$ for every $i > 0$. $\square$

Proof of Theorem 1.5. It is not difficult to prove this theorem by using Corollary 1.4, Kawamata's covering trick, and so on. The reader can find all the details in [Nay, §3]. Hence we omit the details here. $\square$

From now on, we will prove Theorem 1.6. Let us start with the following very important lemma.

Lemma 3.3. *Let X be a complex variety and let $\mathcal{S}$ be a coherent sheaf on X. Let $\mathcal{L}$ be a positive line bundle on X. We put $T := T_1 \cup T_2$, where T_1 is the singular locus of X and $T_2 := \{x \in X \mid \mathcal{S} \text{ is not locally free at } x\}$. Let U be any relatively compact open subset of X. Then, after shrinking X around $\overline{U}$ suitably, we can construct a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ & \searrow^{f} & \downarrow{\scriptstyle h} \\ & & X \end{array}$$

satisfying the following properties:

(i) *$h\colon Z \to X$ is the normalization,*
(ii) *g is a finite composite of blow-ups,*
(iii) *Y is smooth,*
(iv) *f is an isomorphism over $X \setminus T$,*
(v) *$f^*\mathcal{S}/T(f^*\mathcal{S})$ is locally free, where $T(f^*\mathcal{S})$ is the torsion part of $f^*\mathcal{S}$, and*

(vi) *there exists an effective Cartier divisor E on Y such that $\mathcal{O}_Y(-E)$ is f-ample, $\mathcal{O}_Y(-E)\otimes f^*\mathcal{L}^{\otimes a}$ is positive for some integer $a>0$, and $g(E)=h^{-1}(T)$ holds set theoretically.*

Lemma 3.3 will play a crucial role in the proof of Theorem 1.6.

Proof of Lemma 3.3. Let $h\colon Z\to X$ be the normalization. By [R, 3.5. Theorem], we can take a proper bimeromorphic morphism $\rho\colon Z'\to X$ such that ρ is an isomorphism over $X\setminus T$ and that $\rho^*\mathcal{S}/T(\rho^*\mathcal{S})$ is locally free, where $T(\rho^*\mathcal{S})$ is the torsion part of $\rho^*\mathcal{S}$. Let $g_1\colon Z_1\to Z$ be the blow-up along the ideal $\mathcal{J}$, where $\mathcal{J}$ is the defining ideal sheaf of $h^{-1}(T)$ on Z. Let Γ be the graph of $Z' \dashrightarrow Z_1$. By applying the flattening theorem (see [Hi, Corollary 1 and Definition 4.4.3]) to $\Gamma\to Z_1$ and using the desingularization theorem (see, for example, [BM, Theorem 13.3]), after shrinking X around $\overline{U}$ suitably, we get a finite sequence of blow-ups $Z\leftarrow Z_1\leftarrow Z_2\leftarrow\cdots\leftarrow Z_m=:Y$ such that Y is smooth, $f\colon Y\to X$ factors through Z', and $g\colon Y\to Z$ is an isomorphism over $Z\setminus h^{-1}(T)$. Since $g\colon Y\to Z$ is a finite composite of blow-ups, we can construct an effective Cartier divisor E on Y such that $\mathcal{O}_Y(-E)$ is g-ample with $g(E)=h^{-1}(T)$ after shrinking X suitably. Since h is finite, $\mathcal{O}_Y(-E)$ is f-ample. Therefore, by Lemma 3.2, we can take a positive integer a such that $\mathcal{O}_Y(-E)\otimes f^*\mathcal{L}^{\otimes a}$ is positive after shrinking X suitably again. By construction, $f\colon Y\to X$ factors through Z'. Therefore, we can check that $f^*\mathcal{S}/T(f^*\mathcal{S})$ is locally free. Thus, $Y\to Z\to X$ has all the desired properties. □

For the proof of Theorem 1.6, we prepare the following easy two lemmas.

Lemma 3.4. *Let*

$$0\to\mathcal{S}_1\to\mathcal{S}_2\to\mathcal{S}_3\to 0 \tag{3.1}$$

be a short exact sequence of coherent sheaves on X. If Theorem 1.6 holds true for $\mathcal{S}_1$ and $\mathcal{S}_3$ (resp. $\mathcal{S}_1$ and $\mathcal{S}_2$), then it also holds true for $\mathcal{S}_2$ (resp. $\mathcal{S}_3$).

Proof. This is obvious. It is sufficient to consider the long exact sequence in cohomology induced by (3.1). □

Lemma 3.5. *Let*

$$0\to\mathcal{S}_1\to\mathcal{S}_2\to\mathcal{S}_3\to\mathcal{S}_4\to 0 \tag{3.2}$$

be an exact sequence of coherent sheaves on X. If Theorem 1.6 holds true for $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_4$, then it also holds true for $\mathcal{S}_3$.

Proof. We split (3.2) into the following two short exact sequences:

$$0\to\mathcal{S}_1\to\mathcal{S}_2\to\mathcal{S}'\to 0 \tag{3.3}$$

and

$$0\to\mathcal{S}'\to\mathcal{S}_3\to\mathcal{S}_4\to 0. \tag{3.4}$$

By assumption and (3.3), we see that Theorem 1.6 holds true for $\mathcal{S}'$ (see Lemma 3.4). Then, by Lemma 3.4 and (3.4), we obtain that Theorem 1.6 holds for $\mathcal{S}_3$. This is what we wanted. □

Lemma 3.6. *Let X be a weakly 1-complete complex manifold and let $\mathcal{A}$ be a positive line bundle on X. Then, for every $c\in\mathbb{R}$, there exists some positive integer m such that $H^0(X_c,\mathcal{A}^{\otimes m})\neq 0$.*

Proof. We take an arbitrary point $P \in X_c$. Let $f\colon Y \to X$ be the blow-up at P. Then we have $\omega_Y \simeq f^*\omega_X \otimes \mathcal{O}_Y((n-1)E)$, where $n = \dim X$ and $E \simeq \mathbb{P}^{n-1}$ is the f-exceptional divisor on Y. Note that Y is a weakly 1-complete complex manifold. We can take some positive integer m such that

$$\omega_Y^{\otimes -1} \otimes f^*\mathcal{A}^{\otimes m} \otimes \mathcal{O}_Y(-E) \simeq f^*\left(\mathcal{A}^{\otimes m} \otimes \omega_X^{\otimes -1}\right) \otimes \mathcal{O}_Y(-nE)$$

is positive on Y_c (see Lemmas 2.6 and 3.2). Hence, by Corollary 1.2, $H^i(Y_c, f^*\mathcal{A}^{\otimes m} \otimes \mathcal{O}_Y(-E)) = 0$ for every $i > 0$. We note that $f_*\mathcal{O}_Y(-E) \simeq m_P$ holds, where m_P is the ideal sheaf corresponding to P. Therefore, we have $H^1(X_c, \mathcal{A}^{\otimes m} \otimes m_P) = 0$ since it is a subspace of $H^1(Y_c, f^*\mathcal{A}^{\otimes m} \otimes \mathcal{O}_Y(-E)) = 0$. This implies that the natural restriction map

$$H^0(X_c, \mathcal{A}^{\otimes m}) \to \mathcal{A}^{\otimes m} \otimes \mathcal{O}_X/m_P \simeq \mathbb{C}$$

is surjective. In particular, $H^0(X_c, \mathcal{A}^{\otimes m}) \neq 0$. We finish the proof. □

Let us prove Theorem 1.6.

Proof of Theorem 1.6. We use induction on $n := \dim \operatorname{Supp} \mathcal{S}$. If $n = 0$, then it is obvious. From now on, we assume that Theorem 1.6 holds in the lower dimensional case.

Step 1. In this step, we will reduce the proof to the case where X is a variety and $\mathcal{S}$ is a torsion-free coherent sheaf on X. The reduction argument in this step is very well known (see, for example, the proof of [Fn1, Theorem 3.8.1]). We will explain it here for the sake of completeness.

Let $\mathcal{N}_X$ be the nilradical of $\mathcal{O}_X$. By shrinking X around $\overline{X_c}$ suitably, we fave $\mathcal{N}_Z^k = 0$ for some positive integer k. We consider the following short exact sequence:

$$0 \to \mathcal{N}_X^{i+1}\mathcal{S} \to \mathcal{N}_X^{i}\mathcal{S} \to \mathcal{N}_X^{i}\mathcal{S}/\mathcal{N}_X^{i+1}\mathcal{S} \to 0.$$

Then, by Lemma 3.4, we can reduce the proof to the case where X is reduced since $\mathcal{N}_X^{k-1}\mathcal{S}$ and $\mathcal{N}_X^{i}\mathcal{S}/\mathcal{N}_X^{i+1}\mathcal{S}$ are supported on X_{red} for every i. Hence we can assume that X is reduced. Let $X = X^1 \cup \cdots \cup X^k$ be its decomposition into irreducible components after shrinking X around $\overline{X_c}$ suitably. Let $\mathcal{I}$ be the defining ideal sheaf of X^1 on X. We consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{S} \to \mathcal{S} \to \mathcal{S}/\mathcal{I}\mathcal{S} \to 0.$$

The outer terms of the above short exact sequence are supported on $X^2 \cup \cdots \cup X^k$ and X^1, respectively. By induction on the number of irreducible components and Lemma 3.4, we can reduce the proof to the case where X is irreducible. Thus, from now on, we can assume that X is a variety. Finally, we consider the short exact sequence:

$$0 \to T(\mathcal{S}) \to \mathcal{S} \to \mathcal{S}/T(\mathcal{S}) \to 0,$$

where $T(\mathcal{S})$ is the torsion part of $\mathcal{S}$. Since $\dim \operatorname{Supp} T(\mathcal{S}) < n = \dim X$, we can reduce the proof to the case where $\mathcal{S}$ is torsion-free by induction on n and Lemma 3.4. We finish the proof of Step 1.

Hence, from now on, we assume that X is a variety and $\mathcal{S}$ is a torsion-free coherent sheaf on X. We take some $d > c$. We put $U := X_d$ and use Lemma 3.3. We replace X with $X_{d+\varepsilon}$ for $0 < \varepsilon \ll 1$. Then we get a proper bimeromorphic morphism $f\colon Y \to X$ from a complex manifold Y. Note that Y is weakly 1-complete by construction.

Step 2. In this step, we will prove that Theorem 1.6 holds true for $f_*(f^*\mathcal{S} \otimes \mathcal{O}_Y(-kE))$ for some positive integer k.

Let Ψ be a smooth exhaustion function of X. Then Y is a weakly 1-complete complex manifold with an exhaustion function $f^*\Psi$. The line bundle $\mathcal{O}_Y(-E)\otimes f^*\mathcal{L}^{\otimes a}$ is positive (see Lemma 3.3). Since $f^*\mathcal{S}/T(f^*\mathcal{S})$ is locally free, we can find a positive integer k such that

$$H^i(Y_c, f^*\mathcal{S}/T(f^*\mathcal{S})\otimes\mathcal{O}_Y(-kE)\otimes f^*\mathcal{L}^{\otimes ka}\otimes\mathcal{M}')=0 \tag{3.5}$$

for every $i>0$ and for every semipositive line bundle $\mathcal{M}'$ on Y by Lemma 2.6 and Theorem 1.1. Since $T(f^*\mathcal{S})$ is torsion, we may assume that

$$H^i(Y_c, T(f^*\mathcal{S})\otimes\mathcal{O}_Y(-kE)\otimes f^*\mathcal{L}^{\otimes ka}\otimes\mathcal{M}')=0 \tag{3.6}$$

for every $i>0$ and for every semipositive line bundle $\mathcal{M}'$ on Y. Since $\mathcal{O}_Y(-E)$ is f-ample, we may further assume that

$$R^if_*\left(f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\right)=0 \tag{3.7}$$

on X_c for every $i>0$ (see, for example, [BS, Chapter IV. Theorem 2.1]). Hence, by (3.5), (3.6), and (3.7), we have

$$\begin{aligned}&H^i(X_c, f_*\left(f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\right)\otimes\mathcal{L}^{\otimes ka}\otimes\mathcal{M})\\ &\simeq H^i(Y_c, f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\otimes f^*\mathcal{L}^{\otimes ka}\otimes f^*\mathcal{M})=0\end{aligned}$$

for every $i>0$ and for every semipositive line bundle $\mathcal{M}$ on X. This means that Theorem 1.6 holds for $f_*\left(f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\right)$.

Step 3. In this step, we will prove that Theorem 1.6 holds true for $f_*f^*\mathcal{S}$.

We use the positive integer k obtained in Step 2. We consider the exact sequence

$$0\to\mathcal{T}_1\to f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\to f^*\mathcal{S}\to\mathcal{T}_2\to 0 \tag{3.8}$$

induced by the natural map $f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\to f^*\mathcal{S}$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are torsion coherent sheaves on Y. We split (3.8) into the following two short exact sequences:

$$0\to\mathcal{T}_1\to f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\to\mathcal{S}^\dagger\to 0 \tag{3.9}$$

and

$$0\to\mathcal{S}^\dagger\to f^*\mathcal{S}\to\mathcal{T}_2\to 0. \tag{3.10}$$

By (3.9), we obtain

$$0\to\mathcal{T}_3\to f_*\left(f^*\mathcal{S}\otimes\mathcal{O}_Y(-kE)\right)\to f_*\mathcal{S}^\dagger\to\mathcal{T}_4\to 0 \tag{3.11}$$

for some torsion coherent sheaves $\mathcal{T}_3$ and $\mathcal{T}_4$ on X. By (3.10), we have

$$0\to f_*\mathcal{S}^\dagger\to f_*f^*\mathcal{S}\to\mathcal{T}_5\to 0 \tag{3.12}$$

for some torsion coherent sheaf $\mathcal{T}_5$ on X. Since $\mathcal{T}_3$ and $\mathcal{T}_4$ are torsion coherent sheaves on X, by Lemma 3.5, (3.11), and the result in Step 2, we see that Theorem 1.6 holds for $f_*\mathcal{S}^\dagger$. Hence Theorem 1.6 holds true for $f_*f^*\mathcal{S}$ by Lemma 3.4 and (3.12) since $\mathcal{T}_5$ is a torsion coherent sheaf on X. This is what we wanted.

Step 4. In this step, we will prove that Theorem 1.6 holds true for $\mathcal{S}$. More precisely, we will construct the following short exact sequence:

$$0\to f_*f^*\mathcal{S}\to\mathcal{S}\otimes\mathcal{L}^{\otimes l}\to\mathcal{C}\to 0, \tag{3.13}$$

where l is some positive integer.

Once we obtain (3.13), by Lemma 3.4, we see that Theorem 1.6 holds for $\mathcal{S}\otimes\mathcal{L}^{\otimes l}$ since $\mathcal{C}$ is a torsion sheaf on X and Theorem 1.6 holds for $f_*f^*\mathcal{S}$ by Step 3. Thus, Theorem

1.6 holds for $\mathcal{S}$. Hence it is sufficient to construct (3.13). We consider the following short exact sequence:

$$0 \to \mathcal{S} \to f_*f^*\mathcal{S} \to \mathcal{T}' \to 0 \tag{3.14}$$

induced by the natural map $\mathcal{S} \to f_*f^*\mathcal{S}$. By construction, $\mathcal{T}'$ is a torsion coherent sheaf and $\operatorname{Supp}\mathcal{T}'$ is contained in T in Lemma 3.3. Let $\mathcal{A}$ be the sheaf of annihilators of $\mathcal{T}'$. Let $f\colon Y \xrightarrow{g} Z \xrightarrow{h} X$ be as in Lemma 3.3. We put

$$\mathfrak{cond}_X := \mathcal{H}om_{\mathcal{O}_X}(h_*\mathcal{O}_Z, \mathcal{O}_X) \subset \mathcal{O}_X$$

and call it the *conductor ideal sheaf* on X. It is also an ideal sheaf on Z. We write it as $\mathfrak{cond}_Z$ when we view the conductor ideal sheaf as an ideal sheaf on Z. Since $\mathcal{O}_Y(-E) \otimes f^*\mathcal{L}^{\otimes a}$ is positive on Y_d, we can find a non-zero global section φ of $\mathcal{O}_Y(-b_1E) \otimes f^*\mathcal{L}^{\otimes ab_1}$ on Y_d for some positive integer b_1 by Lemma 3.6. We can see φ as a global section of $g_*\mathcal{O}_Y(-b_1E) \otimes h^*\mathcal{L}^{\otimes ab_1}$ on Z_d. We write it as φ_Z when we view φ as a global section of $g_*\mathcal{O}_Y(-b_1E) \otimes h^*\mathcal{L}^{\otimes ab_1}$. Since Z is normal, $g_*\mathcal{O}_Y(-b_1E)$ is an ideal sheaf on Z. We note that $g(E) = h^{-1}(T)$ by construction. Hence, there exists a positive integer b_2 such that $\varphi_Z^{\otimes b_2} \in H^0(Z_{c+\varepsilon}, h^*\mathcal{L}^{\otimes ab_1b_2} \otimes \mathfrak{cond}_Z)$ for $0 < \varepsilon \ll 1$ by the Nullstellenzatz (see, for example, [No, Theorem 6.4.20]). Therefore, we can see $\varphi_Z^{\otimes b_2}$ as a global section φ_X of $\mathcal{L}^{\otimes ab_1b_2} \otimes \mathfrak{cond}_X$ on $X_{c+\varepsilon}$. Since $\varphi \in H^0(Y_d, \mathcal{O}_Y(-b_1E) \otimes f^*\mathcal{L}^{\otimes ab_1})$, φ_X vanishes along $T = f(E)$ over some open neighborhood of $\overline{X_c}$. By the Nullstellenzatz (see, for example, [No, Theorem 6.4.20]) again, we can find a sufficiently large positive integer b_3 such that

$$\psi := \varphi_X^{\otimes b_3} \in H^0(X_c, \mathcal{L}^{\otimes ab_1b_2b_3} \otimes \mathcal{A}).$$

By taking $\otimes\psi$, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & f_*f^*\mathcal{S} & \longrightarrow & \mathcal{T}' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \otimes\psi} & & \downarrow{\scriptstyle \otimes\psi} & & \downarrow{\scriptstyle \otimes 0} & & \\ 0 & \longrightarrow & \mathcal{S}\otimes\mathcal{L}^{\otimes l} & \longrightarrow & f_*f^*\mathcal{S}\otimes\mathcal{L}^{\otimes l} & \longrightarrow & \mathcal{T}'\otimes\mathcal{L}^{\otimes l} & \longrightarrow & 0 \end{array} \tag{3.15}$$

from (3.14), where $l = ab_1b_2b_3$. Thus we get a short exact sequence:

$$0 \to f_*f^*\mathcal{S} \xrightarrow{\otimes\psi} \mathcal{S} \otimes \mathcal{L}^{\otimes l} \to \mathcal{C} \to 0$$

from (3.15). This is what we wanted.

We finish the proof of Theorem 1.6. □

Proof of Theorem 1.8. This is an easy application of Theorem 1.6. For the details, see the proof of [K, Theorem 2.1]. Note that the argument in [K] is well known to algebraic geometers. □

Proof of Theorem 1.9. The following argument is more or less well known to algebraic geometers (see [Ha, Chapter II. Section 7]). Hence we will omit some details. We put $\mathcal{S} = \mathcal{O}_X$ and use Theorem 1.8. Then there exists a positive integer k_0 such that $\mathcal{L}^{\otimes k}$ is generated by finitely many global sections over some open neighborhood of $\overline{X_c}$. We take $P, Q \in \overline{X_c}$. Let m_P (resp. m_Q) be the ideal sheaf corresponding to P (resp. Q). We take some $d > c$. By Theorem 1.6, we may assume that the evaluation map

$$H^0(X_d, \mathcal{L}^{\otimes k}) \to \mathcal{L}^{\otimes k} \otimes \mathcal{O}_X/m_P \oplus \mathcal{L}^{\otimes k} \otimes \mathcal{O}_X/m_Q \tag{3.16}$$

is surjective for every $k \geq k_0$. By Theorem 1.6 again, we may assume that

$$H^0(X_d, m_P \otimes \mathcal{L}^{\otimes k}) \to m_P/m_P^2 \otimes \mathcal{L}^{\otimes k} \tag{3.17}$$

is surjective for every $k \geq k_0$. By (3.16), (3.17), and the compactness of $\overline{X_c}$, we can find a positive integer m_0 with the desired properties (see also [Ha, Chapter II. Proposition 7.3]). □

Corollary 1.11 is obvious by Theorem 1.9.

Proof of Corollary 1.11. If $\mathcal{L}$ is ample, then it is easy to see that $\mathcal{L}$ is positive. On the other hand, if $\mathcal{L}$ is positive, then $\mathcal{L}$ is ample by the embedding theorem: Theorem 1.9. We note that every compact analytic space is automatically weakly 1-complete. □

We prove Theorem 1.12 as an application of Theorem 1.6.

Proof of Theorem 1.12. We can see X as a complex analytic space by Serre's GAGA. By Corollary 1.11, $\mathcal{L}$ is positive. Since X is compact, by Theorem 1.6, we can find m_0 such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{M}') = 0$ holds for $i > 0$, $m \geq m_0$, and for every semipositive line bundle $\mathcal{M}'$ on X. Let $\mathcal{M}$ be a nef line bundle. Then $\mathcal{L} \otimes \mathcal{M}$ is ample. This means that $\mathcal{L} \otimes \mathcal{M}$ is positive. Hence it is obviously semipositive. Therefore, we have $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{L} \otimes \mathcal{M}) = 0$ for $i > 0$ and $m \geq m_0$. Thus, it is sufficient to put $m(\mathcal{F}, \mathcal{L}) := m_0 + 1$. □

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