Foundation of the minimal model program
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Osamu Fujino

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujino@math.kyoto-u.ac.jp

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Abstract. We discuss various vanishing theorems. Then we establish the fundamental theorems, that is, various Kodaira type vanishing theorems, the cone and contraction theorem, and so on, for quasi-log schemes.

Preface

This book is a completely revised version of the author's unpublished manuscript:

- *Osamu Fujino, Introduction to the minimal model program for log canonical pairs, preprint 2008.*

We note that the above unpublished manuscript is an expanded version of the composition of

- *Osamu Fujino, Vanishing and injectivity theorems for LMMP, preprint 2007*

and

- *Osamu Fujino, Notes on the log minimal model program, preprint 2007.*

We also note that this book is not an introductory text book of the minimal model program.

One of the main purposes of this book is to establish the fundamental theorems, that is, various Kodaira type vanishing theorems, the cone and contraction theorem, and so on, for quasi-log schemes. The notion of quasi-log schemes was introduced by Florin Ambro in his epoch-making paper:

- *Florin Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova* ***240*** *(2003), 220–239.*

The theory of quasi-log schemes is extremely powerful. Unfortunately, it has not been popular yet because Ambro's paper has several difficulties. Moreover, the author's paper:

- *Osame Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci.* ***47*** *(2011), no. 3, 727–789*

recovered the main result of Ambro's paper, that is, the cone and contraction theorem for normal pairs, without using the theory of quasi-log schemes. Note that the author's approach in the above paper is sufficient for the fundamental theorems of the minimal model program for log canonical pairs.

Recently, the author proved that every quasi-projective semi log canonical pair has a natural quasi-log structure which is compatible with the original semi log canonical structure in

- *Osamu Fujino, Fundamental theorems for semi log canonical pairs, Algebraic Geometry **1** (2014), no. 2, 194–228.*

This result shows that the theory of quasi-log schemes is indispensable for the study of semi log canonical pairs. Now the importance of the theory of quasi-log schemes is increasing. In this book, we will establish the foundation of quasi-log schemes.

One of the author's main contributions in the above papers is to introduce the theory of mixed Hodge structures on cohomology groups with compact support to the minimal model program systematically. By pursuing this approach, we can naturally obtain a correct generalization of the Fujita–Kawamata semipositivity theorem in

- *Osamu Fujino, Taro Fujisawa, Variations of mixed Hodge structure and semipositivity theorems, to appear in Publ. Res. Inst. Math. Sci.*

This new powerful semipositivity theorem leads to the proof of the projectivity of the coarse moduli spaces of stable varieties in

- *Osamu Fujino, Semipositivity theorems for moduli problems, preprint 2012.*

Note that a stable variety is a projective semi log canonical variety with ample canonical divisor.

Anyway, the theory of quasi-log schemes seems to be indispensable for the study of higher-dimensional algebraic varieties and its importance is increasing now.

On page 57 in

- *János Kollár, Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge University Press, 1998*,

which is a standard text book on the minimal model program, the authors wrote:

> Log canonical: This is the largest class where discrepancy still makes sense. It contains many cases that are rather complicated from the cohomological point of view. Therefore it is very hard to work with.

On page 209, they also wrote:

> The theory of these so-called semi-log canonical (slc for short) pairs is not very much different from the lc case but it needs some foundational work.

By the author's series of papers including this book, we greatly improve the situation around log canonical pairs and semi log canonical pairs from the cohomological point of view.

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Osamu Fujino

Contents

Guide for the reader

In Chapter 1, we start with Mori's cone theorem for smooth projective varieties and his contraction theorem for smooth threefolds. It is one of the starting points of the minimal model program. So the minimal model program is sometimes called Mori's program. We also explain some examples of quasi-log schemes, the motivation of our vanishing theorems, the background of this book, the author's related papers, and so on, for the reader's convenience. Chapter 2 collects several definitions and preliminary results. Almost all the topics in this chapter are well known to the experts and are indispensable for the study of the minimal model program. We recommend the reader to be familiar with them. In Chapter 3, we discuss various Kodaira type vanishing theorems and several applications. Although this chapter contains several new results and arguments, almost all the results are standard and are known to the experts. Chapter 4 is a survey on the minimal model program. We discuss the basic results of the minimal model program, the recent results by Birkar–Cascini–Hacon–M^{c}Kernan, and various results on log canonical pairs, log surfaces, semi log canonical pairs by the author, and so on, without proof. Chapter 5 is devoted to the injectivity, vanishing, and torsion-free theorems for *reducible* varieties. They are generalizations of Kollár's corresponding results from the mixed Hodge theoretic viewpoint and play crucial roles in the theory of quasi-log schemes. Chapter 6 is the main part of this book. We prove the adjunction and the vanishing theorem for quasi-log schemes as applications of the results in Chapter 5. Then we establish the basepoint-free theorem, the rationality theorem, and the cone theorem for quasi-log schemes, and so on. Chapter 7 collects some supplementary results and examples. We recommend the reader who is familiar with the traditional minimal model program and is only interested in the theory of quasi-log schemes to go directly to Chapter 6.

CHAPTER 1

Introduction

The minimal model program is sometimes called *Mori's program* or *Mori theory*. This is because Shigefumi Mori's epoch-making paper [**Mo2**] is one of the starting points of the minimal model program. Therefore, we quickly review Mori's results in [**Mo1**] and [**Mo2**] in Section 1.1. In Section 1.2, we explain some basic examples of quasi-log schemes. By using the theory of quasi-log schemes, we can treat log canonical pairs, non-klt loci of log canonical pairs, semi log canonical pairs, and so on, on an equal footing. By [**F33**], the theory of quasi-log schemes seems to be indispensable for the study of semi log canonical pairs. In Section 1.3, we explain some vanishing theorems, which are much sharper than the usual Kawamata–Viehweg vanishing theorem and the algebraic version of the Nadel vanishing theorem, in order to motivate the reader to read this book. In Section 1.4, we give several historical comments on this book and the recent developments of the minimal model program for the reader's convenience. We explain the reason of the delay of the publication of this book. In Section 1.5, we compare this book with the unpublished manuscript written and circulated in 2008. In Section 1.6, we quickly review the author's related papers and results for the reader's convenience. In the final section: Section 1.7, we fix the notation and some conventions of this book.

1.1. Mori's cone and contraction theorem

In his epoch-making paper [**Mo2**], Shigefumi Mori obtained the cone and contraction theorem. It is one of the starting points of Mori's program or the minimal model program (MMP, for short).

THEOREM 1.1.1 (Cone theorem). *Let X be a smooth projective variety defined over an algebraically closed field. Then we have the following properties.*

(i) *There are at most countably many (possibly singular) rational curves C_i on X such that*
$$0 < -(C_i \cdot K_X) \leq \dim X + 1,$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

Note that $\overline{NE}(X)$ is the Kleiman–Mori cone of X, that is, the closed convex cone spanned by the numerical equivalence classes of effective 1*-cycles on X.*

(ii) *For any positive number ε and any ample Cartier divisor H on X, we have*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\varepsilon H)\geq 0} + \sum_{\mathit{finite}} \mathbb{R}_{\geq 0}[C_i].$$

The proof of Theorem 1.1.1 in [**Mo2**] depends on Mori's bend and break technique (see, for example, [**KoMo**, Chapter 1]). It was invented in [**Mo1**] to prove the Hartshorne conjecture.

Theorem 1.1.2 (Hartshorne conjecture, see [**Mo1**]). *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field. If the tangent bundle T_X is an ample vector bundle, then X is isomorphic to $\mathbb{P}^n$.*

Note that Theorem 1.1.1 contains the following highly nontrivial theorem.

Theorem 1.1.3 (Existence of rational curves). *Let X be a smooth projective variety defined over an algebraically closed field. If K_X is not nef, that is, there exists an irreducible curve C on X such that $K_X \cdot C < 0$, then X contains a* (*possibly singular*) *rational curve.*

There is no known proof of Theorem 1.1.3 which does not use positive characteristic techniques even when the characteristic of the base field is zero.

In [**Mo2**], Shigefumi Mori obtained the contraction theorem for smooth projective threefold defined over $\mathbb{C}$.

Theorem 1.1.4 (Contraction theorem, see [**KoMo**, Theorem 1.32]). *Let X be a smooth projective threefold defined over $\mathbb{C}$. Let R be any K_X-negative extremal ray of $\overline{NE}(X)$. Then there is a contraction morphism $\varphi_R : X \to Y$ associated to R.*

The following is a list of all possibilities for φ_R.

E: (Exceptional). $\dim Y = 3$, *φ_R is birational and there are five types of local behavior near the contracted surfaces.*

E1: *φ_R is the* (*inverse of the*) *blow-up of a smooth curve in the smooth projective threefold Y.*

E2: *φ_R is the* (*inverse of the*) *blow-up of a smooth point of the smooth projective threefold Y.*

E3: *φ_R is the (inverse of the) blow-up of an ordinary double point of Y. Note that an ordinary double point is locally analytically given by the equation*

$$x^2 + y^2 + z^2 + w^2 = 0.$$

E4: *φ_R is the (inverse of the) blow-up of a point of Y which is locally analytically given by the equation*

$$x^2 + y^2 + z^2 + w^3 = 0.$$

E5: *φ_R contracts a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ to a point of multiplicity 4 on Y which is locally analytically the quotient of $\mathbb{C}^3$ by the involution*

$$(x, y, z) \mapsto (-x, -y, -z).$$

C: (Conic bundle). $\dim Y = 2$ *and φ_R is a fibration whose fibers are plane conics. Of course, general fibers are smooth.*

D: (Del Pezzo fibration). $\dim Y = 1$ *and general fibers of φ_R are Del Pezzo surfaces.*

F: (Fano variety). $\dim Y = 0$*, $-K_X$ is ample. Therefore, X is a smooth Fano threefold with the Picard number $\rho(X) = 1$.*

For Mori's bend and break technique, see, for example, [**Ko7**], [**Deb**], and [**KoMo**, Chapter 1]. For the details of the results in this section, see the original papers [**Mo1**] and [**Mo2**]. We also recommend the reader to see a good survey [**Mo6**]. After the epoch-making paper [**Mo2**], Shigefumi Mori classified three-dimensional terminal singularities in [**Mo3**] (see also [**R2**]) and then established the flip theorem for terminal threefolds in [**Mo5**]. By these results with the works of Reid, Kawamata, Shokurov, and others, we obtained the existence theorem of minimal models for $\mathbb{Q}$-factorial terminal threefolds.

Note that a shortest way to prove the existence of minimal models for threefolds is now the combination of Shokurov's proof of 3-fold pl flips described in [**Cor**] and the reduction theorem explained in [**F13**]. By this method, we are released from Mori's deep classification of three-dimensional terminal singularities.

One of the main purposes of this book is to establish the cone and contraction theorem for quasi-log schemes, that is, the cone and contraction theorem for highly singular schemes.

1.2. What is a quasi-log scheme?

In this section, we informally explain why it is natural to consider *quasi-log schemes* (see Section 6.4).

Let (Z, B_Z) be a log canonical pair and let $f : V \to Z$ be a resolution with

$$K_V + S + B = f^*(K_Z + B_Z),$$

where $\mathrm{Supp}(S + B)$ is a simple normal crossing divisor, S is reduced, and $\lfloor B \rfloor \le 0$. It is very important to consider the *non-klt locus* W of the pair (Z, B_Z), that is, $W = f(S)$. We consider the short exact sequence:

$$0 \to \mathcal{O}_V(-S + \lceil -B \rceil) \to \mathcal{O}_V(\lceil -B \rceil) \to \mathcal{O}_S(\lceil -B \rceil) \to 0.$$

We put $K_S + B_S = (K_V + S + B)|_S$. In our case, $B_S = B|_S$. By the Kawamata–Viehweg vanishing theorem, we have

$$R^i f_* \mathcal{O}_V(-S + \lceil -B \rceil) = 0$$

for every $i > 0$. Since $\lceil -B \rceil$ is effective and f-exceptional, we have $f_*\mathcal{O}_V(\lceil -B \rceil) \simeq \mathcal{O}_Z$. Therefore, we obtain the following short exact sequence:

$$0 \to f_*\mathcal{O}_V(-S + \lceil -B \rceil) \to \mathcal{O}_Z \to f_*\mathcal{O}_S(\lceil -B_S \rceil) \to 0.$$

This implies

$$\mathcal{O}_W \simeq f_*\mathcal{O}_S(\lceil -B_S \rceil).$$

Note that the ideal sheaf $f_*\mathcal{O}_V(-S + \lceil -B \rceil)$ is denoted by $\mathcal{J}(Z, B_Z)$ and is called the *multiplier ideal sheaf* of the pair (Z, B_Z).

Therefore, it is natural to introduce the following notion. Precisely speaking, a qlc pair is a quasi-log scheme with only qlc singularities.

Definition 1.2.1 (Qlc pairs). A qlc pair $[X, \omega]$ is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω such that there is a proper morphism $f : (Y, B_Y) \to X$ satisfying the following conditions.

(1) Y is a simple normal crossing divisor on a smooth variety M and there exists an $\mathbb{R}$-divisor D on M such that $\mathrm{Supp}(D + Y)$ is a simple normal crossing divisor, Y and D have no common irreducible components, and $B_Y = D|_Y$.
(2) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(3) B_Y is a subboundary $\mathbb{R}$-divisor, that is, $b_i \le 1$ for every i when $B_Y = \sum b_i B_i$.
(4) $\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$, where $B_Y^{<1} = \sum_{b_i<1} b_i B_i$.

It is easy to see that the pair $[W, \omega]$, where $\omega = (K_X + B)|_W$, with $f : (S, B_S) \to W$ satisfies the definition of qlc pairs. We note that the pair $[Z, K_Z + B_Z]$ with $f : (V, S + B) \to Z$ is also a qlc pair since $f_*\mathcal{O}_V(\lceil -B \rceil) \simeq \mathcal{O}_Z$. Thus, we can treat log canonical pairs and non-klt loci of log canonical pairs in the same framework once we introduce the notion of qlc pairs.

Moreover, we have:

THEOREM 1.2.2. *Let (X, Δ) be a quasi-projective semi log canonical pair. Then $[X, K_X + \Delta]$ is naturally a qlc pair.*

Theorem 1.2.2 is the main theorem of [**F33**], which is highly non-trivial and depends on the recent development of the theory of partial resolution of singularities for reducible varieties (see [**BM**] and [**BVP**]). For the details of Theorem 1.2.2, see [**F33**] (see also Theorem 4.11.9 below).

Anyway, by Theorem 1.2.2, we can treat log canonical pairs, non-klt loci of log canonical pairs, quasi-projective semi log canonical pairs, and so on, on an equal footing by using the theory of quasi-log schemes. The author thinks that Theorem 1.2.2 drastically increased the importance of the theory of quasi-log schemes.

In this book, we establish the fundamental theorems, that is, various Kodaira type vanishing theorems, the cone and contraction theorem, and so on, for quasi-log schemes. For that purpose, we prove the Hodge theoretic injectivity theorem for simple normal crossing pairs (see Theorem 5.1.1) and the injectivity, vanishing, and torsion-free theorems for simple normal crossing pairs (see Theorem 5.1.3). The main ingredient of our framework is the theory of mixed Hodge structures on cohomology with compact support.

1.3. Motivation

The following results will motivate the reader to study our new framework, which is more powerful than the traditional X-method based on the Kawamata–Viehweg vanishing theorem (see, for example, [**KMM**] and [**KoMo**]), and the theory of algebraic multiplier ideal sheaves (see, for example, [**La2**, Part Three]), which depends on the Nadel vanishing theorem.

THEOREM 1.3.1. *Let X be a normal projective variety and let B be an effective $\mathbb{R}$-divisor on X such that (X, B) is log canonical. Let L be a Cartier divisor on X. Assume that $L - (K_X + B)$ is ample. Let $\{C_i\}$ be* any *set of log canonical centers of the pair (X, B). We put $W = \bigcup C_i$ with the reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) = 0$$

for every $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. In particular, the natural restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(W, \mathcal{O}_W(L))$$

is surjective. Therefore, if (X, B) has a zero-dimensional log canonical center, then the linear system $|L|$ is not empty and the base locus of $|L|$ contains no zero-dimensional log canonical centers of (X, B).

More generally, we have:

THEOREM 1.3.2. *Let X be a normal projective variety and let B be an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let L be a Cartier divisor on X. Assume that $L - (K_X + B)$ is nef and log big with respect to the pair (X, B). Let $\mathrm{Nlc}(X, B)$ denote the non-lc locus of the pair (X, B). Let $\{C_i\}$ be* any *set of log canonical centers of the pair (X, B). We put*

$$W = \mathrm{Nlc}(X, B) \cup \bigcup C_i.$$

Then W has a natural scheme structure induced by the pair (X, B), and

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) = 0$$

holds for every $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X.

Although we did not define the scheme structure of W explicitly here, it is natural and Theorem 1.3.2 is a generalization of Theorem 1.3.1. Note that Theorem 1.3.2 is a very special case of Theorem 6.3.4. We also note that the proof of Theorem 1.3.2 is much harder than the proof of Theorem 1.3.1.

1.3.3. In Theorem 1.3.2, if we assume that W is the union of all the log canonical centers of (X, B), then $\mathcal{I}_W$ becomes the multiplier ideal sheaf $\mathcal{J}(X, B)$ of the pair (X, B). In this case, W is the non-klt locus of the pair (X, B) and the vanishing theorem in Theorem 1.3.2 is nothing but the Nadel vanishing theorem:

$$H^i(X, \mathcal{J}(X, B) \otimes \mathcal{O}_X(L)) = 0$$

for every $i > 0$. Therefore, Theorem 1.3.2 is a generalization of the Nadel vanishing theorem. It is obvious that Theorem 1.3.2 is also a generalization of the Kawamata–Viehweg vanishing theorem. Note that $\mathcal{I}_W = \mathcal{O}_X$ when W and $\mathrm{Nlc}(X, B)$ are empty.

Let us see a simple setting to understand the difference between our new framework and the traditional one.

1.3.4. Let X be a smooth projective surface and let C_1 and C_2 be smooth curves on X. Assume that C_1 and C_2 intersect only at a point P transversally. Let L be a Cartier divisor on X such that $L-(K_X+B)$ is ample, where $B=C_1+C_2$. It is obvious that (X,B) is log canonical and P is a log canonical center of (X,B). Then, by Theorem 1.3.1, we can directly obtain

$$H^i(X,\mathcal{I}_P\otimes\mathcal{O}_X(L))=0$$

for every $i>0$, where $\mathcal{I}_P$ is the defining ideal sheaf of P on X.

In the classical framework, we prove it as follows. Let C be a general curve passing through P. We take small positive rational numbers ε and δ such that $(X,(1-\varepsilon)B+\delta C)$ is log canonical and is kawamata log terminal outside P and that P is an isolated log canonical center of $(X,(1-\varepsilon)B+\delta C)$. Since ε and δ are small,

$$L-(K_X+(1-\varepsilon)B+\delta C)$$

is still ample. By the Nadel vanishing theorem, we obtain

$$H^i(X,\mathcal{I}_P\otimes\mathcal{O}_X(L))=0$$

for every $i>0$. We note that $\mathcal{I}_P$ is nothing but the multiplier ideal sheaf associated to the pair $(X,(1-\varepsilon)B+\delta C)$.

By our new vanishing theorems (see, Theorem 1.3.1, Theorem 1.3.2, and so on), the reader will be released from annoyance of perturbing coefficients of boundary divisors.

In Chapter 5, we will generalize Kollár's torsion-free and vanishing theorem (see Theorem 5.1.3). As an application, we will prove Theorem 6.3.4, which contains Theorem 1.3.2. Note that Kollár's torsion-free and vanishing theorem is equivalent to Kollár's injectivity theorem.

Let us try to give a proof of a very special case of Theorem 1.3.1 by using Kollár's torsion-free and vanishing theorem.

THEOREM 1.3.5. *Let S be a normal projective surface which has only one simple elliptic Gorenstein singularity $Q\in S$. We put $X=S\times\mathbb{P}^1$ and $B=S\times\{0\}$. Then the pair (X,B) is log canonical. It is easy to see that $P=(Q,0)\in X$ is a log canonical center of (X,B). Let L be a Cartier divisor on X such that $L-(K_X+B)$ is ample. Then we have*

$$H^i(X,\mathcal{I}_P\otimes\mathcal{O}_X(L))=0$$

for every $i>0$, where $\mathcal{I}_P$ is the defining ideal sheaf of P on X. We note that X is not kawamata log terminal and that P is not an isolated log canonical center of (X,B).

Proof. Let $\varphi : T \to S$ be the minimal resolution. Then we can write $K_T + C = \varphi^* K_S$, where C is the φ-exceptional elliptic curve on T. We put $Y = T \times \mathbb{P}^1$ and $f = \varphi \times \mathrm{id}_{\mathbb{P}^1} : Y \to X$, where $\mathrm{id}_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ is the identity. Then f is a resolution of X and we can write

$$K_Y + B_Y + E = f^*(K_X + B),$$

where B_Y is the strict transform of B on Y and $E \simeq C \times \mathbb{P}^1$ is the exceptional divisor of f. Let $g : Z \to Y$ be the blow-up along $E \cap B_Y$. Then we can write

$$K_Z + B_Z + E_Z + F = g^*(K_Y + B_Y + E) = h^*(K_X + B),$$

where $h = f \circ g$, B_Z (resp. E_Z) is the strict transform of B_Y (resp. E) on Z, and F is the g-exceptional divisor. We note that

$$\mathcal{I}_P \simeq h_* \mathcal{O}_Z(-F) \subset h_* \mathcal{O}_Z \simeq \mathcal{O}_X.$$

Since $-F = K_Z + B_Z + E_Z - h^*(K_X + B)$, we have

$$\mathcal{I}_P \otimes \mathcal{O}_X(L) \simeq h_* \mathcal{O}_Z(K_Z + B_Z + E_Z) \otimes \mathcal{O}_X(L - (K_X + B)).$$

So, it is sufficient to prove that

$$H^i(X, h_* \mathcal{O}_Z(K_Z + B_Z + E_Z) \otimes \mathcal{L}) = 0$$

for every $i > 0$ and any ample line bundle $\mathcal{L}$ on X. We consider the short exact sequence

$$0 \to \mathcal{O}_Z(K_Z) \to \mathcal{O}_Z(K_Z + E_Z) \to \mathcal{O}_{E_Z}(K_{E_Z}) \to 0.$$

We can easily check that

$$0 \to h_* \mathcal{O}_Z(K_Z) \to h_* \mathcal{O}_Z(K_Z + E_Z) \to h_* \mathcal{O}_{E_Z}(K_{E_Z}) \to 0$$

is exact and

$$R^i h_* \mathcal{O}_Z(K_Z + E_Z) \simeq R^i h_* \mathcal{O}_{E_Z}(K_{E_Z})$$

for every $i > 0$ because $R^i h_* \mathcal{O}_Z(K_Z) = 0$ for every $i > 0$. The fact $R^i h_* \mathcal{O}_Z(K_Z) = 0$ for every $i > 0$ is a special case of Kollár's torsion-free theorem since h is birational. We can directly check that

$$R^1 h_* \mathcal{O}_{E_Z}(K_{E_Z}) \simeq R^1 f_* \mathcal{O}_E(K_E) \simeq \mathcal{O}_D(K_D),$$

where $D = Q \times \mathbb{P}^1 \subset X$. Therefore, $R^1 h_* \mathcal{O}_Z(K_Z + E_Z) \simeq \mathcal{O}_D(K_D)$ is a torsion sheaf on X. However, it is torsion-free as a sheaf on D. It is a generalization of Kollár's torsion-free theorem. We consider

$$0 \to \mathcal{O}_Z(K_Z + E_Z) \to \mathcal{O}_Z(K_Z + B_Z + E_Z) \to \mathcal{O}_{B_Z}(K_{B_Z}) \to 0.$$

We note that $B_Z \cap E_Z = \emptyset$. Thus, we have

$$\begin{aligned}0 &\to h_*\mathcal{O}_Z(K_Z+E_Z) \to h_*\mathcal{O}_Z(K_Z+B_Z+E_Z) \to h_*\mathcal{O}_{B_Z}(K_{B_Z})\\ &\xrightarrow{\delta} R^1h_*\mathcal{O}_Z(K_Z+E_Z) \to \cdots .\end{aligned}$$

Since $\operatorname{Supp} h_*\mathcal{O}_{B_Z}(K_{B_Z}) = B$, δ is a zero map by $R^1h_*\mathcal{O}_Z(K_Z+B_Z) \simeq \mathcal{O}_D(K_D)$. Therefore, we know that the following sequence

$$0 \to h_*\mathcal{O}_Z(K_Z+E_Z) \to h_*\mathcal{O}_Z(K_Z+B_Z+E_Z) \to h_*\mathcal{O}_{B_Z}(K_{B_Z}) \to 0$$

is exact. By Kollár's vanishing theorem on B_Z, it is sufficient to prove that $H^i(X, h_*\mathcal{O}_Z(K_Z+E_Z)\otimes\mathcal{L}) = 0$ for every $i>0$ and any ample line bundle $\mathcal{L}$. We have

$$H^i(X, h_*\mathcal{O}_Z(K_Z)\otimes\mathcal{L}) = H^i(X, h_*\mathcal{O}_{E_Z}(K_{E_Z})\otimes\mathcal{L}) = 0$$

for every $i>0$ by Kollár's vanishing theorem. By the following exact sequence

$$\begin{aligned}\cdots &\to H^i(X, h_*\mathcal{O}_Z(K_Z)\otimes\mathcal{L}) \to H^i(X, h_*\mathcal{O}_Z(K_Z+E_Z)\otimes\mathcal{L})\\ &\to H^i(X, h_*\mathcal{O}_{E_Z}(K_{E_Z})\otimes\mathcal{L}) \to \cdots ,\end{aligned}$$

we obtain the desired vanishing theorem. Anyway, we have

$$H^i(X, \mathcal{I}_P\otimes\mathcal{O}_X(L)) = 0$$

for every $i>0$. $\square$

The actual proof of Theorem 1.3.2 (see Theorem 6.3.4) depends on much more sophisticated arguments of the theory of mixed Hodge structures on cohomology groups with compact support.

REMARK 1.3.6. In Theorem 1.3.5, X is log canonical and is not kawamata log terminal. Note that $D = Q\times\mathbb{P}^1 \subset X$ is a one-dimensional log canonical center of X passing through P. Therefore, in order to prove Theorem 1.3.5, we can not apply the traditional perturbation technique as in 1.3.4.

In Chapter 5, we will first generalize Kollár's injectivity theorem (see Theorem 5.1.1 and Theorem 5.1.2). Next, we will obtain a generalization of Kollár's torsion-free and vanishing theorem as an application (see Theorem 5.1.3). Finally, we will apply it to quasi-log schemes in Chapter 6.

1.4. Background

In this section, we give some historical comments on the theory of quasi-log schemes and the recent developments of the minimal model program.

In November 2001, Ambro's preprint:

- *Florin Ambro, Generalized log varieties*

appeared on the archive. It was a preprint version of [**Am1**]. I think that it did not attract so much attention when it appeared on the archive. A preprint version of [**Sh4**], which was first circulated around 2000, attracted much more attention than Ambro's preprint. In February 2002, a working seminar on Shokurov's preprint, which was organized by Alessio Corti, started in the Newton Institute. I stayed at the Newton Institute in February and March to attend the working seminar. The book [**Cored**] is an outcome of this working seminar.

On October 5, 2006, a preprint version of [**BCHM**] appeared on the archive. In November, I invited Hiromichi Takagi to Nagoya from Tokyo and tried to understand the preprint. Although it was much more complicated than the published version, we soon recognized that it is essentially correct. This meant that I lost my goal in life. In December 2006, Christopher Hacon and James McKernan gave talks on [**BCHM**] at Echigo Yuzawa in Japan. In January 2007, Hiromichi Takagi gave a series of lectures on [**BCHM**] for graduate students in Kyoto. I visited Kyoto to attend his lectures. If I remember correctly, Masayuki Kawakita had already understood [**BCHM**] in January 2007. In March 2007, Caucher Birkar visited Japan and gave several talks on his results in Tokyo and Kyoto. In Japan, a preprint version of [**BCHM**] was digested quickly. We note that Hiromichi Takagi, Masayuki Kawakita, and I were the participants of the working seminar on Shokurov's preprint ([**Sh4**]) in the Newton Institute in 2002. After I read a preprint version of [**BCHM**], I decided to establish vanishing theorems sufficient for the theory of quasi-log schemes. We had already known that Ambro's paper [**Am1**] contains various difficulties. In April 2007, I finished a preprint version of [**F14**] and sent it to some experts. Then I visited MSRI to attend a workshop. The title of the workshop is *Hot topics: Minimal and Canonical Models in Algebraic Geometry*. Of course, I tried to publish [**F14**]. Unfortunately, the referees did not understand the importance of [**F14**]. I think that many experts including the referees were busy in reading [**BCHM**] and were not interested in [**F14**] in 2007. So I changed my plan and decided to combine [**F14**] and [**F15**] and publish it as a book. In June 2008, I sent a preliminary version of [**F17**], which is version 2.0, to some experts including János Kollár. He kindly gave me some comments although I did not understand them. After I moved to Kyoto from Nagoya in October, I visited Princeton to ask advice to János Kollár in November. When I visited Princeton, he was preparing [**KoKo**] and gave me a copy of a draft. During my stay at Princeton, he asked Christopher Hacon about the existence of dlt blow-ups by e-mail. He gave me a copy of the e-mail from Hacon which proved the existence of dlt blow-ups. In December 2008, I suddenly came up with a good idea when I attended Professor Hironaka's talk at RIMS on the resolution of singularities. Then I soon got a very short proof of the basepoint-free theorem for log canonical pairs without using the theory of quasi-log schemes (see [**F27**]). By using dlt blow-ups, I succeeded in proving the fundamental theorems for log canonical pairs very easily (see [**F27**]). In [**F28**], I recovered the main result of [**Am1**], that is, the fundamental theorems for normal pairs, and got some generalizations without using the theory of quasi-log schemes. Therefore, I lost much of my interests in the theory of quasi-log schemes. This is the main reason of the delay of the revision and publication of [**F17**]. In May 2011, János Kollár informed me of the development of the theory of partial resolution of singularities for reducible varieties in Kyoto. It looked very attractive for me. In September, a preprint version of [**BVP**] appeared on the archive. By using this new result, in January 2012, I proved that every quasi-projective semi log canonical pair has a natural quasi-log structure with only quasi-log canonical singularities (see [**F33**]). This result shows that the theory of quasi-log schemes is indispensable for the cohomological study of semi log canonical pairs.

After I wrote [**F17**], the minimal model theory for log canonical pairs has developed. For the details, see, for example, [**Bir4**], [**F38**], [**FG1**], [**FG2**], [**HaX1**], [**HaX2**], [**HaMcX**], [**Ko13**], and so on.

1.5. Comparison with the unpublished manuscript

In this section, we compare this book with the author's unpublished manuscript:

- *Osamu Fujino, Introduction to the minimal model program for log canonical pairs, preprint 2008*

for the reader's convenience. The version 6.01 of the above manuscript (see [**F17**]), which was circulated in January 2009, is available from arXiv.org. We think that [**F17**] has already been referred and used in many papers.

This book does not cover Subsections 3.1.4, 3.2.6, and 3.2.7 in [**F17**]. Subsection 3.1.4 in [**F17**] is included in [**F38**, Section 7] with some revisions. Subsections 3.2.6 and 3.2.7 in [**F17**] are essentially contained in [**F39**]. For the details, see [**F38**] and [**F39**].

Chapter 2 of [**F17**] is now the main part of Chapter 5 in this book. Note that we greatly revised the proof of the Hodge theoretic injectivity theorems, which were called the fundamental injectivity theorems in [**F17**]. Please compare [**F17**, Section 2.3] with Section 5.4. Note that the results in Chapter 5 are better than those in Chapter 2 of [**F17**].

Chapter 6 of this book consists of Section 3.2 and Section 3.3 in [**F17**] and Section 4.1 in [**F17**] with several revisions. Of course, the quality of Chapter 6 of this book is much better than that of the corresponding part of [**F17**].

Chapter 3 except Section 3.13 and Section 3.15 is new. Although Chapter 3 contains some new arguments and some new results, almost all results are standard or known to the experts. We wrote Chapter 3 for the reader's convenience.

In this book, we expanded the explanation of the minimal model program compared with [**F17**]. It is Chapter 4 of this book. Chapter 4 contains many results obtained after [**F17**] was written in 2008. We hope that Chapter 4 will help the reader understand the recent developments of the minimal model program.

1.6. Related papers

In this section, we review the author's related papers for the reader's convenience.

In [**F6**, Section 2], we obtained some special cases of the torsion-free theorem for log canonical pairs and Kollár type vanishing theorem for log canonical pairs. The semipositivity theorem in [**F6**] is now completely generalized in [**FF**] (see also [**FFS**]). The paper [**FF**] is in the same framework as [**F32**], [**F36**], and this book. Therefore, we recommend the reader to see [**FF**] after reading this book. The paper [**F23**] is a survey article of the theory of quasi-log schemes. We recommend the reader to see [**F23**] before reading Chapter 6. The two short papers [**F18**] and [**F27**] are almost sufficient for the fundamental theorems for projective log canonical pairs although the paper [**F28**] superseded [**F18**] and [**F27**]. As a nontrivial application of [**F28**], we obtained the minimal model theory for $\mathbb{Q}$-factorial surfaces in [**F29**] (see Section 4.10). The results in [**F29**] are sharper than the traditional minimal model theory for singular surfaces. In [**F19**] and [**F21**], we generalized the effective basepoint-free theorems for log canonical pairs. We can not reach these results by the traditional X-method and the theory of multiplier ideal sheaves. Chapter 5 of this book contains the main results of the papers [**F32**] and [**F36**]. However, this book does not contain applications discussed in [**F32**] and [**F36**]. In this book, we do not prove Theorem 1.2.2 (see also Section 4.11). For the details on semi log canonical pairs, see [**F33**]. For applications to moduli problems of stable varieties, see [**F35**].

In the author's recent preprint [**F39**], we clarify the definition of quasi-log structures and make the theory of quasi-log schemes more flexible and more useful. Note that the definition of quasi-log schemes in this book is slightly different from Ambro's original one although they are equivalent. For the details of the relationship between our definition and Ambro's original one, see [**F39**]. In [**F40**], we introduce various new operations for quasi-log structures. Then we prove the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes as an application (see Section 6.9). We note that the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes was proved under some extra assumptions in [**F17**] and in this book (see Section 6.9).

1.7. Notation and convention

We fix the notation and the convention of this book.

1.7.1 (Schemes and varieties). A *scheme* means a separated scheme of finite type over an algebraically closed field k. A *variety* means a reduced scheme, that is, a reduced separated scheme of finite type over an algebraically closed field k. We note that a variety in this book may be reducible and is not always equidimensional. However, we sometimes implicitly assume that a variety is irreducible without mentioning it explicitly if there is no risk of confusion. If it is not explicitly stated, then the field k is the complex number field $\mathbb{C}$. We note that, by using the Lefschetz principle, we can extend almost all the results over $\mathbb{C}$ in this book to the case when k is an arbitrary algebraically closed field of characteristic zero.

1.7.2 (Birational map). A *birational map* $f : X \dashrightarrow Y$ between schemes means that f is a rational map such that there are Zariski open dense subsets U of X and V of Y with $f : U \xrightarrow{\simeq} V$.

1.7.3 (Exceptional locus). For a birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the set

$$\{x \in X \,|\, f \text{ is not biregular at } x\},$$

that is, the set of points $\{x \in X\}$ where f^{-1} is not a morphism at $f(x)$. We usually see $\mathrm{Exc}(f)$ as a subscheme with the induced reduced structure.

1.7.4 (Pairs). A pair $[X, \omega]$ consists of a scheme X and an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) on X.

1.7.5 (Dualizing complex and dualizing sheaf). The symbol $\omega^\bullet_X$ denotes the *dualizing complex* of X. When X is an equidimensional variety with $\dim X = d$, then we put $\omega_X = \mathcal{H}^{-d}(\omega^\bullet_X)$ and call it the *dualizing sheaf* of X.

1.7.6 (see [**KoMo**, Definition 2.24]). Let X be an equidimensional variety, let $f : Y \to X$ be a (not necessarily proper) birational morphism from a normal variety Y, and let E be a prime divisor on Y. Any such E is called a *divisor over* X. The closure of $f(E) \subset X$ is called the *center* of E on X.

1.7.7 (... for every $m \gg 0$). The expression '... for every $m \gg 0$' means that 'there exists a positive number m_0 such that ... for every $m \geq m_0$.'

1.7.8 ($\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$). The set of integers (resp. rational numbers or real numbers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{Q}$ or $\mathbb{R}$). The set of non-negative (resp. positive) real numbers is denoted by $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{>0}$). Of course, $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Z}_{>0}$) is the set of non-negative (resp. positive) integers.

CHAPTER 2

Preliminaries

In this chapter, we collect the basic definitions of the minimal model program for the reader's convenience.

In Section 2.1, we recall some basic definitions and properties of $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors. The use of $\mathbb{R}$-divisors is indispensable for the recent developments of the minimal model program. Moreover, we have to treat $\mathbb{R}$-divisors on reducible non-normal varieties in this book. In Section 2.2, we recall some basic definitions and properties of the Kleiman–Mori cone. Note that Kleiman's famous ampleness criterion does not always hold for complete *non-projective* singular algebraic varieties. In Section 2.3, we discuss discrepancy coefficients, singularities of pairs, negativity lemmas, and so on. They are very important in the minimal model theory. In Section 2.4, we recall the Iitaka dimension, the numerical Iitaka dimension, movable divisors, pseudo-effective divisors, Nakayama's numerical dimension, and so on.

2.1. Divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors

Let us start with the definition of simple normal crossing divisors and normal crossing divisors.

Definition 2.1.1 (Simple normal crossing divisors and normal crossing divisors). Let X be a smooth algebraic variety. A reduced effective Cartier divisor D on X is said to be a *simple normal crossing divisor* (resp. *normal crossing divisor*) if for each closed point p of X, a local defining equation f of D at p can be written as

$$f = z_1 \cdots z_{j_p}$$

in $\mathcal{O}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{X,p}$), where $\{z_1, \cdots, z_{j_p}\}$ is a part of a regular system of parameters.

Note that the notion of $\mathbb{Q}$-factoriality plays important roles in the minimal model program.

Definition 2.1.2 ($\mathbb{Q}$-factoriality). A normal variety X is said to be $\mathbb{Q}$-*factorial* if every prime divisor D on X is $\mathbb{Q}$-Cartier, that is, some non-zero multiple of D is Cartier.

Example 2.1.3 shows that the notion of $\mathbb{Q}$-factoriality is very subtle.

Example 2.1.3 (cf. [Ka3]). We consider

$$X = \{(x, y, z, w) \in \mathbb{C}^4 \,|\, xy + zw + z^3 + w^3 = 0\}.$$

Claim. *The algebraic variety X is $\mathbb{Q}$-factorial. More precisely, X is factorial, that is,*

$$R = \mathbb{C}[x, y, z, w]/(xy + zw + z^3 + w^3)$$

is a UFD.

Proof of Claim. By Nagata's lemma (see [Mum2, p. 196]), it is sufficient to see that $x \cdot R$ is a prime ideal of R and $R[1/x]$ is a UFD. It is an easy exercise. $\square$

Claim. *Let X^{an} be the associated analytic space of X. Then X^{an} is not analytically $\mathbb{Q}$-factorial.*

Proof of Claim. We consider a germ of X^{an} around the origin. Then X^{an} is local analytically isomorphic to $(xy - uv = 0) \subset \mathbb{C}^4$. Therefore, X^{an} is not $\mathbb{Q}$-factorial since the two divisors $(x = u = 0)$ and $(y = v = 0)$ intersect at a single point. Note that two $\mathbb{Q}$-Cartier divisors must intersect each other in codimension one. $\square$

Lemma 2.1.4 is well known and is sometimes very useful. For other proofs, see [Ka2, Proposition 5.8], [KoMo, Corollary 2.63], and so on.

Lemma 2.1.4. *Let $f : X \to Y$ be a birational morphism between normal varieties. Assume that Y is $\mathbb{Q}$-factorial. Then the exceptional locus $\mathrm{Exc}(f)$ of f is of pure codimension one.*

Proof. Let $x \in \mathrm{Exc}(f)$ be a point. Without loss of generality, we may assume that X is affine by replacing X with an affine neighborhood of x. We assume that $X \subset \mathbb{C}^N$, with coordinates $t_1, \cdots, t_N$, and that $g = f^{-1}$ is the map given by $t_i = g_i$ for $i = 1, \cdots, N$, with $g_i \in \mathbb{C}(Y)$. It is obvious that $g_i = g^* t_i$. We put $y = f(x)$. Since $f^{-1} = g$ is not regular at y, we may assume that g_1 is not regular at y. By assumption, the divisor class group of $\mathcal{O}_{Y,y}$ is torsion. Therefore, we can write

$$g_1^m = \frac{u}{v}$$

for some positive integer m and some relatively prime elements $u, v \in \mathcal{O}_{Y,y}$. Since g_1 is not regular at y, we have $v(y) = 0$. Note that Y is normal. Therefore, $(u = v = 0)$ has codimension two in Y and

$$(f^* u = f^* v = 0) = (t_1^m f^* v = f^* v = 0) \supset (f^* v = 0) \ni x$$

has codimension one at x. $\square$

As is well-known, the notion of $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors is indispensable for the minimal model program.

Definition 2.1.5 ($\mathbb{Q}$-Cartier divisors and $\mathbb{R}$-Cartier divisors). An $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) divisor D on a scheme X is a finite $\mathbb{R}$-linear ($\mathbb{Q}$-linear) combination of Cartier divisors.

Let us recall the definition of ample $\mathbb{R}$-divisors.

Definition 2.1.6 (Ample $\mathbb{R}$-divisors). Let $\pi : X \to S$ be a morphism between schemes. An $\mathbb{R}$-Cartier divisor D on a scheme X is said to be *π-ample* if D is a finite $\mathbb{R}_{>0}$-linear combination of π-ample Cartier divisors on X. We simply say that D is *ample* when S is a point.

We need various operations of $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors in this book.

2.1.7 ($\mathbb{Q}$-divisors and $\mathbb{R}$-divisors). Let B_1 and B_2 be two $\mathbb{R}$-Cartier divisors on a scheme X. Then B_1 is linearly (resp. $\mathbb{Q}$-linearly, or $\mathbb{R}$-linearly) equivalent to B_2, denoted by $B_1 \sim B_2$ (resp. $B_1 \sim_{\mathbb{Q}} B_2$, or $B_1 \sim_{\mathbb{R}} B_2$) if

$$B_1 = B_2 + \sum_{i=1}^{k} r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ and $\mathcal{K}_X^*$ is the sheaf of invertible elements in the sheaf of rings $\mathcal{K}_X$. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$.

Let $f : X \to Y$ be a morphism between schemes. If there is an $\mathbb{R}$-Cartier divisor B on Y such that

$$B_1 \sim_{\mathbb{R}} B_2 + f^*B,$$

then B_1 is said to be *relatively $\mathbb{R}$-linearly equivalent* to B_2. It is denoted by $B_1 \sim_{\mathbb{R},f} B_2$ or $B_1 \sim_{\mathbb{R},Y} B_2$.

When X is complete, B_1 is *numerically equivalent* to B_2, denoted by $B_1 \equiv B_2$, if $B_1 \cdot C = B_2 \cdot C$ for every curve C on X (see also 2.2.1 below).

Let D be a $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor) on an equidimensional variety X, that is, D is a finite formal $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one. We define the *round-up* $\lceil D\rceil = \sum_i \lceil d_i\rceil D_i$ (resp. *round-down* $\lfloor D\rfloor = \sum_i \lfloor d_i\rfloor D_i$), where every real number x, $\lceil x\rceil$ (resp. $\lfloor x\rfloor$) is the integer defined by $x \leq \lceil x\rceil < x+1$ (resp. $x-1 < \lfloor x\rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D\rfloor$. We set

$$D^{<1} = \sum_{d_i<1} d_i D_i, \quad D^{\geq 1} = \sum_{d_i\geq 1} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i=1} D_i.$$

We can define $D^{\leq 1}$, $D^{>1}$, and so on, analogously. We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i.

2.1.8 (Big divisors). Let us collect some basic definitions and properties of big divisors. For the details, [**La1**, Section 2.2], [**Mo4**], [**Nak2**, Chapter II. §3.d], [**U**, Chapter II], and so on. For the details of big $\mathbb{R}$-divisors on (not necessarily normal) irreducible varieties, see [**F33**, Appendix A. Big $\mathbb{R}$-divisors].

Definition 2.1.9 (Big Cartier divisors). Let X be a normal complete irreducible variety and let D be a Cartier divisor on X. Then D is *big* if one of the following equivalent conditions holds.

(1) $\max_{m\in\mathbb{Z}_{>0}} \{\dim \Phi_{|mD|}(X)\} = \dim X$, where $\Phi_{|mD|} : X \dashrightarrow \mathbb{P}^N$ is the rational map associated to the linear system $|mD|$ and $\Phi_{|mD|}(X)$ is the image of $\Phi_{|mD|}$.

(2) There exist a rational number α and a positive integer m_0 such that
$$\alpha m^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(mm_0D))$$
for every $m \gg 0$.

It is well known that we can take $m_0 = 1$ in the condition (2) (see, for example, [**La1**, Corollary 2.1.38], [**Nak2**, Chapter II.3.17. Corollary], and so on).

For non-normal varieties, we need the following definition.

Definition 2.1.10 (Big Cartier divisors on non-normal varieties). Let X be a complete irreducible variety and let D be a Cartier divisor on X. Then D is *big* if ν^*D is big on X^ν, where $\nu : X^\nu \to X$ is the normalization.

Before we define big $\mathbb{R}$-divisors, let us recall the definition of big $\mathbb{Q}$-divisors.

Definition 2.1.11 (Big $\mathbb{Q}$-divisors). Let X be a complete irreducible variety and let D be a $\mathbb{Q}$-Cartier divisor on X. Then D is *big* if mD is a big Cartier divisor for some positive integer m.

We note the following obvious lemma.

LEMMA 2.1.12. *Let $f : W \to V$ be a birational morphism between normal complete irreducible varieties and let D be a $\mathbb{Q}$-Cartier divisor on V. Then D is big if and only if so is f^*D.*

Next, let us define big $\mathbb{R}$-divisors.

DEFINITION 2.1.13 (Big $\mathbb{R}$-divisors on complete varieties). An $\mathbb{R}$-Cartier divisor D on a complete irreducible variety X is *big* if it can be written in the form

$$D = \sum_i a_i D_i$$

where each D_i is a big Cartier divisor and a_i is a positive real number for every i.

REMARK 2.1.14. Definition 2.1.13 is compatible with Definition 2.1.11. This means that if a $\mathbb{Q}$-Cartier divisor D is big in the sense of Definition 2.1.13 then D is big in the sense of Definition 2.1.11. For the details, see [**F33**, Appendix A].

We can check the following proposition.

PROPOSITION 2.1.15 (see [**F33**, Proposition A.14]). *Let D be an $\mathbb{R}$-Cartier divisor on a* normal *complete irreducible variety X. Then the following conditions are equivalent.*

(1) *D is big.*
(2) *There exist a positive rational number α and a positive integer m_0 such that*

$$\alpha m^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(\lfloor m m_0 D \rfloor))$$

for every $m \gg 0$.

Note that we do not assume that X is *projective* in Proposition 2.1.15. We omit the proof of Proposition 2.1.15 here since we do not use it explicitly in this book. For the proof, see [**F33**, Proposition A.14].

DEFINITION 2.1.16 (Big $\mathbb{R}$-divisors on complete reducible varieties). Let X be a complete reducible variety and let D be an $\mathbb{R}$-Cartier divisor on X. Then D is *big* if $D|_{X_i}$ is big for every irreducible component X_i of X.

DEFINITION 2.1.17 (Relative big $\mathbb{R}$-divisors). Let $\pi : X \to S$ be a proper morphism from a variety X to a scheme S and let D be an $\mathbb{R}$-Cartier divisor on X. Then D is called *π-big* or *big over S* if the restriction of D to the geometric generic fiber of every irreducible component of $\pi(X)$ is big.

The following version of Kodaira's lemma is suitable for our purposes.

LEMMA 2.1.18 (Kodaira). *Let X be a projective irreducible variety and let D be a big Cartier divisor on X. Let H be any Cartier divisor on X. Then there exists a positive integer a such that*

$$H^0(X, \mathcal{O}_X(aD - H)) \neq 0.$$

Therefore, we can write $aD \sim H + E$ for some effective Cartier divisor E on X.

PROOF. We put $n = \dim X$, Let $\nu : X^\nu \to X$ be the normalization. We consider the following short exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu} \to \delta \to 0.$$

Note that $\dim \operatorname{Supp} \delta < n$. By taking $\otimes \mathcal{O}_X(mD)$ and taking cohomology, we obtain

$$\begin{aligned} 0 &\to H^0(X, \mathcal{O}_X(mD)) \to H^0(X^\nu, \mathcal{O}_{X^\nu}(m\nu^* D)) \\ &\to H^0(X, \delta \otimes \mathcal{O}_X(mD)) \to \cdots . \end{aligned}$$

Since $\nu^* D$ is big by definition, there is a positive rational number α_1 such that $\dim H^0(X^\nu, \mathcal{O}_{X^\nu}(m\nu^* D)) \geq \alpha_1 m^n$ for every $m \gg 0$. Since $\dim \operatorname{Supp} \delta < n$, there is a positive rational number α_2 such that $\dim H^0(X, \delta \otimes \mathcal{O}_X(mD)) \leq \alpha_2 m^n$ for every $m \gg 0$. Therefore, there is a positive rational number α_3 such that $\dim H^0(X, \mathcal{O}_X(mD)) \geq \alpha_3 m^n$ for every $m \gg 0$. By adding a sufficiently ample divisor to H, we may assume that H is a very ample effective divisor on X. Note that there is a positive rational number α_4 such that $\dim H^0(H, \mathcal{O}_H(mD)) \leq \alpha_4 m^{n-1}$ since $\dim H = n - 1$. Then, by

$$\begin{aligned} 0 &\to H^0(X, \mathcal{O}_X(mD - H)) \to H^0(X, \mathcal{O}_X(mD)) \\ &\to H^0(H, \mathcal{O}_H(mD)) \to \cdots , \end{aligned}$$

we obtain $H^0(X, \mathcal{O}_X(aD - H)) \neq 0$ for some positive integer a. $\square$

We will repeatedly use Kodaira's lemma (Lemma 2.1.18) and its variants throughout this book.

2.1.19 (Semi-ample divisors). Let us recall some basic properties of semi-ample divisors. In this book, we have to deal with semi-ample $\mathbb{R}$-divisors.

DEFINITION 2.1.20 (Semi-ample $\mathbb{R}$-divisors). Let $\pi : X \to S$ be a morphism between schemes. An $\mathbb{R}$-Cartier divisor D on X is π-*semi-ample* if $D \sim_{\mathbb{R}} \sum_i a_i D_i$, where D_i is a π-semi-ample Cartier divisor on

X and a_i is a positive real number for every i. We simply say that D is *semi-ample* when S is a point.

Remark 2.1.21. In Definition 2.1.20, we can replace $D \sim_{\mathbb{R}} \sum_i a_i D_i$ with $D = \sum_i a_i D_i$ since every principal Cartier divisor on X is π-semi-ample.

We note the following two lemmas.

Lemma 2.1.22 (see [F28, Lemma 4.13]). *Let D be an $\mathbb{R}$-Cartier divisor on X and let $\pi : X \to S$ be a morphism between schemes. Then the following conditions are equivalent.*

(1) *D is π-semi-ample.*
(2) *There exists a morphism $f : X \to Y$ over S such that $D \sim_{\mathbb{R}} f^*A$, where A is an $\mathbb{R}$-Cartier divisor on Y which is ample over S.*

Proof. It is obvious that (1) follows from (2). If D is π-semi-ample, then we can write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition 2.1.20. By replacing D_i with its multiple, we may assume that $\pi^*\pi_*\mathcal{O}_X(D_i) \to \mathcal{O}_X(D_i)$ is surjective for every i. Let $f : X \to Y$ be a morphism over S obtained by the surjection $\pi^*\pi_*\mathcal{O}_X(\sum_i D_i) \to \mathcal{O}_X(\sum_i D_i)$. Then it is easy to see that $f : Y \to X$ has the desired property. $\square$

Lemma 2.1.23 (see [F28, Lemma 4.14]). *Let D be a Cartier divisor on X and let $\pi : X \to S$ be a morphism between schemes. If D is π-semi-ample in the sense of Definition 2.1.20, then D is π-semi-ample in the usual sense, that is, $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective for some positive integer m. This means that Definition 2.1.20 is compatible with the usual definition.*

Proof. We write $D \sim_{\mathbb{R}} \sum_i a_i D_i$ as in Definition 2.1.20. Let $f : X \to Y$ be a morphism in Lemma 2.1.22 (2). By taking the Stein factorization, we may assume that f has connected fibers. By construction, $D_i \sim_{\mathbb{Q},f} 0$ for every i. By replacing D_i with its multiple, we may assume that $D_i \sim f^*D'_i$ for some Cartier divisor D'_i on Y for every i. Let U be any Zariski open set of Y on which $D'_i \sim 0$ for every i. On $f^{-1}(U)$, we have $D \sim_{\mathbb{R}} 0$. This implies $D \sim_{\mathbb{Q}} 0$ on $f^{-1}(U)$ since D is Cartier. Therefore, there exists a positive integer m such that $f^*f_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective. By this surjection, we have $mD \sim f^*A$ for a Cartier divisor A on Y which is ample over S. This means that D is π-semi-ample in the usual sense. $\square$

2.2. Kleiman–Mori cone

In this short section, we explain the Kleiman–Mori cone and give some interesting examples.

2.2.1 (Kleiman–Mori cone, see [**Kle**]). Let X be a scheme over $\mathbb{C}$ and let $\pi : X \to S$ be a proper morphism between schemes. Let $\mathrm{Pic}(X)$ be the group of line bundles on X. Take a complete curve on X which is mapped to a point by π. For $\mathcal{L} \in \mathrm{Pic}(X)$, we define the intersection number $\mathcal{L} \cdot C = \deg_{\overline{C}} f^*\mathcal{L}$, where $f : \overline{C} \to C$ is the normalization of C. By this intersection pairing, we introduce a bilinear form

$$\cdot : \mathrm{Pic}(X) \times Z_1(X/S) \to \mathbb{Z},$$

where $Z_1(X/S)$ is the free abelian group generated by integral curves which are mapped to points on S by π.

Now we have the notion of numerical equivalence both in $Z_1(X/S)$ and in $\mathrm{Pic}(X)$, which is denoted by $\equiv$, and we obtain a perfect pairing

$$N^1(X/S) \times N_1(X/S) \to \mathbb{R},$$

where

$$N^1(X/S) = \{\mathrm{Pic}(X)/\equiv\} \otimes \mathbb{R} \quad \text{and} \quad N_1(X/S) = \{Z_1(X/S)/\equiv\} \otimes \mathbb{R},$$

namely $N^1(X/S)$ and $N_1(X/S)$ are dual to each other through this intersection pairing. It is well known that

$$\dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S) < \infty.$$

We write

$$\rho(X/S) = \dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S).$$

We define the Kleiman–Mori cone $\overline{NE}(X/S)$ of $\pi : X \to S$ as the closed convex cone in $N_1(X/S)$ generated by integral curves on X which are mapped to points on S by π. When $S = \mathrm{Spec}\,\mathbb{C}$, we drop $/\,\mathrm{Spec}\,\mathbb{C}$ from the notation, e.g., we simply write $N_1(X)$ instead of $N_1(X/\mathrm{Spec}\,\mathbb{C})$.

Definition 2.2.2. An element $D \in N^1(X/S)$ is called *π-nef* (or *relatively nef for π*), if $D \geq 0$ on $\overline{NE}(X/S)$. When $S = \mathrm{Spec}\,\mathbb{C}$, we simply say that D is *nef*.

Kleiman's ampleness criterion is an important result.

Theorem 2.2.3 (Kleiman's criterion for ampleness, see [**Kle**]). *Let $\pi : X \to S$ be a* projective *morphism between schemes. Then $\mathcal{L} \in \mathrm{Pic}(X)$ is π-ample if and only if the numerical class of $\mathcal{L}$ in $N^1(X/S)$ gives a positive function on $\overline{NE}(X/S) \setminus \{0\}$.*

REMARK 2.2.4. Let $\pi : X \to S$ be a projective morphism between schemes. Let D be a Cartier divisor on X. If D is π-ample in the sense of Definition 2.1.6, then D is a π-ample Cartier divisor in the usual sense by Theorem 2.2.3.

In Theorem 2.2.3, we have to assume that $\pi : X \to S$ is *projective* since there are complete *non-projective* algebraic varieties for which Kleiman's criterion does not hold. We recall the explicit example given in [**F9**] for the reader's convenience. For the details of this example, see [**F9**, Section 3].

EXAMPLE 2.2.5 (see [**F9**, Section 3]). We fix a lattice $N = \mathbb{Z}^3$. We take lattice points

$$\begin{aligned} &v_1 = (1,0,1), && v_2 = (0,1,1), && v_3 = (-1,-1,1),\\ &v_4 = (1,0,-1), && v_5 = (0,1,-1), && v_6 = (-1,-1,-1). \end{aligned}$$

We consider the following fan

$$\Delta = \left\{ \begin{array}{ccc} \langle v_1, v_2, v_4\rangle, & \langle v_2, v_4, v_5\rangle, & \langle v_2, v_3, v_5, v_6\rangle,\\ \langle v_1, v_3, v_4, v_6\rangle, & \langle v_1, v_2, v_3\rangle, & \langle v_4, v_5, v_6\rangle,\\ \text{and their faces} & & \end{array} \right\}.$$

Then the associated toric variety $X = X(\Delta)$ has the following properties.

(i) X is a *non-projective* complete toric variety with $\rho(X) = 1$.
(ii) There exists a Cartier divisor D on X such that D is positive on $\overline{NE}(X) \setminus \{0\}$. In particular, $\overline{NE}(X)$ is a half line.

Therefore, Kleiman's criterion for ampleness (see Theorem 2.2.3) does not hold for this X. We note that X is not $\mathbb{Q}$-factorial and that there is a torus invariant curve $C \simeq \mathbb{P}^1$ on X such that C is numerically equivalent to zero.

If X has only mild singularities, for example, X is $\mathbb{Q}$-factorial, then it is known that Theorem 2.2.3 holds even when $\pi : X \to S$ is *proper*. However, the Kleiman–Mori cone may not have enough informations when π is only proper.

EXAMPLE 2.2.6 (see [**FP**]). There exists a smooth complete toric threefold X such that $\overline{NE}(X) = N_1(X)$.

The description below helps the reader understand examples in [**FP**].

EXAMPLE 2.2.7. Let Δ be the fan in $\mathbb{R}^3$ whose rays are generated by $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1), v_5 = (-1,0,-1), v_6 =$

$(-2,-1,0)$ and whose maximal cones are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_3, v_6\rangle, \langle v_1, v_2, v_5\rangle, \langle v_1, v_5, v_6\rangle, \langle v_2, v_3, v_5\rangle, \langle v_3, v_5, v_6\rangle.$$

Note that $v_1 + v_3 + v_5 = 0$ and $v_2 + v_6 = -1v_1 + 1v_3 + 1v_5$. Thus the associated toric variety $X_1 = X(\Delta)$ is

$$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)).$$

For the details, see, for example, [**Ful**, Exercise in Section 2.4]. We take a sequence of blow-ups

$$Y \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1,$$

where f_1 is the blow-up along the ray $v_4 = (0,-1,-1) = 3v_1 + v_5 + v_6$, f_2 is along

$$v_7 = (-1,-1,-1) = \frac{1}{3}(2v_4 + v_5 + v_6),$$

and the final blow-up f_3 is along the ray

$$v_8 = (-2,-1,-1) = \frac{1}{2}(v_5 + v_6 + v_7).$$

Then we can directly check that Y is a smooth projective toric variety with $\rho(Y) = 5$.

Finally, we remove the wall $\langle v_1, v_5\rangle$ and add the new wall $\langle v_2, v_4\rangle$. Then we obtain a flop $\phi : Y \dashrightarrow X$. We note that $v_2 + v_4 - v_1 - v_5 = 0$. The toric variety X is nothing but $X(\Sigma)$ given in [**FP**, Example 1]. Thus, X is a smooth complete toric variety with $\rho(X) = 5$ and $\overline{NE}(X) = N_1(X)$. Therefore, a simple flop $\phi : Y \dashrightarrow X$ completely destroys the projectivity of Y. Note that every nef line bundle on X is trivial by $\overline{NE}(X) = N_1(X)$.

2.3. Singularities of pairs

We quickly review singularities of pairs in the minimal model program (see, for example, [**F12**], [**Ko8**], [**Ko13**], [**KoMo**], and so on). We also review the negativity lemmas. They are very important in the minimal model program.

First, let us recall the definition of discrepancy and total discrepancy of the pair (X, Δ).

DEFINITION 2.3.1 (Canonical divisor). Let X be a normal variety of dimension n. The *canonical divisor* K_X on X is a Weil divisor such that $\mathcal{O}_{X_{\mathrm{reg}}}(K_X) \simeq \Omega^n_{X_{\mathrm{reg}}}$, where X_{reg} is the smooth locus of X. Note that the canonical divisor K_X is well-defined up to linear equivalence.

DEFINITION 2.3.2 (Discrepancy). Let (X, Δ) be a pair where X is a normal variety and Δ is an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose $f : Y \to X$ is a resolution. We can choose Weil divisors K_Y and K_X such that $f_*K_Y = K_X$. Then, we can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i.$$

This formula means that $\sum_i a(E_i, X, \Delta)E_i$ is defined by

$$\sum_i a(E_i, X, \Delta)E_i = K_Y - f^*(K_X + \Delta).$$

Note that $f^*(K_X + \Delta)$ is a well-defined $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y since $K_X + \Delta$ is $\mathbb{R}$-Cartier. The real number $a(E, X, \Delta)$ is called *discrepancy* of E with respect to (X, Δ). The *discrepancy* of (X, Δ) is given by

$$\mathrm{discrep}(X, \Delta) = \inf_E \{a(E, X, \Delta) \,|\, E \text{ is an exceptional divisor over } X\}.$$

The *total discrepancy* of (X, Δ) is given by

$$\mathrm{totaldiscrep}(X, \Delta) = \inf_E \{a(E, X, \Delta) \,|\, E \text{ is a divisor over } X\}.$$

We note that it is indispensable to understand how to calculate discrepancies for the study of the minimal model program.

LEMMA 2.3.3 ([KoMo, Corollary 2.31 (1)]). *Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then, either*

$$\mathrm{discrep}(X, \Delta) = -\infty$$

or

$$-1 \le \mathrm{totaldiscrep}(X, \Delta) \le \mathrm{discrep}(X, \Delta) \le 1.$$

PROOF. Note that $\mathrm{totaldiscrep}(X, \Delta) \le \mathrm{discrep}(X, \Delta)$ is obvious by definition. By taking a blow-up whose center is of codimension two, intersects the set of smooth points of X, and is not contained in $\operatorname{Supp} \Delta$, we see that $\mathrm{discrep}(X, \Delta) \le 1$. We assume that E is a prime divisor over X such that $a(E, X, \Delta) = -1 - c$ with $c > 0$. We take a birational morphism $f : Y \to X$ from a smooth variety Y such that E is a prime divisor on Y. We put

$$K_Y + \Delta_Y = f^*(K_X + \Delta).$$

Let Z_0 be a codimension two subvariety contained in E but not in any other f-exceptional divisors with $Z_0 \not\subset \operatorname{Supp} f_*^{-1}\Delta$. By shrinking Y, we may assume that E and Z_0 are smooth. Let $g_1 : Y_1 \to Y$ be the blow-up along Z_0 and let E_1 be the exceptional divisor of g_1. Then

$$a(E_1, X, \Delta) = a(E_1, Y, \Delta_Y) = -c.$$

Let $Z_1 \subset Y_1$ be the intersection of E_1 and the strict transform of E. Let $g_2 : Y_2 \to Y_1$ be the blow-up along Z_1 and let $E_2 \subset Y_2$ be the exceptional divisor of g_2. Then

$$a(E_2, X, \Delta) = a(E_2, Y, \Delta_Y) = -2c.$$

By taking the blow-up whose center is the intersection of E_i and the strict transform of E as above for $i \geq 2$ repeatedly, we obtain a prime divisor E_j over X such that

$$a(E_j, X, \Delta) = -jc$$

for every $j \geq 1$. Therefore, we obtain

$$\mathrm{discrep}(X, \Delta) = -\infty$$

when $\mathrm{totaldiscrep}(X, \Delta) < -1$. □

Next, let us recall the basic definition of singularities of pairs.

Definition 2.3.4 (Singularities of pairs). Let (X, Δ) be a pair where X is a normal variety and Δ is an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{cases} \quad \text{if} \quad \mathrm{discrep}(X, \Delta) \quad \begin{cases} > 0, \\ \geq 0, \\ > -1 \quad \text{and} \quad \lfloor \Delta \rfloor = 0, \\ > -1, \\ \geq -1. \end{cases}$$

Here, plt is short for *purely log terminal*, klt is short for *kawamata log terminal*, and lc is short for *log canonical*.

Remark 2.3.5 (Log terminal singularities). If $\Delta = 0$, then the notions *klt*, *plt*, and *dlt* (see Definition 2.3.16 below) coincide. In this case, we say that X has *log terminal* (*lt*, for short) singularities.

For some inductive arguments, the notion of sub klt and sub lc is also useful.

Definition 2.3.6 (Sub klt pairs and sub lc pairs). Let (X, Δ) be a pair where X is a normal variety and Δ is a (not necessarily effective) $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is

$$\begin{cases} \text{sub klt} \\ \text{sub lc} \end{cases} \quad \text{if} \quad \mathrm{totaldiscrep}(X, \Delta) \quad \begin{cases} > -1 \\ \geq -1. \end{cases}$$

Here, sub klt is short for *sub kawamta log terminal* and sub lc is short for *sub log canonical*.

It is obvious that if (X, Δ) is sub lc (resp. sub klt) and Δ is effective then (X, Δ) is lc (resp. klt).

Remark 2.3.7. In [KoMo, Definition 2.34], Δ is not assumed to be *effective* for the definition of terminal, canonical, klt, and plt. Therefore, klt (resp. lc) in [KoMo, Definition 2.34] is nothing but *sub klt* (resp. *sub lc*) in this book.

The following lemma is well known and is very useful.

Lemma 2.3.8. *Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If there exists a resolution $f : Y \to X$ such that $\operatorname{Supp} f_*^{-1}\Delta \cup \operatorname{Exc}(f)$ is a simple normal crossing divisor on Y and that*

$$K_Y = f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i.$$

If $a(E_i, X, \Delta) > -1$ for every i, then (X, Δ) is sub klt. If $a(E_i, X, \Delta) \geq -1$ for every i, then (X, Δ) is sub lc.

Proof. It easily follows from Lemma 2.3.9. $\square$

Lemma 2.3.9 ([KoMo, Corollary 2.31 (3)]). *Let X be a smooth variety and let $\Delta = \sum_{i=1}^m a_i\Delta_i$ be an $\mathbb{R}$-divisor such that $\sum_i \Delta_i$ is a simple normal crossing divisor, $a_i \leq 1$ for every i, and Δ_i is a smooth prime divisor for every i. Then*

$$\operatorname{discrep}(X, \Delta) = \min\{1, \min_i(1 - a_i), \min_{i\neq j, \Delta_i\cap\Delta_j\neq\emptyset}(1 - a_i - a_j)\}$$

Proof. Let $r(X, \Delta)$ be the right hand side of the equality. It is easy to see that $\operatorname{discrep}(X, \Delta) \leq r(X, \Delta)$. Let E be an exceptional divisor for some birational morphism $f : Y \to X$. We have to show $a(E, X, \Delta) \geq r(X, \Delta)$. We note that $r(X, \Delta)$ does not decrease if we shrink X. Without loss of generality, we may assume that f is projective, Y is smooth, and X is affine. By elimination of indeterminacy of the rational map $f^{-1} : X \dashrightarrow Y$, we may assume that E is obtained by a succession of blow-ups along smooth irreducible centers which have simple normal crossings with the union of the exceptional divisors and the inverse image of $\operatorname{Supp}\Delta$ (see, for example, [Ko9, Corollary 3.18 and Theorem 3.35]). We write t to denote the number of the blow-ups.

$$\begin{tikzcd}
X_t \arrow[r] \arrow[d] & X_{t-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r, "f_1"] & X \\
Y \arrow[rrrru, "f"'] & & & &
\end{tikzcd}$$

Let C be the center of the first blow-up $f_1 : X_1 \to X$. After renumbering the Δ_i, we may assume that $\operatorname{codim}_X C = k \geq 2$ and that $C \subset \Delta_i$ if and only if $i \leq b$ for some $b \leq k$. Let E_1 be the exceptional divisor of $f_1 : X_1 \to X$. Then

$$a(E_1, X, \Delta) = k - 1 - \sum_{l \leq b} a_l.$$

(i) If $b \leq 0$, then $a(E_1, X, \Delta) \geq 1 \geq r(X, \Delta)$.
(ii) If $b = 1$, then $a(E_1, X, \Delta) \geq 1 - a_1 \geq r(X, \Delta)$.
(iii) If $b \geq 2$, then we have

$$\begin{aligned} a(E_1, X, \Delta) &\geq (k - b - 1) + \sum_{1 \leq l \leq b} (1 - a_l) \\ &\geq -1 + (1 - a_1) + (1 - a_2) \geq r(X, \Delta). \end{aligned}$$

Thus, the case where $t = 1$ is settled. On the other hand, if we define Δ_1 on X_1 by

$$K_{X_1} + \Delta_1 = {f_1}^*(K_X + \Delta),$$

then

$$\begin{aligned} r(X_1, \Delta_1) &\geq \min\{r(X, \Delta), 1 + a(E_1, X, \Delta) - \max_{\Delta_i \cap C \neq \emptyset} a_i\} \\ &\geq \min\{r(X, \Delta), a(E_1, X, \Delta)\} \geq r(X, \Delta). \end{aligned}$$

Note that $\operatorname{Supp} \Delta_1$ is a simple normal crossing divisor and the coefficient of E_1 in Δ_1 is $-a(E_1, X, \Delta) \leq 1$. Therefore, we have

$$a(E, X, \Delta) \geq r(X_1, \Delta_1) \geq r(X, \Delta)$$

by induction on t. $\square$

Lemma 2.3.10. *Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then there exists the largest nonempty Zariski open set U (resp. V) of X such that $(X, \Delta)|_U$ is sub lc (resp. $(X, \Delta)|_V$ is sub klt).*

Proof. Let $f : Y \to X$ be a resolution such that $\operatorname{Supp} f_*^{-1}\Delta \cup \operatorname{Exc}(f)$ is a simple normal crossing divisor on Y and that

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

We put

$$U = X \setminus \bigcup_{a_i < -1} f(E_i)$$

and

$$V = X \setminus \bigcup_{a_i \leq -1} f(E_i).$$

Then we can check that U and V are the desired Zariski open sets by Lemma 2.3.9. $\square$

2.3.11 (Multiplier ideal sheaf and non-lc ideal sheaf). Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $f:Y\to X$ be a resolution with

$$K_Y+\Delta_Y=f^*(K_X+\Delta)$$

such that $\operatorname{Supp}\Delta_Y$ is a simple normal crossing divisor on Y.

We put

$$\mathcal{J}(X,\Delta)=f_*\mathcal{O}_Y(-\lfloor\Delta_Y\rfloor).$$

Then $\mathcal{J}(X,\Delta)$ is an ideal sheaf on X and is known as the *multiplier ideal sheaf* associated to the pair (X,Δ). For the details, see [**La2**, Part Three]. It is independent of the resolution $f:Y\to X$ by the proof of Proposition 6.3.1. The closed subscheme $\operatorname{Nklt}(X,\Delta)$ defined by $\mathcal{J}(X,\Delta)$ is called the *non-klt locus* of (X,Δ). It is obvious that (X,Δ) is klt if and only if $\mathcal{J}(X,\Delta)=\mathcal{O}_X$.

We put

$$\mathcal{J}_{\mathrm{NLC}}(X,\Delta)=f_*\mathcal{O}_Y(-\lfloor\Delta_Y\rfloor+\Delta_Y^{=1})$$

and call it the *non-lc ideal sheaf* associated to the pair (X,Δ). For the details, see [**F20**]. It is independent of the resolution $f:Y\to X$ by Proposition 6.3.1. The closed subscheme $\operatorname{Nlc}(X,\Delta)$ defined by $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)$ is called the *non-lc locus* of (X,Δ). It is obvious that (X,Δ) is log canonical if and only if $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)=\mathcal{O}_X$.

In the recent minimal model program, the notion of log canonical centers plays an important role.

DEFINITION 2.3.12 (Log canonical centers). Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let U be the Zariski open set as in Lemma 2.3.10. If there exist a resolution $f:Y\to X$ and a divisor E_{i_0} on Y such that $a(E_{i_0},X,\Delta)=-1$ and $f(E_{i_0})\cap U\neq\emptyset$. Then $C=f(E_{i_0})$ is called a *log canonical center* (an *lc center*, for short) of the pair (X,Δ). A log canonical center which is a minimal element with respect to the inclusion is called a *minimal log canonical center* (a *minimal lc center*, for short).

The notion of *log canonical strata* is useful in this book.

DEFINITION 2.3.13 (Log canonical strata). Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. A closed subset W of X is called a *log canonical stratum* (an *lc stratum*, for short) of the pair (X,Δ) if W is X itself or is a log canonical center of the pair (X,Δ).

We note the notion of non-klt centers.

Definition 2.3.14 (Non-klt centers). Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. If there exist a resolution $f: Y \to X$ and a divisor E_{i_0} on Y such that $a(E_{i_0}, X, \Delta) \leq -1$. Then $C = f(E_{i_0})$ is called a *non-klt center* of the pair (X, Δ).

It is obvious that any log canonical center is a non-klt center. However, a non-klt center is not always a log canonical center.

2.3.15 (Divisorial log terminal pairs). Let us recall the definition of divisorial log terminal pairs.

Definition 2.3.16 (Divisorial log terminal pairs). Let X be a normal variety and let Δ be a boundary $\mathbb{R}$-divisor such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. If there exists a resolution $f: Y \to X$ such that

(i) both $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}\Delta)$ are simple normal crossing divisors on Y, and
(ii) $a(E, X, \Delta) > -1$ for every exceptional divisor $E \subset Y$,

then (X, Δ) is called *divisorial log terminal* (*dlt*, for short).

The assumption that $\mathrm{Exc}(f)$ is a divisor in Definition 2.3.16 (i) is very important. See Example 3.13.9 below.

Remark 2.3.17. By Lemma 2.3.9, it is easy to see that a dlt pair (X, Δ) is log canonical.

Remark 2.3.18. In Definition 2.3.16, we can require that f is projective and can further require that there is an f-ample Cartier divisor A on Y whose support coincides with $\mathrm{Exc}(f)$. Moreover, we can make f an isomorphism over the generic point of every log canonical center of (X, Δ). For the details, see the proof of Proposition 2.3.20 below.

Lemma 2.3.19 is very useful and is indispensable for the recent minimal model program. We sometimes call it Szabó's resolution lemma (see [**Sz**] and [**F12**]). For more general results, see [**BM**] and [**BVP**] (see also Theorem 5.2.16 and Theorem 5.2.17). Note that [**Mus**] is a very accessible account of the resolution of singularities.

Lemma 2.3.19 (Resolution lemma). *Let X be a smooth variety and let D be a reduced divisor on X. Then there exists a proper birational morphism $f: Y \to X$ with the following properties:*

(1) *f is a composite of blow-ups of smooth subvarieties,*
(2) *Y is smooth,*

(3) *$f_*^{-1}D \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor, where $f_*^{-1}D$ is the strict transform of D on Y, and*
(4) *f is an isomorphism over U, where U is the largest open set of X such that the restriction $D|_U$ is a simple normal crossing divisor on U.*

Note that f is projective and the exceptional locus $\mathrm{Exc}(f)$ is of pure codimension one in Y since f is a composite of blow-ups.

PROPOSITION 2.3.20 (cf. [Sz]). *Let X be a normal variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then (X, Δ) is dlt if and only if there is a closed subset $Z \subset X$ such that*

(i) *$X \setminus Z$ is smooth and $\mathrm{Supp}\,\Delta|_{X \setminus Z}$ is a simple normal crossing divisor.*
(ii) *If $h : V \to X$ is birational and E is a prime divisor on V such that $h(E) \subset Z$, then $a(E, X, \Delta) > -1$.*

PROOF. We assume the properties (i) and (ii). By using Hironaka's resolution and Szábo's resolution lemma (see Lemma 2.3.19), we can take a resolution $f : Y \to X$ which is a composition of blow-ups and is an isomorphism over $X \setminus Z$ such that $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta$ is a simple normal crossing divisor on Y by (i). By construction, f is projective and $\mathrm{Exc}(f)$ is a divisor. By (ii), $a(E, X, \Delta) > -1$ for every f-exceptional divisor E. Therefore, (X, Δ) is dlt by definition. By construction, it is obvious that f is an isomorphism over the generic point of every log canonical center of (X, Δ). Note that we can take an f-ample Cartier divisor A on Y whose support coincides with $\mathrm{Exc}(f)$ since f is a composition of blow-ups.

Conversely, we assume that (X, Δ) is dlt. Let $f : Y \to X$ be a resolution as in Definition 2.3.16. We put $Z = f(\mathrm{Exc}(f))$. Then Z satisfies the property (i). We put $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Note that $f^{-1}(Z) = \mathrm{Exc}(f)$. Let Δ' be an effective Cartier divisor whose support equals $\mathrm{Exc}(f)$. We note that every irreducible component of Δ' has coefficient < 1 in Δ_Y. Therefore, $(Y, \Delta_Y + \varepsilon\Delta')$ is sub lc for $0 < \varepsilon \ll 1$ by Lemma 2.3.9. If E is any divisor over X whose center is contained in Z, then $c_Y(E)$, the center of E on Y, is contained in $\mathrm{Exc}(f)$. Therefore, we have

$$a(E, X, \Delta) = a(E, Y, \Delta_Y) > a(E, Y, \Delta_Y + \varepsilon\Delta') \geq -1.$$

This implies the property (ii). □

The notion of weak log-terminal singularities was introduced in [KMM, Definition 0-2-10].

DEFINITION 2.3.21 (Weak log-terminal singularities). Let X be a normal variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair (X, Δ) is said to have *weak log-terminal singularities* if the following conditions hold.

(i) There exists a resolution of singularities $f : Y \to X$ such that $\operatorname{Supp} f_*^{-1}\Delta \cup \operatorname{Exc}(f)$ is a normal crossing divisor on Y and that
$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$
with $a_i > -1$ for every exceptional divisor E_i.

(ii) There is an f-ample Cartier divisor A on Y whose support coincides with $\operatorname{Exc}(f)$.

It is easy to see that (X, Δ) is log canonical when (X, Δ) has weak log-terminal singularities. We note that $-A$ is effective by Lemma 2.3.26 below.

REMARK 2.3.22. By Remark 2.3.18, a dlt pair (X, Δ) has weak log-terminal singularities.

Although the notion of weak log-terminal singularities is not necessary for the recent developments of the minimal model program, we include it here for the reader's convenience because [**KMM**] was written by using weak log-terminal singularities.

2.3.23 (Negativity lemmas). The negativity lemmas are very useful in many situations. There were many papers discussing various related topics before the minimal model theory appeared (see, for example, [**Mum1**], [**Gra**], [**Z**], and so on). Here, we closely follow the treatment in [**KoMo**]. Note that Fujita's treatment is also useful (see [**Ft4**, (1.5) Lemma]).

Let us start with Lemma 2.3.24, which is a special case of the Hodge index theorem.

LEMMA 2.3.24 (see [**KoMo**, Lemma 3.40]). *Let $f : Y \to X$ be a proper birational morphism from a smooth surface Y onto a normal surface X with exceptional curves E_i. Assume that $f(E_i) = P$ for every i. Then the intersection matrix $(E_i \cdot E_j)$ is negative definite.*

PROOF. We shrink and compactify X. Then we may assume that X and Y are projective. Let $D = \sum e_i E_i$ be a non-zero linear combination of f-exceptional curves E_i. It is sufficient to prove that $D^2 < 0$. When D is not effective, we write $D = D_+ - D_-$ as a difference of two effective divisors without common irreducible components. Then we have
$$D^2 \leq D_+^2 + D_-^2.$$

Therefore, it is sufficient to consider the case where D is effective. Assume that $D^2 \geq 0$. Let H be an ample Cartier divisor on Y such that $H - K_Y$ is ample. By Serre duality, we have

$$H^2(Y, \mathcal{O}_Y(nD + H)) = 0$$

for every positive integer n. Note that

$$(nD + H \cdot nD + H - K_Y) \geq (nD + H \cdot nD) \geq n(D \cdot H) > 0.$$

By the Riemann–Roch formula, we obtain that

$$\dim H^0(Y, \mathcal{O}_Y(nD + H)) \to \infty$$

when $n \to \infty$. On the other hand,

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(nD + H)) &\subset H^0(X, \mathcal{O}_X(f_*(nD + H))) \\ &= H^0(X, \mathcal{O}_X(f_*H)) \end{aligned}$$

gives a contradiction. Therefore, $D^2 < 0$. $\square$

LEMMA 2.3.25 (see [KoMo, Lemma 3.41]). *Let Y be a smooth surface and let $C = \cup C_i$ be a finite set of proper curves on Y. Assume that the intersection matrix $(C_i \cdot C_j)$ is negative definite. Let $A = \sum a_i C_i$ be an $\mathbb{R}$-linear combination of the curves C_i. Assume that $(A \cdot C_i) \geq 0$ for every i. Then*

(i) *$a_i \leq 0$ for every i.*
(ii) *If C is connected, then either $a_i = 0$ for every i or $a_i < 0$ for every i.*

PROOF. We write $A = A_+ - A_-$ as a difference of two effective $\mathbb{R}$-divisors without common irreducible components. We assume that $A_+ \neq 0$. Since the matrix $(C_i \cdot C_j)$ is negative definite, we have $A_+^2 < 0$. Therefore, there is a curve $C_{i_0} \subset \operatorname{Supp} A_+$ such that $(C_{i_0} \cdot A) < 0$. Then C_{i_0} is not in $\operatorname{Supp} A_-$. Thus $(C_{i_0} \cdot A) < 0$. This is a contradiction. We obtain (i).

We assume that C is connected, $A_- \neq 0$, and $\operatorname{Supp} A_- \neq \operatorname{Supp} C$. Then there is a curve C_i such that $C_i \not\subset \operatorname{Supp} A_-$ but C_i intersects $\operatorname{Supp} A_-$. Then $(C_i \cdot A) = -(C_i \cdot A_-) < 0$. This is a contradiction. We obtain (ii). $\square$

Lemma 2.3.26 is well known as the negativity lemma.

LEMMA 2.3.26 (Negativity lemma, see [KoMo, Lemma 3.39]). *Let $f : V \to W$ be a proper birational morphism between normal varieties. Let $-D$ be an f-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on V. Then*

(i) *D is effective if and only if f_*D is effective.*

(ii) *Assume that D is effective. Then, for every $w \in W$, either $f^{-1}(w) \subset \operatorname{Supp} D$ or $f^{-1}(w) \cap \operatorname{Supp} D = \emptyset$.*

PROOF. Note that if D is effective then so is f_*D. From now on, we assume that f_*D is effective. By Chow's lemma and Hironaka's resolution of singularities, there is a proper birational morphism $p : V' \to V$ such that $V' \to W$ is projective. Note that D is effective if and only if p^*D is effective. Therefore, by replacing V with V', we may assume that f is projective and V is smooth. We may further assume that W is affine by taking an affine cover of W. We write $D = \sum D^k$ where D^k is the sum of those irreducible components D_i of D such that $f(D_i)$ has codimension k in W.

First, we treat the case when $\dim W = 2$. In this case, $D = D^1 + D^2$ and D^1 is f-nef. Note that D^1 is effective by the assumption that f_*D is effective. Therefore, $-D^2$ is f-nef and is a linear combination of f-exceptional curves. By Lemma 2.3.24 and Lemma 2.3.25, D^2 is effective. This implies that D is effective when $\dim W = 2$.

Next, we treat the general case. Let $S \subset W$ be the complete intersection of $\dim W - 2$ general hypersufaces with $T = f^{-1}(S)$. Then $f : T \to S$ is a birational morphism from a smooth surface T onto a normal surface S. Note that $D|_T = D^2|_T + D^1|_T$. Therefore, D^2 is effective. Let $H \subset V$ be a general very ample Cartier divisor. We put $B = D|_H$. Then $-B$ is f-nef, $B^i = D^{i+1}|_H$ for $i \geq 2$ and $B^1 = D^1|_H + D^2|_H$. Note that D^1 is effective by the assumption that f_*D is effective. We have proved that D^2 is effective. Thus B^1 is effective. By induction on the dimension, we can check that B is effective. Therefore, D is effective.

Finally, for $w \in W$, $f^{-1}(w)$ is connected. Thus, if $f^{-1}(w)$ intersects $\operatorname{Supp} D$ but is not contained in it, then there is an irreducible curve $C \subset f^{-1}(w)$ such that $(C \cdot D) > 0$. This is impossible since $-D$ is f-nef. Therefore, either $f^{-1}(w) \subset \operatorname{Supp} D$ or $f^{-1}(w) \cap \operatorname{Supp} D = \emptyset$ holds. □

As an easy application of Lemma 2.3.26, we obtain a very useful lemma. We repeatedly use Lemma 2.3.27 and its proof in the minimal model theory.

LEMMA 2.3.27 (see [KoMo, Lemma 3.38]). *Let us consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

where X, X', and Y are normal varieties, and f and f' are proper birational morphisms. Let Δ (resp. Δ') be an $\mathbb{R}$-divisor on X (resp. X'). Assume the following conditions.

(i) $f_*\Delta = f'_*\Delta'$.
(ii) $-(K_X + \Delta)$ *is $\mathbb{R}$-Cartier and f-nef.*
(iii) $K_{X'} + \Delta'$ *is $\mathbb{R}$-Cartier and f'-nef.*

Then we have

$$a(E, X, \Delta) \leq a(E, X', \Delta')$$

for an arbitrary exceptional divisor E over Y.

If either

(iv) $-(K_X + \Delta)$ *is f-ample and f is not an isomorphism above the generic point of $c_Y(E)$, or*
(v) $K_{X'} + \Delta'$ *is f'-ample and f' is not an isomorphism above the generic point of $c_Y(E)$.*

holds, then we have

$$a(E, X, \Delta) < a(E, X', \Delta').$$

Note that $c_Y(E)$ is the center of E on Y.

Proof. We take a common resolution

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ X \arrow[rr, dashed, "\phi"'] & & X' \end{tikzcd}$$

of X and X' such that $c_Z(E)$, the center of E on Z, is a divisor. We put $h = f \circ g = f' \circ g'$. We have

$$K_Z = g^*(K_X + \Delta) + \sum a(E_i, X, \Delta)$$

and

$$K_Z = g^*(K_{X'} + \Delta') + \sum a(E_i, X', \Delta').$$

We put

$$H = \sum(a(E_i, X', \Delta')E_i - a(E, X, \Delta))E_i.$$

Then $-H$ is h-nef and a sum of h-exceptional divisors by assumption (i). Therefore, H is an effective divisor by Lemma 2.3.26. Moreover, if H is not numerically h-trivial over the generic point of $c_Y(E)$, then the coefficient of E in H is positive by Lemma 2.3.26. $\square$

We will repeatedly use the results and the arguments in this section throughout this book.

2.4. Iitaka dimension, movable and pseudo-effective divisors

Let us start with the definition of the Iitaka dimension and the numerical dimension. For the details, see, for example, [**La1**, Section 2.1], [**Mo4**], [**Nak2**], [**KMM**, Definition 6-1-1], [**U**], and so on.

Definition 2.4.1 (Iitaka dimension and numerical dimension). Let X be a normal complete irreducible variety and let D be a $\mathbb{Q}$-Cartier divisor on X. Assume that m_0D is Cartier for a positive integer m_0. Let

$$\Phi_{|mm_0D|} : X \dashrightarrow \mathbb{P}^{\dim |mm_0D|}$$

be rational mappings given by linear systems $|mm_0D|$ for positive integers m. We define the *Iitaka dimension* or the *D-dimension*

$$\kappa(X, D) = \begin{cases} \max\limits_{m>0} \dim \Phi_{|mm_0D|}(X), & \text{if } \ |mm_0D| \neq \emptyset \ \text{ for some } \ m > 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

In case D is nef, we can also define the *numerical dimension* or the *numerical Iitaka dimension*

$$\nu(X, D) = \max\{\, e \mid D^e \not\equiv 0\},$$

where $\equiv$ denotes numerical equivalence. We note that

$$\nu(X, D) \geq \kappa(X, D)$$

always holds. We also note that the numerical dimension $\nu(X, D)$ also makes sense for nef $\mathbb{R}$-Cartier divisors D.

2.4.2 (Movable divisors and movable cone). We quickly review the notion of movable divisors. It sometimes plays important roles in the minimal model program.

Definition 2.4.3 (Movable divisors and movable cone, see [**Ka3**, Section 2]). Let $f : X \to Y$ be a projective morphism from a normal variety X onto a variety Y. A Cartier divisor D on X is called *f-movable* or *movable over Y* if $f_*\mathcal{O}_X(D) \neq 0$ and if the cokernel of the natural homomorphism

$$f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$$

has a support of codimension ≥ 2.

Let M be an $\mathbb{R}$-Cartier divisor on X. Then M is called *f-movable* or *movable over Y* if and only if $M = \sum_i a_i D_i$ where a_i is a positive real number and D_i is an f-movable Cartier divisor for every i.

We define $\overline{\mathrm{Mov}}(X/Y)$ as the closed convex cone in $N^1(X/Y)$, which is called the *movable cone* of $f : X \to Y$, generated by the numerical classes of f-movable Cartier divisors.

Lemma 2.4.4 is a variant of the negativity lemma (see Lemma 2.3.26). We will use Lemma 2.4.4 in the proof of dlt blow-ups (see Theorem 4.4.21).

LEMMA 2.4.4 (see [F26, Lemma 4.2]). *Let $f : X \to Y$ be a projective birational morphism from a normal $\mathbb{Q}$-factorial variety X onto a normal variety Y. Let E be an $\mathbb{R}$-divisor on X such that* $\operatorname{Supp} E$ *is f-exceptional and $E \in \overline{\mathrm{Mov}}(X/Y)$. Then $-E$ is effective.*

PROOF. We write $E = E_+ - E_-$ such that E_+ and E_- are effective $\mathbb{R}$-divisors and have no common irreducible components. We assume that $E_+ \neq 0$. By taking a resolution of singularities, we may assume that X is smooth. Without loss of generality, we may assume that Y is affine by taking an affine open covering of Y. Let A be an ample Cartier divisor on Y and let H be an ample Cartier divisor on X. Then we can find an irreducible component E_0 of E_+ such that

$$E_0 \cdot (f^*A)^k \cdot H^{n-k-2} \cdot E < 0$$

when $\dim X = n$ and $\mathrm{codim}_Y f(E_+) = k$ by Lemma 2.3.24. This is a contradiction. Note that

$$E_0 \cdot (f^*A)^k \cdot H^{n-k-2} \cdot E \geq 0$$

since $E \in \overline{\mathrm{Mov}}(X/Y)$. Therefore, $-E$ is effective. □

2.4.5 (Pseudo-effective divisors). Let us recall the definition of pseudo-effective divisors. The notion of pseudo-effective divisors is indispensable for the recent developments of the minimal model program, although it is not so important in this book.

DEFINITION 2.4.6 (Pseudo-effective divisors). Let X be a complete variety. We define $\mathrm{PE}(X)$ as the closed convex cone in $N^1(X)$, which is called the *pseudo-effective cone* of X, generated by the numerical classes of effective Cartier divisors on X. Let D be an $\mathbb{R}$-Cartier divisor on X. Then D is called *pseudo-effective* if the numerical class of D is contained in $\mathrm{PE}(X)$.

Let $f : X \to Y$ be a proper morphism from a variety X to a scheme Y. Let D be an $\mathbb{R}$-Cartier divisor on X. Then D is called *f-pseudo-effective* or *pseudo-effective over Y* if the restriction of D to the geometric generic fiber of every irreducible component of $f(X)$ is pseudo-effective.

Although we do not need the following lemma explicitly in this book, it may help us understand the notion of pseudo-effective divisors. So we include it here for the reader's convenience.

LEMMA 2.4.7. *Let $f : X \to Y$ be a projective surjective morphism between normal irreducible varieties with connected fibers. Let D be an $\mathbb{R}$-Cartier divisor on X. Then D is pseudo-effective over Y if and only if $D + A$ is big over Y for any f-ample $\mathbb{R}$-Cartier divisor A on X.*

PROOF. First, we prove 'if' part. Let D be an $\mathbb{R}$-Cartier divisor on X and let H be an f-ample Cartier divisor on X. Then $D + \frac{1}{n}H$ is f-big for every positive integer n by assumption. Then the restriction of $D + \frac{1}{n}H$ to the geometric generic fiber of f is big. By taking $n \to \infty$, we see that the restriction of D to the geometric generic fiber of f is pseudo-effective. Therefore, D is pseudo-effective over Y.

Next, we prove 'only if' part. Let A be an f-ample $\mathbb{R}$-Cartier divisor on X and let D be an f-pseudo-effective $\mathbb{R}$-Cartier divisor on X. Then the restriction of $D + A$ to the geometric generic fiber of f is obviously big. Therefore, $D + A$ is big over Y. $\square$

Nakayama's numerical dimension for pseudo-effective divisors plays crucial roles in the recent developments of the minimal model program. So we include it for the reader's convenience.

DEFINITION 2.4.8 (Nakayama's numerical dimension, see [**Nak2**, Chapter V.2.5. Definition]). Let D be a pseudo-effective $\mathbb{R}$-Cartier divisor on a normal projective variety X and let A be a Cartier divisor on X. If $H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) \neq 0$ for infinitely many positive integers m, then we set

$$\sigma(D; A) = \max\left\{ k \in \mathbb{Z}_{\geq 0} \;\middle|\; \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A))}{m^k} > 0 \right\}.$$

If $H^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) \neq 0$ only for finitely many $m \in \mathbb{Z}_{\geq 0}$, then we set $\sigma(D; A) = -\infty$. We define *Nakayama's numerical dimension* κ_σ by

$$\kappa_\sigma(X, D) = \max\{\sigma(D; A) \,|\, A \text{ is a Cartier divisor on } X\}.$$

If D is a nef $\mathbb{R}$-Cartier divisor on a normal projective variety X, then D is pseudo-effective and

$$\kappa_\sigma(X, D) = \nu(X, D).$$

We close this section with an easy remark.

REMARK 2.4.9. Let X be a normal projective irreducible variety and let D be a Cartier divisor on X. Then we have the following properties.

- If D is ample, then D is nef.
- If D is semi-ample, then D is nef.
- If D is ample, then D is semi-ample.

- If D is nef, then D is pseudo-effective.
- If D is effective, then D is pseudo-effective.
- If D is movable, then D is linearly equivalent to an effective Cartier divisor.
- If D_1 is a big $\mathbb{R}$-divisor on X and D_2 is a pseudo-effective $\mathbb{R}$-Cartier divisor on X, then $D_1 + D_2$ is big.

Let Y be a (not necessarily normal) projective irreducible variety.

- If B_1 is a big $\mathbb{R}$-divisor on Y and B_2 is any $\mathbb{R}$-Cartier divisor on Y, then $B_1 + \varepsilon B_2$ is big for any $0 < \varepsilon \ll 1$.

CHAPTER 3

Classical vanishing theorems and some applications

In this chapter, we discuss various classical vanishing theorems, for example, the Kodaira vanishing theorem, the Kawamata–Viehweg vanishing theorem, the Fujita vanishing theorem, and so on. They play crucial roles for the study of higher-dimensional algebraic varieties. We also treat some applications. Although this chapter contains some new arguments and some new results, almost all results are standard or known to the experts. Of course, our choice of topics is biased and reflects the author's personal taste.

In Section 3.1, we give a proof of the Kodaira vanishing theorem for smooth projective varieties based on the theory of mixed Hodge structures on cohomology with compact support. It is a slightly different from the usual one but suits for our framework discussed in Chapter 5. In Sections 3.2, 3.3, and 3.4, we prove the Kawamata–Viehweg vanishing theorem, the Viehweg vanishing theorem, and the Nadel vanishing theorem. They are generalizations of the Kodaira vanishing theorem. In Section 3.5, we prove the Miyaoka vanishing theorem as an application of the Kawamata–Viehweg–Nadel vanishing theorem. Note that the Miyaoka vanishing theorem is the first vanishing theorem for the integral part of $\mathbb{Q}$-divisors. Section 3.6 is a quick review of Kollár's injectivity, torsion-free, and vanishing theorems without proof. We will prove complete generalizations in Chapter 5. In Section 3.7, we treat Enoki's injectivity theorem, which is a complex analytic counterpart of Kollár's injectivity theorem. In Sections 3.8 and 3.9, we discuss Fujita's vanishing theorem and its applications. In Section 3.10, we quickly review Tanaka's vanishing theorems without proof. They are relatively new and are Kodaira type vanishing theorems in positive characteristic. In Section 3.11, we prove Ambro's vanishing theorem as an application of the argument in Section 3.1. In Section 3.12, we discuss Kovács's characterization of rational singularities. Kovács's result and its proof are very useful. In Section 3.13, we prove some basic properties of divisorial log terminal pairs. In particular, we show that every divisorial log terminal pair has only rational singularities as an application of Kovács's characterization of rational singularities. Section 3.14 is devoted to the Elkik–Fujita vanishing theorem and its application. We give a simplified proof of the Elkik–Fujita vanishing theorem due to Chih-Chi Chou. In Section 3.15, we explain the method of two spectral sequences of local cohomology groups. Section 3.16 is an introduction to our new vanishing theorems. We will discuss the details and more general results in Chapters 5 and 6.

3.1. Kodaira vanishing theorem

In this section, we give a proof of Kodaira's vanishing theorem for projective varieties based on the theory of *mixed* Hodge structures.

Let us start with the following easy lemma (see [**Am2**] and [**F36**]). We will prove generalizations of Lemma 3.1.1 in Chapter 5 (see Theorem 5.4.1 and Theorem 5.4.2).

LEMMA 3.1.1. *Let X be a smooth projective variety and let Δ be a reduced simple normal crossing divisor on X. Let D be an effective Cartier divisor on X such that* $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$. *Then the map*

$$H^i(X, \mathcal{O}_X(-D-\Delta)) \to H^i(X, \mathcal{O}_X(-\Delta)),$$

which is induced by the natural inclusion $\mathcal{O}_X(-D) \subset \mathcal{O}_X$, is surjective for every i. Equivalently, by Serre duality,

$$H^i(X, \mathcal{O}_X(K_X+\Delta)) \to H^i(X, \mathcal{O}_X(K_X+\Delta+D)),$$

which is induced by the natural inclusion $\mathcal{O}_X \subset \mathcal{O}_X(D)$, is injective for every i.

PROOF. In this proof, we use the classical topology and Serre's GAGA. We consider the following Hodge to de Rham type spectral sequence:

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log \Delta) \otimes \mathcal{O}_X(-\Delta)) \Rightarrow H_c^{p+q}(X \setminus \Delta, \mathbb{C}).$$

It is well known that it degenerates at E_1 by the theory of mixed Hodge structures (see Remark 3.1.4 and Remark 3.1.5 below). This implies that the natural inclusion

$$\iota_! \mathbb{C}_{X\setminus\Delta} \subset \mathcal{O}_X(-\Delta),$$

where $\iota : X \setminus \Delta \to X$, induces the surjections

$$H_c^i(X\setminus\Delta, \mathbb{C}) = H^i(X, \iota_!\mathbb{C}_{X\setminus\Delta}) \xrightarrow{\alpha_i} H^i(X, \mathcal{O}_X(-\Delta))$$

for all i. Note that

$$\iota_!\mathbb{C}_{X\setminus\Delta} \subset \mathcal{O}_X(-D-\Delta) \subset \mathcal{O}_X(-\Delta).$$

Therefore, α_i factors as

$$\alpha_i : H^i(X, \iota_! \mathbb{C}_{X \setminus \Delta}) \to H^i(X, \mathcal{O}_X(-D-\Delta)) \to H^i(X, \mathcal{O}_X(-\Delta))$$

for every i. This implies that

$$H^i(X, \mathcal{O}_X(-D-\Delta)) \to H^i(X, \mathcal{O}_X(-\Delta))$$

is surjective for every i. □

As an obvious application, we have:

COROLLARY 3.1.2. *Let X be a smooth projective variety and let Δ be a reduced simple normal crossing divisor on X. Assume that there is an ample Cartier divisor D on X such that* $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$. *Then*

$$H^i(X, \mathcal{O}_X(-\Delta)) = 0$$

for every $i < \dim X$, equivalently,

$$H^i(X, \mathcal{O}_X(K_X + \Delta)) = 0$$

for every $i > 0$.

PROOF. By Serre duality and Serre's vanishing theorem, we have

$$H^i(X, \mathcal{O}_X(-aD-\Delta)) = 0$$

for a sufficiently large and positive integer a and for every $i < \dim X$. By Lemma 3.1.1, we obtain that $H^i(X, \mathcal{O}_X(-\Delta)) = 0$ for every $i < \dim X$. By Serre duality, we see that $H^i(X, \mathcal{O}_X(K_X + \Delta)) = 0$ for every $i > 0$. □

For an application of Corollary 3.1.2, we will prove Ambro's vanishing theorem: Theorem 3.11.1

By using a standard covering trick, we can recover Kodaira's vanishing theorem for projective varieties from Lemma 3.1.1. We will treat the Kodaira vanishing theorem for compact complex manifold in Theorem 3.7.4.

THEOREM 3.1.3 (Kodaira vanishing theorem). *Let X be a smooth projective variety and let H be an ample Cartier divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + H)) = 0$$

for every $i > 0$, equivalently, by Serre duality,

$$H^i(X, \mathcal{O}_X(-H)) = 0$$

for every $i < \dim X$.

PROOF. We take a smooth divisor $B \in |mH|$ for some positive integer m. Let $f : V \to X$ be the m-fold cyclic cover ramifying along B. Then

$$f_*\mathcal{O}_V = \bigoplus_{k=0}^{m-1} \mathcal{O}_X(-kH).$$

Therefore, it is sufficient to prove that $H^i(V, \mathcal{O}_V(-f^*H)) = 0$ for every $i < \dim V = \dim X$. This is because $\mathcal{O}_X(-H)$ is a direct summand of $f_*\mathcal{O}_V(-f^*H)$. Note that $\mathcal{O}_V(f^*H)$ has a section, which is the reduced preimage of B, by construction. By iterating this process, we obtain a tower of cyclic covers:

$$V_n \to \cdots \to V_0 \to X.$$

By suitable choice of the ramification divisors, we may assume that the pull-back of H on V_n has no base points. Therefore, we reduced Theorem 3.1.3 to the case when the linear system $|H|$ has no base points. Let $\Delta \in |H|$ be a reduced smooth divisor on X. Then

$$H^i(X, \mathcal{O}_X(-l\Delta)) \to H^i(X, \mathcal{O}_X(-\Delta))$$

is surjective for every i and every $l \geq 1$ by Lemma 3.1.1. Therefore, $H^i(X, \mathcal{O}_X(-\Delta)) = 0$ for $i < \dim X$ by Serre duality and Serre's vanishing theorem. $\square$

For the reader's convenience, we give remarks on the E_1-degeneration of the Hodge to de Rham type spectral sequence in the proof of Lemma 3.1.1.

REMARK 3.1.4. For the proof of Theorem 3.1.3, it is sufficient to assume that Δ is smooth in Lemma 3.1.1. When Δ is smooth, we can easily construct the mixed Hodge complex of sheaves on X giving a natural mixed Hodge structure on $H^\bullet_c(X \setminus \Delta, \mathbb{Z})$. From now on, we use the notation and the framework in [**PS**, §3.3 and §3.4]. Let $\mathcal{H}dg^\bullet(X)$ (resp. $\mathcal{H}dg^\bullet(\Delta)$) be a Hodge complex of sheaves on X (resp. Δ) giving a natural pure Hodge structure on $H^\bullet(X, \mathbb{Z})$ (resp. $H^\bullet(\Delta, \mathbb{Z})$). Then the mixed cone

$$\mathcal{H}dg^\bullet(X, \Delta) := \mathrm{Cone}(\mathcal{H}dg^\bullet(X) \to i_*\mathcal{H}dg^\bullet(\Delta))[-1],$$

where $i : \Delta \to X$ is the natural inclusion, gives a natural mixed Hodge structure on $H^\bullet_c(X \setminus \Delta, \mathbb{Z})$. For the details, see [**PS**, Example 3.24]. We note that

$$0 \to \Omega^p_X(\log \Delta) \otimes \mathcal{O}_X(-\Delta) \to \Omega^p_X \to \Omega^p_\Delta \to 0$$

is exact for every p. Therefore, we can easily see that

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log \Delta) \otimes \mathcal{O}_X(-\Delta)) \Rightarrow H^{p+q}_c(X \setminus \Delta, \mathbb{C})$$

degenerates at E_1 by the theory of mixed Hodge structures.

REMARK 3.1.5. We put $d = \dim X$. In the proof of Lemma 3.1.1,

$$H^q(X, \Omega_X^p(\log \Delta) \otimes \mathcal{O}_X(-\Delta))$$

is dual to

$$H^{d-q}(X, \Omega_X^{d-p}(\log \Delta))$$

by Serre duality. By Poincaré duality,

$$H_c^{p+q}(X \setminus \Delta, \mathbb{C})$$

is dual to

$$H^{2d-(p+q)}(X \setminus \Delta, \mathbb{C}).$$

By Deligne (see [Del]), it is well known that

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log \Delta)) \Rightarrow H^{p+q}(X \setminus \Delta, \mathbb{C})$$

degenerates at E_1. This implies that

$$\dim H^k(X \setminus \Delta, \mathbb{C}) = \sum_{p+q=k} H^q(X, \Omega_X^p(\log \Delta))$$

for every k. Therefore, by the above observation, we have

$$\dim H_c^k(X \setminus \Delta, \mathbb{C}) = \sum_{p+q=k} H^q(X, \Omega_X^p(\log \Delta) \otimes \mathcal{O}_X(-\Delta))$$

for every k. Thus the Hodge to de Rham type spectral sequence

$$E_1^{p,q} = H^q(X \setminus \Delta, \Omega_X^p(\log \Delta) \otimes \mathcal{O}_X(-\Delta)) \Rightarrow H_c^{p+q}(X \setminus \Delta, \mathbb{C})$$

degenerates at E_1.

Anyway, we will completely generalize Lemma 3.1.1 in Chapter 5.

REMARK 3.1.6. It is well known that the Kodaira vanishing theorem for projective varieties follows from the theory of *pure* Hodge structures. For the details, see, for example, [KoMo, 2.4 The Kodaira vanishing theorem].

By using a covering trick, we have a slight but very important generalization of Kodaira's vanishing theorem. Theorem 3.1.7 is usually called Kawamata–Viehweg vanishing theorem.

THEOREM 3.1.7 (Kawamata–Viehweg vanishing theorem). *Let X be a smooth projective variety and let D be an ample $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp}\{D\}$ *is a simple normal crossing divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$$

for every $i > 0$.

REMARK 3.1.8. In this book, there are various formulations of the Kawamata–Viehweg vanishing theorem in order to make it useful for many applications. See, for example, Theorem 3.1.7, Theorem 3.2.1, Theorem 3.2.8, Theorem 3.2.9, Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.4, Theorem 3.3.7, Theorem 4.1.1, Corollary 5.7.7, and so on.

Before we start the proof of Theorem 3.1.7, let us recall an easy lemma.

LEMMA 3.1.9. *Let $f : Y \to X$ be a finite morphism between n-dimensional normal irreducible varieties. Then the natural inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is a split injection.*

PROOF. It is easy to see that $\frac{1}{n}\mathrm{Trace}_{Y/X}$ splits the natural inclusion $\mathcal{O}_X \to f_*\mathcal{O}_Y$, where $\mathrm{Trace}_{Y/X}$ is the trace map. □

PROOF OF THEOREM 3.1.7. We put $L = \lceil D \rceil$ and $\Delta = L - D$. We put $\Delta = \sum_j a_j \Delta_j$ such that Δ_j is a reduced and smooth (possibly disconnected) divisor on X for every j and that a_j is a positive rational number for every j. It is sufficient to prove $H^i(X, \mathcal{O}_X(-L)) = 0$ for every $i < \dim X$ by Serre duality. We use induction on the number of divisors Δ_j. If $\Delta = 0$, then Theorem 3.1.7 is nothing but Kodaira's vanishing theorem: Theorem 3.1.3. We put $a_1 = b/m$ such that b is an integer and m is a positive integer. We can construct a finite surjective morphism $p_1 : X_1 \to X$ such that X_1 is smooth and that $p_1^*\Delta_1 \sim mB$ for some Cartier divisor B on X_1 (see Lemma 3.1.10). We may further assume that every $p_1^*\Delta_j$ is smooth and $\sum_j p_1^*\Delta_j$ is a simple normal crossing divisor on X_1 (see Lemma 3.1.10). It is easy to see that $H^i(X, \mathcal{O}_X(-L))$ is a direct summand of $H^i(X_1, p_1^*\mathcal{O}_X(-L))$ by Lemma 3.1.9. By construction, $p_1^*\Delta_1$ is a member of $|mB|$. Let $p_2 : X_2 \to X_1$ be the corresponding cyclic cover. Then X_2 is smooth, $p_2^*p_1^*\Delta_j$ is smooth for every j, and $\sum_j p_2^*p_1^*\Delta_j$ is a simple normal crossing divisor on X_2. Since

$$p_{2*}\mathcal{O}_{X_2} = \bigoplus_{k=0}^{m-1} \mathcal{O}_{X_1}(-kB),$$

we obtain

$$\begin{aligned} &H^i(X_2, p_2^*(p_1^*\mathcal{O}_X(-L) \otimes \mathcal{O}_{X_1}(bB))) \\ &= \bigoplus_{k=0}^{m-1} H^i(X_1, p_1^*\mathcal{O}_X(-L) \otimes \mathcal{O}_{X_1}((b-k)B)). \end{aligned}$$

The $k = b$ case shows that $H^i(X_1, p_1^*\mathcal{O}_X(-L))$ is a direct summand of $H^i(X_2, p_2^*(p_1^*\mathcal{O}_X(-L) \otimes \mathcal{O}_{X_1}(bB)))$. Note that

$$p_2^*(p_1^*L - bB) \sim p_2^*p_1^*D + \sum_{j>1} a_i p_2^*p_1^*\Delta_j.$$

Therefore, by induction on the number of divisors Δ_j, we obtain

$$H^i(X_2, p_2^*(p_1^*\mathcal{O}_X(-L) \otimes \mathcal{O}_{X_1}(bB))) = 0$$

for every $i < \dim X_2 = \dim X$. Thus we obtain the desired vanishing theorem. $\square$

The following covering trick is due to Bloch–Gieseker (see [**BlGi**]) and is well known (see, for example, [**KoMo**, Proposition 2.67]).

LEMMA 3.1.10. *Let X be a projective variety, let D be a Cartier divisor on X, and let m be a positive integer. Then there is a normal variety Y, a finite surjective morphism $f : Y \to X$, and a Cartier divisor D' on Y such that $f^*D \sim mD'$.*

Furthermore, if X is smooth and $\sum_j F_j$ is a simple normal crossing divisor on X, then we can choose Y to be smooth such that f^*F_j is smooth for every j and $\sum_j f^*F_j$ is a simple normal crossing divisor on Y.

PROOF. Let $\pi : \mathbb{P}^n \to \mathbb{P}^n$ be the morphism given by

$$(x_0 : x_1 : \cdots , x_n) \mapsto (x_0^m : x_1^m : \cdots : x_n^m).$$

Then $\pi^*\mathcal{O}_{\mathbb{P}^n}(1) \simeq \mathcal{O}_{\mathbb{P}^n}(m)$.

Let L be a very ample Cartier divisor on X. Then there is a morphism $h : X \to \mathbb{P}^n$ such that $\mathcal{O}_X(L) \simeq h^*\mathcal{O}_{\mathbb{P}^n}(1)$. Let Y be the normalization of the fiber product $X \times_{\mathbb{P}^n} \mathbb{P}^n$ sitting in the diagram:

$$\begin{tikzcd} Y \arrow[r, "h_Y"] \arrow[d, "f"'] & \mathbb{P}^n \arrow[d, "\pi"] \\ X \arrow[r, "h"'] & \mathbb{P}^n. \end{tikzcd}$$

If D is very ample, then we put $L = D$. In this case,

$$f^*\mathcal{O}_X(D) \simeq h_Y^*(\pi^*\mathcal{O}_{\mathbb{P}^n}(1)) \simeq h_Y^*\mathcal{O}_{\mathbb{P}^n}(m).$$

If X is smooth, then we consider $\pi' : \mathbb{P}^n \to \mathbb{P}^n$ which is the composition of π with a general automorphism of the target space $\mathbb{P}^n$. By Kleiman's Bertini type theorem (see, for example, [**Har4**, Chapter III Theorem 10.8]), we can make Y smooth and $\sum_j f^*F_j$ a simple normal crossing divisor on Y.

In general, we can write $D \sim L_1 - L_2$ where L_1 and L_2 are both very ample Cartier divisors. By using the above argument twice, we obtain $f : Y \to X$ such that $f^*L_i \sim mL'_i$ for some Cartier divisors L'_i for $i = 1, 2$. Thus we obtain the desired morphism $f : Y \to X$. □

3.2. Kawamata–Viehweg vanishing theorem

In this section, we generalize Theorem 3.1.7 for the latter usage. The following theorem is well known as the Kawamata–Viehweg vanishing theorem.

THEOREM 3.2.1 (see [KMM, Theorem 1-2-3]). *Let X be a smooth variety and let $\pi : X \to S$ be a proper surjective morphism onto a variety S. Assume that a $\mathbb{Q}$-divisor D on X satisfies the following conditions*:

(i) *D is π-nef and π-big, and*
(ii) *$\{D\}$ has support with only normal crossings.*

Then $R^i\pi_*\mathcal{O}_X(K_X + \lceil D \rceil) = 0$ for every $i > 0$.

PROOF. We divide the proof into two steps.

STEP 1. In this step, we treat a special case.

We prove the theorem under the conditions:

(1) D is π-ample, and
(2) $\{D\}$ has support with only simple normal crossings.

We may assume that S is affine since the statement is local. Then, by Lemma 3.2.3 below, we may assume that X and S are projective and D is ample by replacing D with $D + \pi^*A$, where A is a sufficiently ample Cartier divisor on S.

We take an ample Cartier divisor H on S and a positive integer m. Let us consider the following spectral sequence

$$\begin{aligned} E_2^{p,q} =& H^p(S, R^q\pi_*\mathcal{O}_X(K_X + \lceil D \rceil + m\pi^*H)) \\ \simeq& H^p(S, R^q\pi_*\mathcal{O}_X(K_X + \lceil D \rceil) \otimes \mathcal{O}_S(mH)) \\ &\Rightarrow H^{p+q}(X, \mathcal{O}_X(K_X + \lceil D \rceil + m\pi^*H)). \end{aligned}$$

For every sufficiently large integer m, we have $E_2^{p,q} = 0$ for $p > 0$ by Serre's vanishing theorem. Therefore, $E_2^{0,q} = E_\infty^q$ holds for every q. Thus, we obtain

$$\begin{aligned} &H^0(S, R^q\pi_*\mathcal{O}_X(K_X + \lceil D \rceil + m\pi^*H)) \\ &= H^q(X, \mathcal{O}_X(K_X + \lceil D \rceil + m\pi^*H)) = 0 \end{aligned}$$

for $q>0$ by Theorem 3.1.7. Since H is ample on S and m is sufficiently large,

$$\begin{aligned}&R^q\pi_*\mathcal{O}_X(K_X+\lceil D\rceil+m\pi^*H)\\&\simeq R^q\pi_*\mathcal{O}_X(K_X+\lceil D\rceil)\otimes\mathcal{O}_S(mH)\end{aligned}$$

is generated by global sections. Therefore, we obtain

$$R^i\pi_*\mathcal{O}_X(K_X+\lceil D\rceil)=0$$

for every $i>0$.

STEP 2. In this step, we treat the general case by using the result obtained in Step 1.

Now we prove the theorem under the conditions (i) and (ii). We may assume that S is affine since the statement is local. By Kodaira's lemma (see Lemma 2.1.18) and Hironaka's resolution theorem, we can construct a projective birational morphism $f:Y\to X$ from another smooth variety Y which is projective over S and divisors F_α's on Y such that $\operatorname{Supp} f^*D\cup(\underset{\alpha}{\cup}F_\alpha)\cup\operatorname{Exc}(f)$ is a simple normal crossing divisor on Y and that $f^*D-\sum\delta_\alpha F_\alpha$ is $\pi\circ f$-ample for some $\delta_\alpha\in\mathbb{Q}$ with $0<\delta_\alpha\ll 1$ (see also [KMM, Corollary 0-3-6]). Then by applying the result proved in Step 1 to f, we obtain

$$0=R^if_*\mathcal{O}_Y(K_Y+\lceil f^*D-\sum\delta_\alpha F_\alpha\rceil)=R^if_*\mathcal{O}_Y(K_Y+\lceil f^*D\rceil)$$

for every $i>0$. We can also see that

$$f_*\mathcal{O}_Y(K_Y+\lceil f^*D\rceil)\simeq\mathcal{O}_X(K_X+\lceil D\rceil)$$

by Lemma 3.2.2 below. So, we have, by the special case treated in Step 1,

$$\begin{aligned}0&=R^i(\pi\circ f)_*\mathcal{O}_Y(K_Y+\lceil f^*D-\sum\delta_\alpha F_\alpha\rceil)\\&=R^i\pi_*(f_*\mathcal{O}_Y(K_Y+\lceil f^*D\rceil))\\&=R^i\pi_*\mathcal{O}_X(K_X+\lceil D\rceil)\end{aligned}$$

for every $i>0$. □

LEMMA 3.2.2. *Let X be a smooth variety and let D be an $\mathbb{R}$-divisor on X such that* $\operatorname{Supp}\{D\}$ *is a simple normal crossing divisor on X. Let $f:Y\to X$ be a proper birational morphism from a smooth variety Y such that* $\operatorname{Supp} f^*\{D\}\cup\operatorname{Exc}(f)$ *is a simple normal crossing divisor on Y. Then we have*

$$f_*\mathcal{O}_Y(K_Y+\lceil f^*D\rceil)\simeq\mathcal{O}_X(K_X+\lceil D\rceil).$$

Proof. We put $\Delta = \lceil D\rceil - \lfloor D\rfloor$. Then Δ is a reduced simple normal crossing divisor on X. We can write

$$K_Y + f_*^{-1}\Delta = f^*(K_X+\Delta) + \sum_{E_i\colon f\text{-exceptional}} a(E_i, X, \Delta)E_i.$$

We have $a(E_i, X, \Delta) \in \mathbb{Z}$ and $a(E_i, X, \Delta) \geq -1$ for every i. Then

$$K_Y + f_*^{-1}\Delta + f^*\lfloor D\rfloor = f^*(K_X+\lceil D\rceil) + \sum_{E_i\colon f\text{-exceptional}} a(E_i, X, \Delta)E_i.$$

We can easily check that

$$\mathrm{mult}_{E_i}\left(\lceil f^*D\rceil - (f_*^{-1}\Delta + f^*\lfloor D\rfloor)\right) \geq 1$$

for every f-exceptional divisor E_i with $a(E_i, X, \Delta) = -1$. Thus we can write

$$K_Y + \lceil f^*D\rceil = f^*(K_X+\lceil D\rceil) + F,$$

where F is an effective f-exceptional Cartier divisor on Y. Therefore, we have $f_*\mathcal{O}_Y(K_Y+\lceil f^*D\rceil) \simeq \mathcal{O}_X(K_X+\lceil D\rceil)$. $\square$

We used the following lemma in the proof of Theorem 3.2.1. We give a detailed proof for the reader's convenience (see also Lemma 5.5.2).

Lemma 3.2.3. *Let $\pi : X \to S$ be a projective surjective morphism from a smooth variety X to an affine variety S. Let D be a $\mathbb{Q}$-divisor on X such that D is π-ample and* Supp$\{D\}$ *is a simple normal crossing divisor on X. Then there exist a completion $\overline{\pi} : \overline{X} \to \overline{S}$ of $\pi : X \to S$ where $\overline{X}$ and $\overline{S}$ are both projective with $\overline{\pi}|_X = \pi$ and a $\overline{\pi}$-ample $\mathbb{Q}$-divisor $\overline{D}$ on $\overline{X}$ with $\overline{D}|_X = D$ such that* Supp$\{\overline{D}\}$ *is a simple normal crossing divisor on $\overline{X}$.*

Proof. Let m be a sufficiently large and divisible positive integer such that the natural surjection

$$\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$$

induces an embedding of X into $\mathbb{P}_S(\pi_*\mathcal{O}_X(mD))$ over S. Let $\pi' : X' \to \overline{S}$ be an arbitrary completion of $\pi : X \to S$ such that X' and $\overline{S}$ are both projective and X' is smooth. We can construct such $\pi' : X' \to \overline{S}$ by Hironaka's resolution theorem. Let D' be the closure of D on X'. We consider the natural map

$$\pi'^*\pi'_*\mathcal{O}_{X'}(mD') \to \mathcal{O}_{X'}(mD').$$

The image of the above map can be written as

$$\mathcal{J}\otimes\mathcal{O}_{X'}(mD') \subset \mathcal{O}_{X'}(mD'),$$

where $\mathcal{J}$ is an ideal sheaf on X' such that $\operatorname{Supp}\mathcal{O}_{X'}/\mathcal{J} \subset X'\setminus X$. Let X'' be the normalization of the blow-up of X' by $\mathcal{J}$ and $f : X'' \to X'$

the natural map. We note that f is an isomorphism over $X \subset X'$. We can write $f^{-1}\mathcal{J} \cdot \mathcal{O}_{X''} = \mathcal{O}_{X''}(-E)$ for some effective Cartier divisor E on X''. By replacing X' with X'' and mD' with $mf^*D' - E$, we may assume that mD' is π-very ample over S and is π-generated over $\overline{S}$. Therefore, we can consider the morphism $\varphi : X' \to X''$ over $\overline{S}$ associated to the surjection

$$\pi'^{*}\pi'_{*}\mathcal{O}_{X'}(mD') \to \mathcal{O}_{X'}(mD') \to 0.$$

We note that φ is an isomorphsim over S by construction. By replacing X' with X'' again, we may assume that D' is π'-ample. By using Hironaka's resolution theorem, we may further assume that X' is smooth. By Szabó's resolution lemma (see Lemma 2.3.19), we can make $\mathrm{Supp}\{D'\}$ a simple normal crossing divisor. Thus, we obtain desired completions $\overline{\pi} : \overline{X} \to \overline{S}$ and $\overline{D}$. ☐

Remark 3.2.4. In Lemma 3.2.3, we used Szabó's resolution lemma (see Lemma 2.3.19), which was obtained after [**KMM**] was written. See 3.2.5 below.

3.2.5 (Kawamata–Viehweg vanishing theorem without using Szabó's resolution lemma). Here, we explain how to prove the Kawamata–Viehweg vanishing theorem (see Theorem 3.2.1) without using Szabó's resolution lemma (see Lemma 2.3.19). The following proof is due to Noboru Nakayama.

Proof of Theorem 3.2.1 without Szabó's lemma. It is sufficient to prove Step 1 in the proof of Theorem 3.2.1. Let $\pi : X \to S$ be a projective surjective morphism from a smooth variety X to an affine variety S and let D be a $\mathbb{Q}$-divisor on X such that D is π-ample and that $\mathrm{Supp}\{D\}$ is a simple normal crossing divisor on X. Then, by taking completions, we have projective varieties $\overline{X}$ and $\overline{S}$ with a projective morphism $\overline{\pi} : \overline{X} \to \overline{S}$ such that

- X is a Zariski open dense subset of $\overline{X}$,
- S is a Zariski open dense subset of $\overline{S}$, and
- $\overline{\pi}|_X$ is the composition of π and the open immersion $S \to \overline{S}$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\pi"'] & \overline{X} \arrow[d, "\overline{\pi}"] \\ S \arrow[r, hook] & \overline{S} \end{tikzcd}$$

Note that $\overline{\pi}^{-1}(S) = X$ since π is proper.

Claim. *In the above setting, there exist a birational morphism $\overline{\mu} : \overline{Y} \to \overline{X}$ from another smooth projective variety $\overline{Y}$ and a $\mathbb{Q}$-divisor $\overline{C}$ on $\overline{Y}$ such that*

(i) $\overline{C}$ *is relatively nef and relatively big over* $\overline{S}$,
(ii) $\operatorname{Supp}\{\overline{C}\} \cup \operatorname{Exc}(\overline{\mu})$ *is a simple normal crossing divisor on* $\overline{Y}$, *and*
(iii) $\overline{C}|_Y = \mu^* D$, *where* $Y = \overline{\mu}^{-1}(X)$ *and* $\mu = \overline{\mu}|_Y : Y \to X$ *is the induced birational morphism.*

Here, we have an isomorphism

$$(\clubsuit) \qquad \mathcal{O}_X(K_X + \lceil D \rceil) \simeq \mu_* \mathcal{O}_Y(K_Y + \lceil C \rceil).$$

Proof of Claim. Note that, by Lemma 3.2.2, we can check the isomorphism ($\clubsuit$) by (ii) and (iii). By taking a resolution of $\overline{X}$, we may assume that $\overline{X}$ is smooth. Then the closure $\overline{D}$ of D in $\overline{X}$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Let us consider the natural homomorphism

$$\varphi_m : \overline{\pi}^* \overline{\pi}_* \mathcal{O}_{\overline{X}}(m\overline{D}) \to \mathcal{O}_{\overline{X}}(m\overline{D})$$

for a sufficiently large positive fixed integer m such that mD is Cartier. Then φ_m is surjective on X since D is π-ample. By taking some further blow-ups, we may assume that the image of φ_m is expressed as $\mathcal{O}_{\overline{X}}(m\overline{D} - E)$ for an effective Cartier divisor E on $\overline{X}$ with $E \cap X = \emptyset$. Thus, we have a projective variety $\overline{P}$ over $\overline{S}$ and a morphism $f : \overline{X} \to \overline{P}$ over $\overline{S}$ such that

$$f^* H \sim m\overline{D} - E$$

for a Cartier divisor H on $\overline{P}$ which is relatively ample over $\overline{S}$. Since D is π-ample and $E \cap X = \emptyset$, the induced morphism

$$f|_X : X = \overline{X} \times_{\overline{S}} S \to P := \overline{P} \times_{\overline{S}} S$$

is finite. In particular, f is a generically finite morphism. We set

$$D' := \overline{D} - \frac{1}{m} E.$$

Then D' is $\overline{\pi}$-nef and $\overline{\pi}$-big, and $D'|_X = D$. We can take a birational morphism $\overline{\mu} : \overline{Y} \to \overline{X}$ from another smooth projective variety $\overline{Y}$ such that the union of the $\overline{\mu}$-exceptional locus and $\operatorname{Supp} \overline{\mu}^*(\{D'\})$ is a simple normal crossing divisor on $\overline{Y}$. We set $\overline{C} := \overline{\mu}^* D'$. Then we have a desired $\overline{\mu} : \overline{Y} \to \overline{X}$ with $\overline{C}$. $\square$

We can easily see that we can prove Theorem 3.2.1 without using Lemma 3.2.3 when S is *projective* (see the proof of Theorem 3.2.1). Note that we do not have to shrink S and assume that S is affine in the proof of Theorem 3.2.1 if S is *projective*. From now on, we will freely use Theorem 3.2.1 when the target space is projective.

By applying Theorem 3.2.1, we obtain

$$R^i \overline{\mu}_* \mathcal{O}_{\overline{Y}}(K_{\overline{Y}} + \lceil \overline{C} \rceil) = 0$$

for every $i > 0$ since $\overline{X}$ is projective. By applying Theorem 3.2.1 again, we obtain

$$R^i(\overline{\pi} \circ \overline{\mu})_* \mathcal{O}_{\overline{Y}}(K_{\overline{Y}} + \lceil \overline{C} \rceil) = 0$$

for every $i > 0$ since $\overline{S}$ is projective. Hence, by the Leray spectral sequence, we have

$$R^i \overline{\pi}_* (\overline{\mu}_* \mathcal{O}_{\overline{Y}}(K_{\overline{Y}} + \lceil \overline{C} \rceil)) = 0$$

for every $i > 0$. By considering the restriction to S and by the isomorphism ($\clubsuit$), we have

$$R^i \pi_* \mathcal{O}_X(K_X + \lceil D \rceil) = 0$$

for every $i > 0$. It is the desired Kawamata–Viehweg vanishing theorem. $\square$

REMARK 3.2.6. In [**Nak1**, Theorem 3.7], Nakayama proved a generalization of the Kawamata–Viehweg vanishing theorem (see Theorem 3.2.1) in the analytic category. Of course, the proof of [**Nak1**, Theorem 3.7] does not need Szabó's resolution lemma. For several related results in the analytic category, we recommend the reader to see [**F31**].

As a very special case of Theorem 3.2.1, we have:

THEOREM 3.2.7 (Grauert–Riemenschneider vanishing theorem). *Let $f : X \to Y$ be a generically finite morphism from a smooth variety X. Then $R^i f_* \mathcal{O}_X(K_X) = 0$ for every $i > 0$.*

PROOF. Note that $K_X - K_X$ is f-nef and f-big since f is generically finite. Therefore, we obtain Theorem 3.2.7 as a special case of Theorem 3.2.1. $\square$

For a related result, see Lemma 3.8.7, Remark 3.8.8, and Theorem 3.8.9 below.

Viehweg's formulation of the Kawamata–Viehweg vanishing theorem is slightly different from Theorem 3.2.1.

THEOREM 3.2.8 (Viehweg). *Let X be a smooth variety and let $\pi : X \to S$ be a proper surjective morphism onto a variety S. Assume that a $\mathbb{Q}$-divisor D on X satisfies the following conditions*:

(i′) *D is π-nef and $\lceil D \rceil$ is π-big, and*
(ii) *$\{D\}$ has support with only normal crossings.*

Then $R^i \pi_* \mathcal{O}_X(K_X + \lceil D \rceil) = 0$ for every $i > 0$.

We note that the condition (i′) in Theorem 3.2.8 is slightly weaker than (i) in Theorem 3.2.1. We discuss a generalization of Theorem 3.2.8 in Section 3.3.

Let us generalize Theorem 3.2.1 for $\mathbb{R}$-divisors. We will repeatedly use it in the subsequent chapters.

THEOREM 3.2.9 (Kawamata–Viehweg vanishing theorem for $\mathbb{R}$-divisors). *Let X be a smooth variety and let $\pi : X \to S$ be a proper surjective morphism onto a variety S. Assume that an $\mathbb{R}$-divisor D on X satisfies the following conditions*:

(i) *D is π-nef and π-big, and*
(ii) *$\{D\}$ has support with only normal crossings.*

Then $R^i\pi_*\mathcal{O}_X(K_X + \lceil D\rceil) = 0$ for every $i > 0$.

PROOF. When D is π-ample, we perturb the coefficients of D and may assume that D is a $\mathbb{Q}$-divisor. Then, by Theorem 3.2.1, we obtain $R^i\pi_*\mathcal{O}_X(K_X + \lceil D\rceil) = 0$ for every $i > 0$. By using this special case, Step 2 in the proof of Theorem 3.2.1 works without any changes. So, we obtain this theorem. □

As a corollary of Theorem 3.2.9, we obtain the vanishing theorem of Reid–Fukuda type. It will play important roles in the subsequent chapters. Before we state it, we prepare the following definition.

DEFINITION 3.2.10 (Nef and log big divisors). Let $f : V \to W$ be a proper surjective morphism from a smooth variety V to a variety W and let B be a boundary $\mathbb{R}$-divisor on V such that $\operatorname{Supp} B$ is a simple normal crossing divisor. We put $T = \lfloor B\rfloor$. Let $T = \sum_{i=1}^{m} T_i$ be the irreducible decomposition. Let G be an $\mathbb{R}$-divisor on V. We say that G is *f-nef and f-log big with respect to* (V, B) if and only if G is f-nef, f-big, and $G|_C$ is $f|_C$-big for every C, where C is an irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k}$ for some $\{i_1, \cdots, i_k\} \subset \{1, \cdots, m\}$.

Of course, Definition 3.2.10 is compatible with Definition 5.7.2 below.

THEOREM 3.2.11 (Vanishing theorem of Reid–Fukuda type). *Let V be a smooth variety and let B be a boundary $\mathbb{R}$-divisor on V such that* $\operatorname{Supp} B$ *is a simple normal crossing divisor. Let $f : V \to W$ be a proper morphism onto a variety W. Assume that D is a Cartier divisor on V such that $D - (K_V + B)$ is f-nef and f-log big with respect to (V, B). Then $R^if_*\mathcal{O}_V(D) = 0$ for every $i > 0$.*

PROOF. We use induction on the number of irreducible components of $\lfloor B\rfloor$ and on the dimension of V. If $\lfloor B\rfloor = 0$, then Theorem 3.2.11 follows from the Kawamata–Viehweg vanishing theorem: Theorem 3.2.9. Therefore, we may assume that there is an irreducible divisor $S \subset \lfloor B\rfloor$.

We consider the following short exact sequence

$$0 \to \mathcal{O}_V(D-S) \to \mathcal{O}_V(D) \to \mathcal{O}_S(D) \to 0.$$

By induction, we see that $R^i f_*\mathcal{O}_V(D-S) = 0$ and $R^i f_*\mathcal{O}_S(D) = 0$ for every $i > 0$. Thus, we have $R^i f_*\mathcal{O}_V(D) = 0$ for every $i > 0$. □

Note that Theorem 3.2.11 contains Norimatsu's vanishing theorem.

THEOREM 3.2.12 (Norimatsu vanishing theorem). *Let X be a smooth projective variety and let Δ be a reduced simple normal crossing divisor on X. Let D be an ample Cartier divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + \Delta + D)) = 0$$

for every $i > 0$.

Of course, we can obtain Theorem 3.2.12 easily as an easy consequence of Kodaira's vanishing theorem (see Theorem 3.1.3) by using induction on the number of irreducible components of Δ and on the dimension of X (see the proof of Theorem 3.2.11). Note that Kawamata [**Ka1**] used Theorem 3.2.12 for the proof of the Kawamata–Viehweg vanishing theorem.

3.3. Viehweg vanishing theorem

Viehweg sometimes used the following formulation of the Kawamata–Viehweg vanishing theorem (see, for example, [**V2**, Theorem 2.28] and Theorem 3.2.8). In this book, we call it the Viehweg vanishing theorem.

THEOREM 3.3.1 (Viehweg vanishing theorem). *Let X be a smooth proper variety. Let $\mathcal{L}$ be a line bundle, let N be a positive integer, and let D be an effective Cartier divisor on X whose support is a simple normal crossing divisor. Assume that $\mathcal{L}^N(-D)$ is nef and that the sheaf*

$$\mathcal{L}^{(1)} = \mathcal{L}(-\lfloor \frac{D}{N} \rfloor)$$

is big. Then

$$H^i(X, \mathcal{L}^{(1)} \otimes \omega_X) = 0$$

for every $i > 0$.

In this section, we quickly give a proof of a slightly generalized Viehweg vanishing theorem as an application of the usual Kawamata–Viehweg vanishing theorem. For the original approach to Theorem 3.3.1, see [**EsVi2**, (2.13) Theorem], [**EsVi3**, Corollary 5.12 d)], and so on. Our proof is different from the proofs given in [**EsVi2**] and [**EsVi3**].

THEOREM 3.3.2. *Let $\pi : X \to S$ be a proper surjective morphism from a smooth variety X, let $\mathcal{L}$ be an invertible sheaf on X, and let D be an effective Cartier divisor on X such that* $\operatorname{Supp} D$ *is normal crossing. Assume that $\mathcal{L}^N(-D)$ is π-nef for some positive integer N and that $\kappa(X_\eta, (\mathcal{L}^{(1)})_\eta) = m$, where X_η is the generic fiber of π, $(\mathcal{L}^{(1)})_\eta = \mathcal{L}^{(1)}|_{X_\eta}$, and*

$$\mathcal{L}^{(1)} = \mathcal{L}(-\lfloor \frac{D}{N} \rfloor).$$

Then we have

$$R^i\pi_*(\mathcal{L}^{(1)} \otimes \omega_X) = 0$$

for $i > \dim X - \dim S - m$.

We note that $\operatorname{Supp} D$ is not necessarily *simple* normal crossing. We only assume that $\operatorname{Supp} D$ is normal crossing.

REMARK 3.3.3. In Theorem 3.3.2, we assume that S is a point for simplicity. We note that $\kappa(X, \mathcal{L}^{(1)}) = m$ does not necessarily imply $\kappa(X, \mathcal{L}^{(i)}) = m$ for $2 \le i \le N-1$, where

$$\mathcal{L}^{(i)} = \mathcal{L}^{\otimes i}(-\lfloor \frac{iD}{N} \rfloor).$$

Therefore, Viehweg's original arguments in [**V1**] depending on Bogomolov's vanishing theorem do not seem to work in our setting.

Let us reformulate Theorem 3.2.1 for the proof of Theorem 3.3.2.

THEOREM 3.3.4 (Kawamata–Viehweg vanishing theorem). *Let $f : Y \to X$ be a proper surjective morphism from a smooth variety Y and let M be a Cartier divisor on Y. Let Δ be an effective $\mathbb{Q}$-divisor on Y such that* $\operatorname{Supp}\Delta$ *is normal crossing and* $\lfloor \Delta \rfloor = 0$. *Assume that $M - (K_Y + \Delta)$ is f-nef and f-big. Then*

$$R^i f_* \mathcal{O}_Y(M) = 0$$

for every $i > 0$.

PROOF. We put $D = M - (K_Y + \Delta)$. Then D is an f-nef and f-big $\mathbb{Q}$-divisor on Y such that $\{D\} = \lceil \Delta \rceil - \Delta$ and $\lceil D \rceil = M - K_Y$. By Theorem 3.2.1, we obtain $R^i f_* \mathcal{O}_Y(K_X + \lceil D \rceil) = 0$ for every $i > 0$. Therefore, $R^i f_* \mathcal{O}_Y(M) = 0$ for every $i > 0$. □

REMARK 3.3.5. It is obvious that Theorem 3.3.4 is a special case of Theorem 3.3.2. By applying Theorem 3.3.2, the assumption in Theorem 3.3.4 can be weaken as follows: $M - (K_X + \Delta)$ is f-nef and $M - K_X$ is f-big. We note that $M - K_X$ is f-big if $M - (K_X + \Delta)$ is f-big. In this section, we give a quick proof of Theorem 3.3.2 only by using

Theorem 3.3.4 and Hironaka's resolution. Therefore, Theorem 3.3.2 is essentially the same as Theorem 3.3.4.

Let us start the proof of Theorem 3.3.2.

Proof of Theorem 3.3.2. Without loss of generality, we may assume that S is affine. Let $f : Y \to X$ be a proper birational morphism from a smooth quasi-projective variety Y such that $\operatorname{Supp} f^*D \cup \operatorname{Exc}(f)$ is a simple normal crossing divisor. We write

$$K_Y = f^*(K_X + (1-\varepsilon)\{\frac{D}{N}\}) + E_\varepsilon.$$

Then $F = \lceil E_\varepsilon \rceil$ is an effective exceptional Cartier divisor on Y and independent of ε for $0 < \varepsilon \ll 1$. Therefore, the coefficients of $F - E_\varepsilon$ are continuous for $0 < \varepsilon \ll 1$. Let L be a Cartier divisor on X such that $\mathcal{L} \simeq \mathcal{O}_X(L)$. We may assume that $\kappa(X_\eta, (L - \lfloor \frac{D}{N} \rfloor)_\eta) = m \geq 0$. Let $\Phi : X \dashrightarrow Z$ be the relative Iitaka fibration over S with respect to $l(L - \lfloor \frac{D}{N} \rfloor)$, where l is a sufficiently large and divisible positive integer. We may further assume that

$$f^*(L - \lfloor \frac{D}{N} \rfloor) \sim_{\mathbb{Q}} \varphi^* A + E,$$

where E is an effective $\mathbb{Q}$-divisor such that $\operatorname{Supp} E \cup \operatorname{Supp} f^*D \cup \operatorname{Exc}(f)$ is simple normal crossing, $\varphi = \Phi \circ f : Y \to Z$ is a morphism, and A is a ψ-ample $\mathbb{Q}$-divisor on Z with $\psi : Z \to S$.

$$\begin{tikzcd}
Y \arrow[d, "f"'] \arrow[dr, "\varphi"] & \\
X \arrow[r, dashed, "\Phi"] \arrow[dr, "\pi"'] & Z \arrow[d, "\psi"] \\
& S
\end{tikzcd}$$

Let

$$\sum_i E_i = \operatorname{Supp} E \cup \operatorname{Supp} f^*D \cup \operatorname{Exc}(f)$$

be the irreducible decomposition. We can write $E_\varepsilon = \sum_i a_i^\varepsilon E_i$ and $E = \sum_i b_i E_i$. We note that a_i^ε is continuous for $0 < \varepsilon \ll 1$. We put

$$\Delta_\varepsilon = F - E_\varepsilon + \varepsilon E.$$

By definition, we can see that every coefficient of Δ_ε is in $[0, 2)$ for $0 < \varepsilon \ll 1$. Thus, $\lfloor \Delta_\varepsilon \rfloor$ is reduced. If $a_i^\varepsilon < 0$, then $a_i^\varepsilon \geq -1 + \frac{1}{N}$ for $0 < \varepsilon \ll 1$. Therefore, if $\lceil a_i^\varepsilon \rceil - a_i^\varepsilon + \varepsilon b_i \geq 1$ for $0 < \varepsilon \ll 1$, then $a_i^\varepsilon > 0$.

Thus, $F' = F - \lfloor \Delta_\varepsilon \rfloor$ is effective and f-exceptional for $0 < \varepsilon \ll 1$. On the other hand, $(Y, \{\Delta_\varepsilon\})$ is obviously klt for $0 < \varepsilon \ll 1$. We note that

$$\begin{aligned} & f^*(K_X + L - \lfloor \frac{D}{N} \rfloor) + F' - (K_Y + \{\Delta_\varepsilon\}) \\ &= f^*(K_X + L - \lfloor \frac{D}{N} \rfloor) + F - f^*(K_X + (1-\varepsilon)\{\frac{D}{N}\}) - E_\varepsilon \\ &\quad - (F - E_\varepsilon + \varepsilon E) \\ &\sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N}) + \varepsilon \varphi^* A \end{aligned}$$

for a rational number ε with $0 < \varepsilon \ll 1$. We put

$$M = f^*(K_X + L - \lfloor \frac{D}{N} \rfloor) + F'.$$

Let H be a p-ample general smooth Cartier divisor on Y, where $p = \psi \circ \varphi = \pi \circ f : Y \to S$. Since

$$(M + H) - (K_Y + \{\Delta_\varepsilon\}) \sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N}) + \varepsilon \varphi^* A + H,$$

is p-ample, we obtain

$$R^i p_* \mathcal{O}_Y(M + H) = 0$$

for every $i > 0$ by Theorem 3.3.4. By the long exact sequence

$$\cdots \to R^i p_* \mathcal{O}_Y(M) \to R^i p_* \mathcal{O}_Y(M + H) \to R^i p_* \mathcal{O}_H(M + H) \to \cdots$$

obtained from

$$0 \to \mathcal{O}_Y(M) \to \mathcal{O}_Y(M + H) \to \mathcal{O}_H(M + H) \to 0,$$

we obtain

$$R^i p_* \mathcal{O}_H(M + H) \simeq R^{i+1} p_* \mathcal{O}_Y(M)$$

for every $i > 0$. We note that

$$M - (K_Y + \{\Delta_\varepsilon\}) \sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N}) + \varepsilon \varphi^* A$$

and

$$(M + H)|_H - (K_H + \{\Delta_\varepsilon\}|_H) \sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N})|_H + \varepsilon \varphi^* A|_H.$$

We also note that $(H, \{\Delta_\varepsilon\}|_H)$ is klt and

$$\kappa(H_\eta, (\varphi^* A)|_{H_\eta}) \geq \min\{m, \dim H_\eta\}.$$

By repeating the above argument, that is, taking a general smooth hyperplane cut, and by Theorem 3.3.4, we obtain

$$R^i p_* \mathcal{O}_Y(M) = R^i p_* \mathcal{O}_Y(f^*(K_X + L - \lfloor \frac{D}{N} \rfloor) + F') = 0$$

for every $i > \dim Y - \dim S - m = \dim X - \dim S - m$ (see also [V1, Remark 0.2]). On the other hand,

$$R^i f_* \mathcal{O}_Y(M) = R^i f_* \mathcal{O}_Y(f^*(K_X + L - \lfloor \frac{D}{N} \rfloor) + F') = 0$$

for every $i > 0$ by Theorem 3.3.4. We note that

$$f_* \mathcal{O}_Y(f^*(K_X + L - \lfloor \frac{D}{N} \rfloor) + F') \simeq \mathcal{O}_X(K_X + L - \lfloor \frac{D}{N} \rfloor)$$

by the projection formula because F' is effective and f-exceptional. Therefore, we obtain

$$R^i \pi_* \mathcal{O}_X(K_X + L - \lfloor \frac{D}{N} \rfloor) = R^i p_* \mathcal{O}_Y(M) = 0$$

for every $i > \dim X - \dim S - m$. $\square$

We give an obvious corollary of Theorem 3.3.2.

COROLLARY 3.3.6. *Let X be an n-dimensional smooth complete variety and let $\mathcal{L}$ be an invertible sheaf on X. Assume that $D \in |\mathcal{L}^N|$ for some positive integer N and that* $\operatorname{Supp} D$ *is a simple normal crossing divisor on X. Then we have*

$$H^i(X, \mathcal{L}^{(1)} \otimes \omega_X) = 0$$

for $i > n - \kappa(X, \{\frac{D}{N}\})$.

We think that Theorem 3.3.7, which is similar to Theorem 3.3.2 and easily follows from the usual Kawamata–Viehweg vanishing theorem: Theorem 3.2.1 (see also Theorem 3.3.4), is easier to use than Theorem 3.3.2. So we contain it for the reader's convenience.

THEOREM 3.3.7 (Kawamata–Viehweg vanishing theorem). *Let $f : Y \to X$ be a projective morphism from a smooth variety Y onto a variety X. Let Δ be an effective $\mathbb{Q}$-divisor on Y such that* $\operatorname{Supp} \Delta$ *is a normal crossing divisor and that $\lfloor \Delta \rfloor = 0$. Let M be a Cartier divisor on Y such that $M - (K_Y + \Delta)$ is f-nef and $\nu(X_\eta, (M - (K_Y + \Delta))|_{X_\eta}) = m$, where X_η is the generic fiber of f. Then $R^i f_* \mathcal{O}_Y(M) = 0$ for every $i > \dim Y - \dim X - m$.*

PROOF. We use induction on $\dim Y - \dim X$. If $\dim Y - \dim X = 0$, then $M - (K_Y + \Delta)$ is f-big. Therefore, Theorem 3.3.7 is a special case of Theorem 3.3.4 when $\dim Y - \dim X = 0$. When $m = \dim Y - \dim X$, Theorem 3.3.7 follows from Theorem 3.3.4. Thus, we may assume that $m < \dim Y - \dim X$. Without loss of generality, we may assume that X is affine by shrinking X. Let A be an f-very ample Cartier divisor on Y. We take a general member H of $|A|$. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(M) \to \mathcal{O}_Y(M+H) \to \mathcal{O}_H(M+H) \to 0.$$

Since $M + H - (K_Y + \Delta)$ is f-ample, we obtain $R^i f_* \mathcal{O}_Y(M+H) = 0$ for every $i > 0$ by Theorem 3.3.4. This implies that

$$R^i f_* \mathcal{O}_H(M+H) \simeq R^{i+1} f_* \mathcal{O}_Y(M)$$

holds for every $i \geq 1$. Since $(M+H)|_H - (K_H + \Delta|_H)$ is f-nef and

$$\nu(H_\eta, ((M+H)|_H - (K_H + \Delta|_H))|_{H_\eta}) \geq m,$$

where H_η is the generic fiber of $H \to f(H)$, we obtain

$$R^i f_* \mathcal{O}_H(M+H) = 0$$

for $i > \dim H - \dim X - m = \dim Y - \dim X - m - 1$ by induction on $\dim Y - \dim X$. Therefore, we have

$$R^i f_* \mathcal{O}_Y(M) = 0$$

for $i > \dim Y - \dim X - m$. □

3.4. Nadel vanishing theorem

Let us recall the definition of the multiplier ideal sheaf of a pair (X, Δ) (see also 2.3.11).

Definition 3.4.1 (Multiplier ideal sheaves). Let X be a normal variety and let Δ be a (not necessarily effective) $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

and that $\operatorname{Supp} \Delta_Y$ is a simple normal crossing divisor on Y. We put

$$\mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor)$$

and call it the *multiplier ideal sheaf* of the pair (X, Δ). It is easy to see that $\mathcal{J}(X, \Delta)$ is independent of the resolution $f : Y \to X$ by the proof of Proposition 6.3.1. When Δ is effective, we have $\mathcal{J}(X, \Delta) \subset \mathcal{O}_X$.

The following (algebraic version of) Nadel vanishing theorem is very important for the recent developments of the higher-dimensional algebraic geometry (see, for example, [HaKo, Chapter 6, Multiplier ideal sheaves]). It is a variant of the Kawamata–Viehweg vanishing theorem (see, for example, Theorem 3.2.9).

THEOREM 3.4.2 (Nadel vanishing theorem). *Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let D be a Cartier divisor on X such that $D-(K_X+\Delta)$ is π-nef and π-big, where $\pi : X \to S$ is a proper surjective morphism onto a variety S. Then*

$$R^i\pi_*(\mathcal{O}_X(D)\otimes \mathcal{J}(X,\Delta))=0$$

for every $i>0$.

PROOF. Let $f : Y \to X$ be a resolution as in Definition 3.4.1. Then

$$f^*D-\lfloor \Delta_Y\rfloor-(K_Y+\{\Delta_Y\})=f^*(D-(K_X+\Delta))$$

is $\pi\circ f$-nef and $\pi \circ f$-big. In particular, it is f-nef and f-big. By the Kawamata–Viehweg vanishing theorem (see Theorem 3.2.9), we have

$$R^if_*\mathcal{O}_Y(f^*D-\lfloor \Delta_Y\rfloor)=0$$

for every $i>0$. By the projection formula, we obtain

$$f_*\mathcal{O}_Y(f^*D-\lfloor \Delta_Y\rfloor)=\mathcal{O}_X(D)\otimes \mathcal{J}(X,\Delta).$$

By the Kawamata–Viehweg vanishing theorem (see Theorem 3.2.9) again, we have

$$R^i(\pi\circ f)_*\mathcal{O}_Y(f^*D-\lfloor \Delta_Y\rfloor)=0$$

for every $i>0$. Therefore, $R^i\pi_*(\mathcal{O}_X(D)\otimes \mathcal{J}(X,\Delta))=0$ for every $i>0$. $\square$

REMARK 3.4.3. Let X be a smooth variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $\operatorname{Supp}\Delta$ is a normal crossing divisor and that $\lfloor \Delta \rfloor=0$. Then we can easily check that $\mathcal{J}(X,\Delta)=\mathcal{O}_X$. Therefore, Theorem 3.4.2 contains the usual Kawamata–Viehweg vanishing theorem (see, for example, Theorem 3.3.4).

For the details of the theory of (algebraic) multiplier ideal sheaves, we recommend the reader to see [**La2**, Part Three].

3.5. Miyaoka vanishing theorem

Let us recall Miyaoka's vanishing theorem (see [**Mi**, Proposition 2.3]). Miyaoka's vanishing theorem is the first vanishing theorem for the integral part of $\mathbb{Q}$-divisors. So, it is a historically important result.

THEOREM 3.5.1 (Miyaoka vanishing theorem). *Let X be a smooth projective surface and let D be a big Cartier divisor on X. Let $D = P+N$ be its Zariski decomposition, where P (resp. N) is the positive (resp. negative) part of the Zariski decomposition. Assume that $\lfloor N\rfloor = 0$. Then $H^1(X,\mathcal{O}_X(-D))=0$.*

Let us quickly recall the Zariski decomposition on smooth projective surfaces (see [Z]). For the details, see, for example, [Bă, Chapter 14].

THEOREM 3.5.2 (Zariski decomposition). *Let D be a big Cartier divisor on a smooth projective surface X. Then there exists a unique decomposition*

$$D = P + N$$

which satisfies the following conditions:

(i) *P is nef and big;*
(ii) *N is an effective $\mathbb{Q}$-divisor;*
(iii) *$P \cdot C = 0$ for every irreducible component C of* $\operatorname{Supp} N$.

This decomposition is called the Zariski decomposition of D. We usually call P (resp. N) the positive (resp. negative) part of the Zariski decomposition of D.

The following statement is a correct formulation of Miyaoka's vanishing theorem (see Theorem 3.5.1) from our modern viewpoint.

THEOREM 3.5.3. *Let X be a smooth complete variety with* $\dim X \geq 2$ *and let D be a Cartier divisor on X. Assume that D is numerically equivalent to $M + B$, where M is a nef $\mathbb{Q}$-divisor on X with $\nu(X, M) \geq 2$ and B is an effective $\mathbb{Q}$-divisor with $\lfloor B \rfloor = 0$. Then $H^1(X, \mathcal{O}_X(-D)) = 0$.*

PROOF. By Serre duality, it is sufficient to see that

$$H^{n-1}(X, \mathcal{O}_X(K_X + D)) = 0,$$

where $n = \dim X$. Let $\mathcal{J}(X, B)$ be the multiplier ideal sheaf of (X, B) (see Definition 3.4.1). We consider

$$\begin{aligned} \cdots \to H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{J}(X, B)) \to H^{n-1}(X, \mathcal{O}_X(K_X + D)) \\ \to H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_X/\mathcal{J}(X, B)) \to \cdots . \end{aligned}$$

Since $\lfloor B \rfloor = 0$, we see that $\dim \operatorname{Supp} \mathcal{O}_X/\mathcal{J}(X, B) \leq n - 2$. Therefore,

$$H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_X/\mathcal{J}(X, B)) = 0.$$

Thus, it is enough to see that

$$H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{J}(X, B)) = 0.$$

Let $f : Y \to X$ be a resolution such that $\operatorname{Supp} f^*B$ is a simple normal crossing divisor. Then we have

$$\mathcal{J}(X, B) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*B \rfloor)$$

and

$$R^i f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*B \rfloor) = 0$$

for every $i > 0$ (see, for example, Theorem 3.2.1). So, we obtain

$$\begin{aligned} &H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{J}(X, B)) \\ &\simeq H^{n-1}(Y, \mathcal{O}_Y(K_Y + f^*D - \lfloor f^*B \rfloor)) = 0 \end{aligned}$$

by Theorem 3.3.7. □

REMARK 3.5.4. In Theorem 3.5.3, we can replace the assumption $\nu(X, M) \geq 2$ with $\kappa(Y, f^*D - \lfloor f^*B \rfloor) \geq 2$ by Theorem 3.3.2.

PROOF OF THEOREM 3.5.1. Since $D = P + N$ is the Zariski decomposition, P is a nef and big $\mathbb{Q}$-divisor on X and N is an effective $\mathbb{Q}$-divisor on X. By assumption, $\lfloor N \rfloor = 0$. Thus, by Theorem 3.5.3, we obtain Theorem 3.5.1. □

3.6. Kollár injectivity theorem

In this section, we quickly review Kollár's injectivity theorem, torsion-free theorem, and vanishing theorem without proof.

In [**Tank**], Tankeev proved:

THEOREM 3.6.1 ([**Tank**, Proposition 1]). *Let X be a smooth projective variety with $\dim X \geq 2$. Assume that the complete linear system $|H|$ has no base points and determines a morphism $\Phi_{|H|} : X \to Y$ onto a variety Y with $\dim Y \geq 2$. Then*

$$H^0(X, \mathcal{O}_X(K_X + 2D)) \to H^0(D, \mathcal{O}_D((K_X + 2D)|_D))$$

is surjective for almost all divisors $D \in |H|$.

PROOF. By Bertini, D is smooth. Therefore, by Lemma 3.1.1, we obtain that

$$H^1(X, \mathcal{O}_X(K_X + D)) \to H^1(X, \mathcal{O}_X(K_X + 2D))$$

is injective. Thus we obtain the desired surjection. □

In [**Ko2**], Kollár obtained Theorem 3.6.2 as a generalization of Theorem 3.6.1. We call it Kollár's injectivity theorem.

THEOREM 3.6.2 ([**Ko2**, Theorem 2.2]). *Let X be a smooth projective variety and let L be a semi-ample Cartier divisor on X. Let D be a member of $|kL|$. Then*

$$H^i(X, \mathcal{O}_X(K_X + nL)) \to H^i(X, \mathcal{O}_X(K_X + (n + k)L)),$$

which is induced by the natural inclusion $\mathcal{O}_X \subset \mathcal{O}_X(D) \simeq \mathcal{O}_X(kL)$, is injective for every i and every positive integer n.

He also obtained Theorem 3.6.3 in [**Ko2**].

THEOREM 3.6.3 ([**Ko2**, Theorem 2.1]). *Let X be a smooth projective variety, let Y be an arbitrary projective variety, and let $\pi : X \to Y$ be a surjective morphism. Then we have the following properties.*

(i) *$R^i\pi_*\mathcal{O}_X(K_X)$ is torsion-free for every i.*

(ii) *Let L be an ample Cartier divisor on Y, then*

$$H^j(Y, \mathcal{O}_Y(L) \otimes R^i\pi_*\mathcal{O}_X(K_X)) = 0$$

for every $j > 0$ and every i.

We usually call Theorem 3.6.3 (i) (resp. (ii)) Kollár's torsion-free theorem (resp. Kollár's vanishing theorem). Note that Theorem 3.6.3 (ii) contains the Kodaira vanishing theorem for projective varieties: Theorem 3.1.3. We also note that Theorem 3.6.3 (i) generalizes the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7.

In [**Ko2**], Kollár proved Theorem 3.6.2 and Theorem 3.6.3 simultaneously. Therefore, the relationship between Theorem 3.6.2 and Theorem 3.6.3 is not clear by the proof in [**Ko2**]. Now it is well known that Theorem 3.6.2 and Theorem 3.6.3 are equivalent by the works of Kollár himself and Esnault–Viehweg (see, for example, [**EsVi3**] and [**Ko5**]). For the proof of Theorem 3.6.2 and Theorem 3.6.3, see also [**EsVi1**].

We do not prove Kollár's theorems here. We will prove complete generalizations in Chapter 5.

3.7. Enoki injectivity theorem

In this section, we discuss Enoki's injectivity theorem (see [**Eno**, Theorem 0.2]), which contains Kollár's original injectivity theorem: Theorem 3.6.2. We recommend the reader to compare the proof of Theorem 3.7.1 with the arguments in [**Ko2**, Section 2] and [**Ko6**, Chapter 9].

THEOREM 3.7.1 (Enoki's injectivity theorem). *Let X be a compact Kähler manifold and let L be a semi-positive line bundle on X. Then, for any non-zero holomorphic section s of $L^{\otimes k}$ with some positive integer k, the multiplication homomorphism*

$$\times s : H^q(X, \omega_X \otimes L^{\otimes l}) \longrightarrow H^q(X, \omega_X \otimes L^{\otimes (l+k)}),$$

which is induced by $\otimes s$, is injective for every $q \geq 0$ and $l > 0$.

Let us recall the basic notion of the complex geometry. For details, see, for example, [**Dem**].

DEFINITION 3.7.2 (Chern connection and its curvature form). Let X be a complex manifold and let (E, h) be a holomorphic hermitian vector bundle on X. Then there exists the *Chern connection* $D = D_{(E,h)}$, which can be split in a unique way as a sum of a $(1, 0)$ and of a

$(0,1)$-connection, $D = D'_{(E,h)} + D''_{(E,h)}$. By the definition of the Chern connection, $D'' = D''_{(E,h)} = \bar{\partial}$. We obtain the *curvature form*

$$\Theta_h(E) := D^2_{(E,h)}.$$

The subscripts might be suppressed if there is no risk of confusion.

Let L be a holomorphic line bundle on X. We say that L is *positive* (reps. *semi-positive*) if there exists a smooth hermitian metric h_L on L such that $\sqrt{-1}\Theta_{h_L}(L)$ is a positive (resp. semi-positive) $(1,1)$-form on X.

Definition 3.7.3 (Inner product). Let X be an n-dimensional complex manifold with the hermitian metric g. We denote by ω the *fundamental form* of g. Let (E,h) be a holomorphic hermitian vector bundle on X, and u, v are E-valued (p,q)-forms with measurable coefficients, we set

$$\|u\|^2 = \int_X |u|^2 dV_\omega, \ \langle\langle u, v\rangle\rangle = \int_X \langle u, v\rangle dV_\omega,$$

where $|u|$ (resp. $\langle u, v\rangle$) is the pointwise norm (resp. inner product) induced by g and h on $\Lambda^{p,q}T^*_X \otimes E$, and $dV_\omega = \frac{1}{n!}\omega^n$.

Let us prove Theorem 3.7.1.

Proof of Theorem 3.7.1. Throughout this proof, we fix a Kähler metric g on X. Let h be a smooth hermitian metric on L such that the curvature $\sqrt{-1}\Theta_h(L) = \sqrt{-1}\bar{\partial}\partial \log h$ is a smooth semi-positive $(1,1)$-form on X. We put $n = \dim X$. We introduce the space of $L^{\otimes l}$-valued harmonic (n,q)-forms as follows,

$$\mathcal{H}^{n,q}(X, L^{\otimes l}) := \{u \in C^{n,q}(X, L^{\otimes l}) | \Delta'' u = 0\}$$

for every $q \geq 0$, where

$$\Delta'' := \Delta''_{(L^{\otimes l}, h^l)} := D''^*_{(L^{\otimes l}, h^l)}\bar{\partial} + \bar{\partial} D''^*_{(L^{\otimes l}, h^l)}$$

and $C^{n,q}(X, L^{\otimes l})$ is the space of $L^{\otimes l}$-valued smooth (n,q)-forms on X. We note that $D''_{(L^{\otimes l}, h^l)} = \bar{\partial}$ and that $D''^*_{(L^{\otimes l}, h^l)}$ is the formal adjoint of $D''_{(L^{\otimes l}, h^l)}$. It is easy to see that $\Delta'' u = 0$ if and only if

$$D''^*_{(L^{\otimes l}, h^l)} u = \bar{\partial} u = 0$$

for $u \in C^{n,q}(X, L^{\otimes l})$ since X is compact. It is well known that

$$C^{n,q}(X, L^{\otimes l}) = \mathrm{Im}\bar{\partial} \oplus \mathcal{H}^{n,q}(X, L^{\otimes l}) \oplus \mathrm{Im} D''^*_{(L^{\otimes l}, h^l)}$$

and

$$\mathrm{Ker}\bar{\partial} = \mathrm{Im}\bar{\partial} \oplus \mathcal{H}^{n,q}(X, L^{\otimes l}).$$

Therefore, we have the following isomorphisms,

$$H^q(X, \omega_X \otimes L^{\otimes l}) \simeq H^{n,q}(X, L^{\otimes l}) = \frac{\mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}} \simeq \mathcal{H}^{n,q}(X, L^{\otimes l}).$$

We obtain $H^q(X, \omega_X \otimes L^{\otimes(l+k)}) \simeq \mathcal{H}^{n,q}(X, L^{\otimes(l+k)})$ similarly.

Claim. *The multiplication map*

$$\times s : \mathcal{H}^{n,q}(X, L^{\otimes l}) \longrightarrow \mathcal{H}^{n,q}(X, L^{\otimes(l+k)})$$

is well-defined.

If the claim is true, then the theorem is obvious. This is because $su = 0$ in $\mathcal{H}^{n,q}(X, L^{\otimes(l+k)})$ implies $u = 0$ for $u \in \mathcal{H}^{n,q}(X, L^{\otimes l})$. This implies the desired injectivity. Thus, it is sufficient to prove the above claim.

Proof of Claim. By the Nakano identity (see, for example, [Dem, (4.6)]), we have

$$\|D''^{*}_{(L^{\otimes l},h^l)}u\|^2 + \|D''u\|^2 = \|D'^{*}u\|^2 + \langle\langle\sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle\rangle$$

holds for $L^{\otimes l}$-valued smooth (n,q)-form u, where Λ is the adjoint of $\omega \wedge \cdot$ and ω is the fundamental form of g. If $u \in \mathcal{H}^{n,q}(X, L^{\otimes l})$, then the left hand side is zero by the definition of $\mathcal{H}^{n,q}(X, L^{\otimes l})$. Thus we obtain $\|D'^{*}u\|^2 = \langle\langle\sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle\rangle = 0$ since

$$\sqrt{-1}\Theta_{h^l}(L^{\otimes l}) = \sqrt{-1}l\Theta_h(L)$$

is a smooth semi-positive $(1,1)$-form on X. Therefore, $D'^{*}u = 0$ and $\langle\sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle_{h^l} = 0$, where $\langle\ ,\ \rangle_{h^l}$ is the pointwise inner product with respect to h^l and g. By Nakano's identity again,

$$\begin{aligned}&\|D''^{*}_{(L^{\otimes(l+k)},h^{l+k})}(su)\|^2 + \|D''(su)\|^2 \\ &= \|D'^{*}(su)\|^2 + \langle\langle\sqrt{-1}\Theta_{h^{l+k}}(L^{\otimes(l+k)})\Lambda su, su\rangle\rangle\end{aligned}$$

Note that we assumed $u \in \mathcal{H}^{n,q}(X, L^{\otimes l})$. Since s is holomorphic, $D''(su) = \bar{\partial}(su) = 0$ by the Leibnitz rule. We know that

$$D'^{*}(su) = - * \bar{\partial} * (su) = sD'^{*}u = 0$$

since s is a holomorphic $L^{\otimes k}$-valued $(0,0)$-form and $D'^{*}u = 0$, where $*$ is the Hodge star operator with respect to g. Note that D'^{*} is independent of the fiber metrics. So, we have

$$\|D''^{*}_{(L^{\otimes(l+k)},h^{l+k})}(su)\|^2 = \langle\langle\sqrt{-1}\Theta_{h^{l+k}}(L^{\otimes(l+k)})\Lambda su, su\rangle\rangle.$$

We note that

$$\begin{aligned}&\langle\sqrt{-1}\Theta_{h^{l+k}}(L^{\otimes(l+k)})\Lambda su, su\rangle_{h^{l+k}}\\&=\frac{l+k}{l}|s|^2_{h^k}\langle\sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle_{h^l}=0\end{aligned}$$

where $\langle\ ,\ \rangle_{h^{l+k}}$ (resp. $|s|_{h^k}$) is the pointwise inner product (resp. the pointwise norm of s) with respect to h^{l+k} and g (resp. with respect to h^k). Thus, we obtain $D''^{*}_{(L^{\otimes(l+k)},h^{l+k})}(su)=0$. Therefore, we know that $\Delta''_{(L^{\otimes(l+k)},h^{l+k})}(su)=0$, equivalently, $su\in\mathcal{H}^{n,q}(X,L^{\otimes(l+k)})$. We finish the proof of the claim. $\square$

Thus we obtain the desired injectivity theorem. $\square$

The above proof of Theorem 3.7.1, which is due to Enoki, is arguably simpler than Kollár's original proof of his injectivity theorem (see Theorem 3.6.2) in [Ko2].

We include Kodaira's vanishing theorem for compact complex manifolds and its proof based on Bochner's technique for the reader's convenience.

Theorem 3.7.4 (Kodaira vanishing theorem for complex manifolds). *Let X be a compact complex manifold and let L be a positive line bundle on X. Then $H^q(X,\omega_X\otimes L)=0$ for every $q>0$.*

Proof. We take a smooth hermitian metric h on L such that $\sqrt{-1}\Theta_h(L)=\sqrt{-1}\bar\partial\partial\log h$ is a smooth positive $(1,1)$-form on X. We define a Kähler metric g on X associated to $\omega:=\sqrt{-1}\Theta_h(L)$. As we saw in the proof of Theorem 3.7.1, we have

$$H^q(X,\omega_X\otimes L)\simeq\mathcal{H}^{n,q}(X,L)$$

where $n=\dim X$ and $\mathcal{H}^{n,q}(X,L)$ is the space of L-valued harmonic (n,q)-forms on X. We take $u\in\mathcal{H}^{n,q}(X,L)$. By Nakano's identity, we have

$$\begin{aligned}0&=\|D''^{*}_{(L,h)}u\|^2+\|D''u\|^2\\&=\|D'^{*}u\|^2+\langle\!\langle\sqrt{-1}\Theta_h(L)\Lambda u,u\rangle\!\rangle.\end{aligned}$$

On the other hand, we have

$$\langle\sqrt{-1}\Theta_h(L)\Lambda u,u\rangle_h=q|u|^2_h.$$

Therefore, we obtain $0=\|u\|^2$ when $q\geq 1$. Thus, we have $u=0$. This means that $\mathcal{H}^{n,q}(X,L)=0$ for every $q\geq1$. Therefore, we have $H^q(X,\omega_X\otimes L)=0$ for every $q\geq 1$. $\square$

It is a routine work to prove Theorem 3.7.5 by using Theorem 3.7.1. More precisely, Theorem 3.6.2 for compact Kähler manifolds, which is a special case of Theorem 3.7.1, induces Theorem 3.7.5 by the usual argument as in [EsVi3] and [Ko6].

THEOREM 3.7.5 (Torsion-freeness and vanishing theorem). *Let X be a compact Kähler manifold and let Y be a projective variety. Let $\pi : X \to Y$ be a surjective morphism. Then we obtain the following properties.*

(i) *$R^i\pi_*\omega_X$ is torsion-free for every $i \geq 0$.*
(ii) *If H is an ample line bundle on Y, then*
$$H^j(Y, H \otimes R^i\pi_*\omega_X) = 0$$
for every $i \geq 0$ and $j > 0$.

For related topics, see [Take2], [Oh], [F30], and [F31]. See also [F37]. We close this section with a conjecture.

CONJECTURE 3.7.6. *Let X be a compact Kähler manifold (or a smooth projective variety) and let D be a reduced simple normal crossing divisor on X. Let L be a semi-positive line bundle on X and let s be a non-zero holomorphic section of $L^{\otimes k}$ on X for some positive integer k. Assume that $(s = 0)$ contains no strata of D, that is, $(s = 0)$ contains no log canonical centers of (X, D). Then the multiplication homomorphism*
$$\times s : H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes l}) \to H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes (l+k)}),$$
which is induced by $\otimes s$, is injective for every $q \geq 0$ and $l > 0$.

3.8. Fujita vanishing theorem

The following theorem was obtained by Takao Fujita (see [Ft1, Theorem (1)] and [Ft2, (5.1) Theorem]). See also [La1, Theorem 1.4.35].

THEOREM 3.8.1 (Fujita vanishing theorem). *Let X be a projective scheme defined over a field k and let H be an ample Cartier divisor on X. Given any coherent sheaf $\mathcal{F}$ on X, there exists an integer $m(\mathcal{F}, H)$ such that*
$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$
for all $i > 0$, $m \geq m(\mathcal{F}, H)$, and any nef Cartier divisor D on X.

PROOF. Without loss of generality, we may assume that k is algebraically closed. By replacing X with $\operatorname{Supp} \mathcal{F}$, we may assume that $X = \operatorname{Supp} \mathcal{F}$.

REMARK 3.8.2. Let $\mathcal{F}$ be a coherent sheaf on X. In the proof of Theorem 3.8.1, we always define a subscheme structure on $\operatorname{Supp}\mathcal{F}$ by the $\mathcal{O}_X$-ideal $\operatorname{Ker}(\mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}))$.

We use induction on the dimension.

STEP 1. When $\dim X = 0$, Theorem 3.8.1 obviously holds.

From now on, we assume that Theorem 3.8.1 holds in the lower dimensional case.

STEP 2. We can reduce the proof to the case where X is reduced.

PROOF OF STEP 2. We assume that Theorem 3.8.1 holds for reduced schemes. Let $\mathcal{N}$ be the nilradical of $\mathcal{O}_X$, so that $\mathcal{N}^r = 0$ for some $r > 0$. Consider the filtration

$$\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \supset \cdots \supset \mathcal{N}^r \cdot \mathcal{F} = 0.$$

The quotients $\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}$ are coherent $\mathcal{O}_{X_{\mathrm{red}}}$-modules, and therefore, by assumption,

$$H^j(X, (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}, H)$ thanks to the amplitude of $\mathcal{O}_{X_{\mathrm{red}}}(H)$. Twisting the exact sequences

$$0 \to \mathcal{N}^{i+1}\mathcal{F} \to \mathcal{N}^i\mathcal{F} \to \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \to 0$$

by $\mathcal{O}_X(mH + D)$ and taking cohomology, we then find by decreasing induction on i that

$$H^j(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{N}^i\mathcal{F}, H)$. When $i = 0$ this gives the desired vanishings. $\square$

From now on, we assume that X is reduced.

STEP 3. We can reduce the proof to the case where X is irreducible.

PROOF OF STEP 3. We assume that Theorem 3.8.1 holds for reduced and irreducible schemes. Let $X = X_1 \cup \cdots \cup X_k$ be its decomposition into irreducible components and let $\mathcal{I}$ be the ideal sheaf of X_1 in X. We consider the exact sequence

$$0 \to \mathcal{I} \cdot \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I} \cdot \mathcal{F} \to 0.$$

The outer terms of the above exact sequence are supported on $X_2 \cup \cdots \cup X_k$ and X_1 respectively. So by induction on the number of irreducible components, we may assume that

$$H^j(X, \mathcal{I}\mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{IF}, H|_{X_2\cup\cdots\cup H_k})$ and

$$H^j(X, (\mathcal{F}/\mathcal{IF}) \otimes \mathcal{O}_X(mH + D)) = 0$$

for $j > 0$ and $m \geq m(\mathcal{F}/\mathcal{IF}, H|_{X_1})$. It then follows from the above exact sequence that

$$H^j(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

when $j > 0$ and

$$m \geq m(\mathcal{F}, H) := \max\{m(\mathcal{IF}, H|_{X_2\cup\cdots\cup H_k}), m(\mathcal{F}/\mathcal{IF}, H|_{X_1})\},$$

as required. □

From now on, we assume that X is reduced and irreducible.

Step 4. We can reduce the proof to the case where H is very ample.

Proof of Step 4. Let l be a positive integer such that lH is very ample. We assume that Theorem 3.8.1 holds for lH. Apply Theorem 3.8.1 to $\mathcal{F} \otimes \mathcal{O}_X(nH)$ for $0 \leq n \leq l-1$ with lH. Then we obtain $m(\mathcal{F} \otimes \mathcal{O}_X(nH), lH)$ for $0 \leq n \leq l-1$. We put

$$m(\mathcal{F}, H) = l\left(\max_n m(\mathcal{F} \otimes \mathcal{O}_X(nH), lH) + 1\right).$$

Then we can easily check that $m(\mathcal{F}, H)$ satisfies the desired property. □

From now on, we assume that H is very ample.

Step 5. It is sufficient to find $m(\mathcal{F}, H)$ such that

$$H^1(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0$$

for all $m \geq m(\mathcal{F}, H)$ and any nef Cartier divisor D on X.

Proof of Step 5. We take a general member A of $|H|$ and consider the exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(-A) \to \mathcal{F} \to \mathcal{F}_A \to 0.$$

Since $\dim \operatorname{Supp} \mathcal{F}_A < \dim X$, we can find $m(\mathcal{F}_A, H|_A)$ such that

$$H^i(A, \mathcal{F}_A \otimes \mathcal{O}_A(mH + D)) = 0$$

for all $i > 0$ and $m \geq m(\mathcal{F}_A, H|_A)$ by induction. Therefore,

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X((m-1)H + D)) = H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D))$$

for every $i \geq 2$ and $m \geq m(\mathcal{F}_A, H|_A)$. By Serre's vanishing theorem, we obtain

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X((m-1)H + D)) = 0$$

for every $i \geq 2$ and $m \geq m(\mathcal{F}_A, H|_A)$. □

STEP 6. We can reduce the proof to the case where $\mathcal{F} = \mathcal{O}_X$.

PROOF OF STEP 6. We assume that Theorem 3.8.1 holds for $\mathcal{F} = \mathcal{O}_X$. There is an injective homomorphism

$$\alpha : \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{O}_X(aH)$$

for some large integer a. We consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{O}_X(aH) \to \mathrm{Coker}\alpha \to 0$$

and use the induction on $\mathrm{rank}\,\mathcal{F}$. Then we can find $m(\mathcal{F}, H)$. □

From now on, we assume $\mathcal{F} = \mathcal{O}_X$.

STEP 7. If the characteristic of k is zero, then Theorem 3.8.1 holds.

PROOF OF STEP 7. Let $f : Y \to X$ be a resolution. Then we obtain the following exact sequence

$$0 \to f_*\omega_Y \to \mathcal{O}_X(bH) \to \mathcal{C} \to 0$$

for some integer b, where $\dim \mathrm{Supp}\, \mathcal{C} < \dim X$. Note that $f_*\omega_Y$ is torsion-free and $\mathrm{rank} f_*\omega_Y$ is one. On the other hand,

$$H^j(X, f_*\omega_Y \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m > 0$ and $j > 0$ by Kollár's vanishing theorem (see Theorem 3.6.3). Therefore,

$$H^j(X, \mathcal{O}_X((b + m)H + D)) = 0$$

for every positive integer $m \geq m(\mathcal{C}, H)$ and $j > 0$. □

If we do not like to use Kollár's vanishing theorem (see Theorem 3.6.3) in Step 7, which was not proved in Section 3.6, then we can use the following easy lemma.

LEMMA 3.8.3. *Let X be an irreducible proper variety and let $\mathcal{L}$ be a nef and big line bundle on X. Let $f : Y \to X$ be a resolution of singularities. Then $H^i(X, f_*\omega_Y \otimes \mathcal{L}) = 0$ for every $i > 0$.*

PROOF. By the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7, we have $H^i(X, f_*\omega_Y \otimes \mathcal{L}) \simeq H^i(Y, \omega_Y \otimes f^*\mathcal{L})$ for every i. By the Kawamata–Viehweg vanishing theorem: Theorem 3.2.1, we obtain $H^i(Y, \omega_Y \otimes f^*\mathcal{L}) = 0$ for every $i > 0$. Therefore, we obtain the desired vanishing theorem. □

STEP 8. We can reduce the proof to the case where $\mathcal{F} = \omega_X$, where ω_X is the dualizing sheaf of X.

REMARK 3.8.4. The dualizing sheaf ω_X is denoted by ω_X° in [Har4, Chapter III §7]. We know that $\omega_X^\circ \simeq \mathcal{E}xt^{N-\dim X}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ when $X \subset \mathbb{P}^N$. For details, see the proof of Proposition 7.5 in [Har4, Chapter III §7].

PROOF OF STEP 8. We assume that Theorem 3.8.1 holds for $\mathcal{F} = \omega_X$. There is an injective homomorphism

$$\beta : \omega_X \to \mathcal{O}_X(cH)$$

for some positive integer c. Note that ω_X is torsion-free. We consider the exact sequence

$$0 \to \omega_X \to \mathcal{O}_X(cH) \to \operatorname{Coker} \beta \to 0.$$

We note that $\dim \operatorname{Supp} \operatorname{Coker} \beta < \dim X$ because

$$\operatorname{rank} \omega_X = \operatorname{rank} \mathcal{O}_X(cH) = 1.$$

Therefore, we can find $m(\mathcal{O}_X, H)$ by induction on the dimension and Theorem 3.8.1 for ω_X. $\square$

From now on, we assume that $\mathcal{F} = \omega_X$ and that the characteristic of k is positive.

STEP 9. Theorem 3.8.1 holds when the characteristic of k is positive.

PROOF OF STEP 9. Let $X \to \mathbb{P}^N$ be the embedding induced by H. Let

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d] & X \arrow[d] \\ \mathbb{P}^N \arrow[r, "F"'] & \mathbb{P}^N \end{tikzcd}$$

be the commutative diagram of the Frobenius morphisms. By taking $R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\underline{\quad}, \omega^\bullet_{\mathbb{P}^N})$ to $\mathcal{O}_X \to F_*\mathcal{O}_X$, we obtain

$$R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(F_*\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N}) \to R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N}).$$

By Grothendieck duality (see [Har1] and [Con]),

$$R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(F_*\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N}) \simeq F_*R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N}).$$

Therefore, we obtain

$$\gamma : F_*\omega_X \to \omega_X.$$

Note that $\omega_X = \mathcal{E}xt^{N-\dim X}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$. Let U be a non-empty Zariski open set of X such that U is smooth. We can easily check that

$$\gamma : F_*\omega_X \to \omega_X$$

is surjective on U. Note that the cokernel $\mathcal{A}$ of $\mathcal{O}_X \to F_*\mathcal{O}_X$ is locally free on U. Then $\mathcal{E}xt^k_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{A}, \omega_{\mathbb{P}^N}) = 0$ for $k > N - \dim X$ on U. We consider the exact sequences

$$0 \to \mathrm{Ker}\gamma \to F_*\omega_X \to \mathrm{Im}\gamma \to 0$$

and

$$0 \to \mathrm{Im}\gamma \to \omega_X \to \mathcal{C} \to 0.$$

Then $\dim \mathrm{Supp}\, \mathcal{C} < \dim X$. Note that there is an integer m_1 such that

$$H^2(X, \mathrm{Ker}\gamma \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m \geq m_1$ by Step 5. By applying induction on the dimension to $\mathcal{C}$, we obtain some positive integer m_0 such that

$$H^1(X, F_*\omega_X \otimes \mathcal{O}_X(mH + D)) \to H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D))$$

is surjective for every $m \geq m_0$. We note that

$$H^1(X, F_*\omega_X \otimes \mathcal{O}_X(mH + D)) \simeq H^1(X, \omega_X \otimes \mathcal{O}_X(p(mH + D)))$$

by the projection formula, where p is the characteristic of k. By repeating the above process, we obtain that

$$H^1(X, \omega_X \otimes \mathcal{O}_X(p^e(mH + D))) \to H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D))$$

is surjective for every $e > 0$ and $m \geq m_0$. Note that m_0 is independent of the nef divisor D. Therefore, by Serre's vanishing theorem, we obtain

$$H^1(X, \omega_X \otimes \mathcal{O}_X(mH + D)) = 0$$

for every $m \geq m_0$. □

We finish the proof of Theorem 3.8.1. □

In Step 9, we can use the following elementary lemma to construct a generically surjective homomorphism $F_*\omega_X \to \omega_X$.

LEMMA 3.8.5 (see [**Ft2**, (5.7) Corollary]). *Let $f : V \to W$ be a projective surjective morphism between projective varieties defined over an algebraically closed field k with $\dim V = \dim W = n$. Then there is a generically surjective homomorphism $\varphi : f_*\omega_V \to \omega_W$.*

PROOF. By definition (see [**Har4**, Chapter III §7]), $H^n(V, \omega_V) \neq 0$. We consider the Leray spectral sequence

$$E_2^{p,q} = H^p(W, R^q f_*\omega_W) \Rightarrow H^{p+q}(V, \omega_V).$$

Note that $\mathrm{Supp}\, R^q f_*\omega_V$ is contained in the set

$$W_q := \{w \in W \mid \dim f^{-1}(w) \geq q\}.$$

Since $\dim f^{-1}(W_q) < n$ for every $q > 0$, we have $\dim W_q < n - q$ for every $q > 0$. Therefore, $E_2^{n-q,q} = 0$ unless $q = 0$. Thus we obtain

$E_2^{n,0} = H^n(W, f_*\omega_V) \neq 0$ since $H^n(V, \omega_V) \neq 0$. By the definition of ω_W, $\mathrm{Hom}(f_*\omega_V, \omega_W) \neq 0$. We take a non-zero element $\varphi \in \mathrm{Hom}(f_*\omega_V, \omega_W)$ and consider $\mathrm{Im}(\varphi) \subset \omega_W$. Since $\mathrm{Hom}(\mathrm{Im}(\varphi), \omega_W) \neq 0$, we have $H^n(W, \mathrm{Im}(\varphi)) \neq 0$ (see [**Har4**, Chapter III §7]). This implies that $\dim \mathrm{Supp}\, \mathrm{Im}(\varphi) = n$. Therefore, $\varphi : f_*\omega_V \to \omega_W$ is generically surjective since $\mathrm{rank}\, \omega_W = 1$. $\square$

REMARK 3.8.6. In Lemma 3.8.5, if $R^q f_*\omega_V = 0$ for every $q > 0$, then we obtain $H^n(W, f_*\omega_V) \simeq H^n(V, \omega_V)$. We note that $H^n(V, \omega_V) \simeq k$ since k is algebraically closed. Therefore, $\mathrm{Hom}(f_*\omega_V, \omega_W) \simeq k$. This means that, for any nontrivial homomorphism $\psi : f_*\omega_V \to \omega_W$, there is some $a \in k \setminus \{0\}$ such that $\psi = a\varphi$, where φ is given in Lemma 3.8.5. Note that $R^q f_*\omega_V = 0$ for every $q > 0$ if f is finite. We also note that $R^q f_*\omega_V = 0$ for every $q > 0$ if the characteristic of k is zero and V has only rational singularities by the Grauert–Riemenschneider vanishing theorem (see Theorem 3.2.7) or by Kollár's torsion-free theorem: Theorem 3.6.3 (see also Lemma 3.8.7 below).

Although the following lemma is a special case of Kollár's torsion-freeness (see Theorem 3.6.3), it easily follows from the Kawamata–Viehweg vanishing theorem (see Theorem 3.2.1).

LEMMA 3.8.7 (cf. [**Ft2**, (4.13) Proposition]). *Let $f : V \to W$ be a projective surjective morphism from a smooth projective variety V to a projective variety W, which is defined over an algebraically closed field k of characteristic zero. Then $R^q f_*\omega_V = 0$ for every $q > \dim V - \dim W$.*

PROOF. Let A be a sufficiently ample Cartier divisor on W such that
$$H^0(W, R^q f_*\omega_V \otimes \mathcal{O}_W(A)) \simeq H^q(V, \omega_V \otimes \mathcal{O}_V(f^*A))$$
and that $R^q f_*\omega_V \otimes \mathcal{O}_W(A)$ is generated by global sections for every q. We note that the numerical dimension $\nu(V, f^*A)$ of f^*A is $\dim W$. Therefore, we obtain
$$H^q(V, \omega_V \otimes \mathcal{O}_V(f^*A)) = 0$$
for $q > \dim V - \dim W = \dim V - \nu(V, f^*A)$ by the Kawamata–Viehweg vanishing theorem: Theorem 3.3.7. Thus, we obtain $R^q f_*\omega_V = 0$ for $q > \dim V - \dim W$. $\square$

REMARK 3.8.8. In [**Ft2**, Section 4], Takao Fujita proves Lemma 3.8.7 for a proper surjective morphism $f : V \to W$ from a complex manifold V in Fujiki's class $\mathcal{C}$ to a projective variety W. His proof uses the theory of harmonic forms. For the details, see [**Ft2**, Section 4]. See also Theorem 3.8.9 below. Note that [**Ft2**, (4.12) Conjecture] was completely solved by Kensho Takegoshi (see [**Take1**]). See also [**F31**].

The following theorem is a weak generalization of Kodaira's vanishing theorem: Theorem 3.7.4. We need no new ideas to prove Theorem 3.8.9. The proof of Kodaira's vanishing theorem based on Bochner's method works.

THEOREM 3.8.9 (A weak generalization of Kodaira's vanishing theorem). *Let X be an n-dimensional compact Kähler manifold and let $\mathcal{L}$ be a line bundle on X whose curvature form $\sqrt{-1}\Theta(\mathcal{L})$ is semi-positive and has at least k positive eigenvalues on a dense open subset of X. Then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for $i > n-k$.*

We note that $H^i(X, \omega_X \otimes \mathcal{L})$ is isomorphic to $\mathcal{H}^{n,i}(X, \mathcal{L})$, which is the space of $\mathcal{L}$-valued harmonic (n,i)-forms on X. By Nakano's formula, we can easily check that $\mathcal{H}^{n,i}(X, \mathcal{L}) = 0$ for $i + k \geq n+1$.

We close this section with a slight generalization of Kollár's result (cf. [**Ko2**, Proposition 7.6]), which is related to Lemma 3.8.5. For a related result, see also [**FF**, Theorem 7.5].

PROPOSITION 3.8.10. *Let $f : V \to W$ be a proper surjective morphism between normal algebraic varieties with connected fibers, which is defined over an algebraically closed field k of characteristic zero. Assume that V and W have only rational singularities. Then $R^d f_*\omega_V \simeq \omega_W$ where $d = \dim V - \dim W$.*

PROOF. We can construct a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ Y \arrow[r, "p"'] & W \end{tikzcd}$$

with the following properties.

(i) X and Y are smooth algebraic varieties.
(ii) π and p are projective birational.
(iii) g is projective, and smooth outside a simple normal crossing divisor Σ on Y.

We note that $R^j g_*\omega_X$ is locally free for every j (see, for example, [**Ko3**, Theorem 2.6]). By Grothendieck duality, we have

$$Rg_*\mathcal{O}_X \simeq R\mathcal{H}om_{\mathcal{O}_Y}(Rg_*\omega^{\bullet}_X, \omega^{\bullet}_Y).$$

Therefore, we have

$$\mathcal{O}_Y \simeq \mathcal{H}om_{\mathcal{O}_Y}(R^d g_*\omega_X, \omega_Y).$$

Thus, we obtain $R^d g_*\omega_X \simeq \omega_Y$. By applying p_*, we have

$$p_*R^d g_*\omega_X \simeq p_*\omega_Y \simeq \omega_W.$$

We note that $p_*R^dg_*\omega_X \simeq R^d(p\circ g)_*\omega_X$ since $R^ip_*R^dg_*\omega_X = 0$ for every $i > 0$ (see, for example, [**Ko2**, Theorem 3.8 (i)], [**Ko3**, Theorem 2.14, Theorem 3.4 (iii)], or Theorem 5.7.3 (ii) below). On the other hand,

$$R^d(p\circ g)_*\omega_X \simeq R^d(f\circ \pi)_*\omega_X \simeq R^df_*\omega_V$$

since $R^i\pi_*\omega_X = 0$ for every $i > 0$ and $\pi_*\omega_X \simeq \omega_V$. Therefore, we obtain $R^df_*\omega_V \simeq \omega_W$. $\square$

3.9. Applications of Fujita vanishing theorem

In this section, we discuss some applications of Theorem 3.8.1. For more general statements and other applications, see [**Ft2**, Section 6].

THEOREM 3.9.1 (cf. [**Ft1**, Theorem (4)] and [**Ft2**, (6.2) Theorem]). *Let $\mathcal{F}$ be a coherent sheaf on a scheme X which is proper over an algebraically closed field k. Let $\mathcal{L}$ be a nef line bundle on X. Then*

$$\dim H^q(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}) \leq O(t^{m-q})$$

where $m = \dim \operatorname{Supp}\mathcal{F}$.

PROOF. First, we assume that X is projective. We use induction on q. We put $q = 0$. Let H be an effective ample Cartier divisor on X such that $\mathcal{L}\otimes\mathcal{O}_X(H)$ is ample. Since

$$H^0(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}) \subset H^0(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}\otimes\mathcal{O}_X(tH))$$

for every positive integer t, we may assume that $\mathcal{L}$ is ample by replacing $\mathcal{L}$ with $\mathcal{L}\otimes\mathcal{O}_X(H)$. In this case, $\dim H^0(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}) \leq O(t^m)$ because

$$\dim H^0(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}) = \chi(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t})$$

for every $t \gg 0$ by Serre's vanishing theorem. When $q > 0$, by Theorem 3.8.1, we have a very ample Cartier divisor A on X such that

$$H^q(X, \mathcal{F}\otimes\mathcal{O}_X(A)\otimes\mathcal{L}^{\otimes t}) = 0$$

for every $t \geq 0$. Let D be a general member of $|A|$ such that the induced homomorphism $\alpha : \mathcal{F}\otimes\mathcal{O}_X(-D) \to \mathcal{F}$ is injective. Then

$$\begin{aligned}\dim H^q(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}) &\leq \dim H^{q-1}(D, \operatorname{Coker}(\alpha)\otimes\mathcal{O}_D(A)\otimes\mathcal{L}^{\otimes t})\\ &\leq O(t^{m-q})\end{aligned}$$

by the induction hypothesis. Therefore, we obtain the theorem when X is projective.

Next, we consider the general case. We use the noetherian induction on $\operatorname{Supp}\mathcal{F}$. By the same arguments as in Step 2 and Step 3 in the proof of Theorem 3.8.1, we may assume that $X = \operatorname{Supp}\mathcal{F}$ is a variety, that is, X is reduced and irreducible. By Chow's lemma, there is a birational morphism $f : V \to X$ from a projective variety V. We put $\mathcal{G} = f^*\mathcal{F}$

and consider the natural homomorphism $\beta : \mathcal{F} \to f_*\mathcal{G}$. Since β is an isomorphism on a non-empty Zariski open subset of X. We consider the following short exact sequences

$$0 \to \mathrm{Ker}(\beta) \to \mathcal{F} \to \mathrm{Im}(\beta) \to 0$$

and

$$0 \to \mathrm{Im}(\beta) \to f_*\mathcal{G} \to \mathrm{Coker}(\beta) \to 0.$$

By induction, we obtain

$$\dim H^q(X, \mathrm{Ker}(\beta) \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q})$$

and

$$\dim H^{q-1}(X, \mathrm{Coker}(\beta) \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

Therefore, it is sufficient to see that

$$\dim H^q(X, f_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

We consider the Leray spectral sequence

$$E_2^{i,j} = H^i(X, R^jf_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \Rightarrow H^{i+j}(V, \mathcal{G} \otimes (f^*\mathcal{L})^{\otimes t}).$$

Then we have

$$\begin{aligned}\dim H^q(X, f_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \leq & \sum_{j \geq 1} \dim H^{q-j-1}(X, R^jf_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \\ & + \dim H^q(V, \mathcal{G} \otimes (f^*\mathcal{L})^{\otimes t}).\end{aligned}$$

Note that

$$\dim H^q(V, \mathcal{G} \otimes (f^*\mathcal{L})^{\otimes t}) \leq O(t^{m-q})$$

since V is projective. On the other hand, we have

$$\dim \mathrm{Supp}\, R^jf_*\mathcal{G} \leq \dim X - j - 1$$

for every $j \geq 1$ as in the proof of Lemma 3.8.5. Therefore,

$$\dim H^{q-j-1}(X, R^jf_*\mathcal{G} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q})$$

by the induction hypothesis. Thus, we obtain

$$\dim H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

We complete the proof. □

As an application of Theorem 3.9.1, we can prove Fujita's numerical characterization of nef and big line bundles. We note that the characteristic of the base field is arbitrary in Corollary 3.9.2. Corollary 3.9.2 in characteristic zero, which is due to Sommese, is well known. See, for example, [Ka3, Lemma 3].

Corollary 3.9.2 (cf. [Ft1, Theorem (6)] and [Ft2, (6.5) Corollary]). *Let $\mathcal{L}$ be a nef line bundle on a proper algebraic irreducible variety V defined over an algebraically closed field k with $\dim V = n$. Then $\mathcal{L}$ is big if and only if the self-intersection number $\mathcal{L}^n$ is positive.*

Proof. Let $\nu : X^\nu \to X$ be the normalization. By replacing X and $\mathcal{L}$ with X^ν and $\nu^*\mathcal{L}$, we may assume that X is normal. It is well known that

$$\chi(V, \mathcal{L}^{\otimes t}) - \frac{\mathcal{L}^n}{n!} t^n \leq O(t^{n-1}).$$

By Theorem 3.9.1, we have

$$\dim H^0(V, \mathcal{L}^{\otimes t}) - \chi(V, \mathcal{L}^{\otimes t}) \leq O(t^{n-1}).$$

Therefore, $\mathcal{L}$ is big if and only if $\mathcal{L}^n > 0$. Note that $\mathcal{L}^n \geq 0$ since $\mathcal{L}$ is nef. $\square$

Corollary 3.9.3 (cf. [Ft1, Corollary (7)] and [Ft2, (6.7) Corollary]). *Let $\mathcal{L}$ be a nef and big line bundle on a projective irreducible variety V defined over an algebraically closed field k with $\dim V = n$. Then, for any coherent sheaf $\mathcal{F}$ on V, we have*

$$\dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q-1})$$

for every $q \geq 1$. In particular, $H^n(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) = 0$ for every $t \gg 0$.

Proof. Let A be an ample Cartier divisor such that

$$H^i(V, \mathcal{F} \otimes \mathcal{O}_V(A) \otimes \mathcal{L}^{\otimes t}) = 0$$

for every $i > 0$ and $t \geq 0$ (see Theorem 3.8.1). Since $\mathcal{L}$ is big, there is a positive integer m such that $|\mathcal{L}^{\otimes m} \otimes \mathcal{O}_V(-A)| \neq \emptyset$ by Kodaira's lemma (see Lemma 2.1.18). We take $D \in |\mathcal{L}^{\otimes m} \otimes \mathcal{O}_V(-A)|$ and consider the homomorphism $\gamma : \mathcal{F} \otimes \mathcal{O}_V(-D) \to \mathcal{F}$ induced by γ. Then we have

$$\begin{aligned} \dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq & \dim H^q(V, \mathrm{Coker}(\gamma) \otimes \mathcal{L}^{\otimes t}) \\ & + \dim H^q(V, \mathrm{Im}(\gamma) \otimes \mathcal{L}^{\otimes t}), \end{aligned}$$

and

$$\begin{aligned} \dim H^q(V, \mathrm{Im}(\gamma) \otimes \mathcal{L}^{\otimes t}) &\leq \dim H^q(V, \mathcal{F} \otimes \mathcal{O}_V(-D) \otimes \mathcal{L}^{\otimes t}) \\ &\quad + \dim H^{q+1}(V, \mathrm{Ker}(\gamma) \otimes \mathcal{L}^{\otimes t}) \\ &= \dim H^{q+1}(V, \mathrm{Ker}(\gamma) \otimes \mathcal{L}^{\otimes t}) \end{aligned}$$

for every $t \geq m$. This is because

$$\begin{aligned} & H^q(V, \mathcal{F} \otimes \mathcal{O}_V(-D) \otimes \mathcal{L}^{\otimes t}) \\ & \simeq H^q(V, \mathcal{F} \otimes \mathcal{O}_V(A) \otimes \mathcal{L}^{\otimes (t-m)}) = 0 \end{aligned}$$

for every $t \geq m$. Note that

$$\dim H^q(V, \mathrm{Coker}(\gamma) \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-1-q})$$

by Theorem 3.9.1 since $\mathrm{Supp}\,\mathrm{Coker}(\gamma)$ is contained in D. On the other hand,

$$\dim H^{q+1}(V, \mathrm{Ker}(\gamma) \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q-1})$$

by Theorem 3.9.1. By combining there estimates, we obtain the desired estimate. $\square$

3.10. Tanaka vanishing theorems

In this section, we discuss Tanaka's vanishing theorems. It is well known that Kodaira's vanishing theorem does not always hold even for surfaces when the characteristic of the base filed is positive. In [**Tana2**], Hiromu Tanaka obtained the following vanishing theorem as an application of Fujita's vanishing theorem: Theorem 3.8.1. They are sufficient for X-method for surfaces in positive characteristic.

THEOREM 3.10.1 (Kodaira type vanishing theorem). *Let X be a smooth projective surface defined over an algebraically closed filed k of positive characteristic. Let A be an ample Cartier divisor on X and let N be a nef Cartier divisor on X which is not numerically trivial. Then*

$$H^i(X, \mathcal{O}_X(K_X + A + mN)) = 0$$

for every $i > 0$ and every $m \gg 0$.

More generally, Hiromu Tanaka proved the following vanishing theorems.

THEOREM 3.10.2 (Kawamata–Viehweg type vanishing theorem). *Let X be a smooth projective surface defined over an algebraically closed filed k of positive characteristic. Let A be an ample $\mathbb{R}$-divisor on X whose fractional part has a simple normal crossing support and let N be a nef Cartier divisor on X which is not numerically trivial. Then*

$$H^i(X, \mathcal{O}_X(K_X + \lceil A \rceil + mN)) = 0$$

for every $i > 0$ and every $m \gg 0$.

THEOREM 3.10.3 (Nadel type vanishing theorem). *Let X be a normal projective surface defined over an algebraically closed field k of positive characteristic. Let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let N be a nef Cartier divisor on X which is not numerically trivial. Let L be a Cartier divisor on X such that $L - (K_X + \Delta)$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(L + mN) \otimes \mathcal{J}(X, \Delta)) = 0$$

for every $i > 0$ *and every* $m \gg 0$, *where* $\mathcal{J}(X, \Delta)$ *is the multiplier ideal sheaf of the pair* (X, Δ).

For the details of Theorems 3.10.1, 3.10.2, and 3.10.3, and some related topics, see [Tana2].

3.11. Ambro vanishing theorem

In this section, we prove Ambro's vanishing theorem in [Am2], which is an application of Corollary 3.1.2.

THEOREM 3.11.1. *Let* X *be a smooth projective variety and let* Δ *be a reduced simple normal crossing divisor on* X. *Assume that* $X \setminus \Delta$ *is contained in an affine Zariski open set* U *of* X. *Then we have*

$$H^i(X, \mathcal{O}_X(K_X + \Delta)) = 0$$

for every $i > 0$.

Our proof of Theorem 3.11.1 is based on Corollary 3.1.2 and the weak factorization theorem in [AKMW].

PROOF. We may assume that $U \subset \mathbb{C}^n$. By taking the closure of U in $\mathbb{P}^n$ and taking some suitable blow-ups outside U, we can construct a smooth projective variety X' with the following properties (cf. Goodman's criterion in [Har3, Chapter II Theorem 6.1]).

(i) $U \subset X'$ and $\Sigma = X' \setminus U$ is a simple normal crossing divisor on X'.
(ii) There is a simple normal crossing divisor Δ' on X' such that $\Sigma \leq \Delta'$ and that $(X', \Delta')|_U = (X, \Delta)|_U$. In particular, $X' \setminus \Delta' = X \setminus \Delta$.
(iii) There is an effective ample Cartier divisor D' on X' such that $\operatorname{Supp} D' \subset \operatorname{Supp} \Delta'$.

By Corollary 3.1.2, we obtain that

$$H^i(X', \mathcal{O}_{X'}(K_{X'} + \Delta')) = 0$$

for every $i > 0$.

CLAIM. *We have*

$$H^i(X, \mathcal{O}_X(K_X + \Delta)) \simeq H^i(X', \mathcal{O}_{X'}(K_{X'} + \Delta'))$$

for every i.

PROOF OF CLAIM. By the weak factorization theorem (see [AKMW, Theorem 0.3.1]), we may assume that $f : X' \to X$ is a blow-up whose center C, which is contained in Δ, is smooth and has simple normal crossings with Δ. It is sufficient to check that

$$R^j f_*\mathcal{O}_{X'}(K_{X'}+\Delta')=0$$

for every $j>0$ and

$$f_*\mathcal{O}_{X'}(K_{X'}+\Delta')\simeq \mathcal{O}_X(K_X+\Delta).$$

By shrinking X', we may assume that C is irreducible. Then we have

$$K_{X'}+\Delta'=f^*(K_X+\Delta)+(c-m)E$$

and

$$K_{X'}=f^*K_X+(c-1)E$$

where $c=\mathrm{codim}_X C$, $m=\mathrm{mult}_C\Delta$, and E is the exceptional divisor of f. Since $c-m\geq 0$ and E is f-exceptional, we obtain

$$f_*\mathcal{O}_{X'}(K_{X'}+\Delta')\simeq \mathcal{O}_X(K_X+\Delta).$$

Since

$$K_{X'}+\Delta'-K_{X'}\sim_f (1-m)E$$

is f-nef and f-big,

$$R^j f_*\mathcal{O}_{X'}(K_{X'}+\Delta')=0$$

for every $j>0$ by the Kawamata–Viehweg vanishing theorem: Theorem 3.2.1. Of course, we can directly check the above vanishing statement because $f:X'\to X$ is a blow-up whose center is smooth. $\square$

Therefore, we obtain the desired vanishing theorem. $\square$

We learned the following example from Takeshi Abe.

EXAMPLE 3.11.2. There is a projective birational morphism $f:X\to Y$ from a smooth projective variety X to a normal projective variety Y with the following properties.

(i) The exceptional locus $\mathrm{Exc}(f)$ of f is an irreducible curve C on X.

(ii) There is a prime Weil divisor H on Y with $P:=f(C)\in H$ which is an ample Cartier divisor on Y.

Then $U:=X\setminus f^{-1}(H)\simeq Y\setminus H$ is an affine Zariski open set of X. In this case, $D:=X\setminus U$ is a prime Weil divisor on X which is a nef and big Cartier divisor on X such that $D\cdot C=0$. Therefore, D is not ample. Note that we can choose $f:X\to Y$ to be a three-dimensional flopping contraction.

For some related results, see [**Har3**, Chapter II].

3.12. Kovács's characterization of rational singularities

In this section, we discuss Kovács's characterization of rational singularities in [Kv3].

Let us recall the definition of rational singularities.

DEFINITION 3.12.1 (Rational singularities). Let X be a variety. If there exists a resolution of singularities $f : Y \to X$ such that $R^i f_*\mathcal{O}_Y = 0$ for every $i > 0$ and $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$, equivalently, the natural map $\mathcal{O}_X \to Rf_*\mathcal{O}_Y$ is a quasi-isomorphism, then X is said to have only rational singularities.

LEMMA 3.12.2 ([KKMS]). *Let X be a variety and let $f : Y \to X$ be a resolution of singularities. Then the natural map $\mathcal{O}_X \to Rf_*\mathcal{O}_Y$ is a quasi-isomorphism if and only if X is Cohen–Macaulay and $f_*\omega_Y \simeq \omega_X$.*

PROOF. Assume that X is Cohen–Macaulay and $f_*\omega_Y \simeq \omega_X$. Then $Rf_*\omega^\bullet_Y \simeq \omega^\bullet_X$ by the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7. By Grothendieck duality, we obtain

$$\begin{aligned}\mathcal{O}_X &\simeq R\mathcal{H}om(\omega^\bullet_X, \omega^\bullet_X) \simeq R\mathcal{H}om(Rf_*\omega^\bullet_Y, \omega^\bullet_X)\\ &\simeq Rf_*R\mathcal{H}om(\omega^\bullet_Y, \omega^\bullet_Y) \simeq Rf_*\mathcal{O}_Y.\end{aligned}$$

Assume that the natural map $\mathcal{O}_X \to Rf_*\mathcal{O}_Y$ is a quasi-isomorphism. By Grothendieck duality,

$$\begin{aligned}Rf_*\omega^\bullet_Y &\simeq R\mathcal{H}om(Rf_*\mathcal{O}_Y, \omega^\bullet_X)\\ &\simeq R\mathcal{H}om(\mathcal{O}_X, \omega^\bullet_X) \simeq \omega^\bullet_X.\end{aligned}$$

Note that $\omega^\bullet_Y \simeq \omega_Y[d]$ where $d = \dim X = \dim Y$. Then $h^i(\omega^\bullet_X) = R^{i+d}\omega_Y = 0$ for $i > -d$ by the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7. Therefore, X is Cohen–Macaulay and $\omega^\bullet_X \simeq \omega_X[d]$. Thus we obtain $f_*\omega_Y \simeq \omega_X$. $\square$

We can easily check the following lemmas.

LEMMA 3.12.3. *Let X be a smooth variety. Then X has only rational singularities.*

PROOF. Note that the identity map $\mathrm{id}_X : X \to X$ is a resolution of singularities of X since X itself is smooth. $\square$

LEMMA 3.12.4. *Assume that X has only rational singularities. Let $f : Y \to X$ be any resolution of singularities. Then $R^i f_*\mathcal{O}_Y = 0$ for every $i > 0$ and $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$.*

Proof. By Lemma 3.12.2, it is sufficient to see $f_*\omega_Y \simeq \omega_X$. On the other hand, it is easy to see that $f_*\omega_Y$ is independent of the resolution $f : Y \to X$. Therefore, we have $f_*\omega_Y \simeq \omega_X$ by Lemma 3.12.2 and Definition 3.12.1. $\square$

The following theorem is Kovács's characterization of rational singularities.

Theorem 3.12.5 ([Kv3, Theorem 1]). *Let $f : Y \to X$ be a morphism between varieties and let $\alpha : \mathcal{O}_X \to Rf_*\mathcal{O}_Y$ be the associated natural morphism. Assume that Y has only rational singularities and there exists a morphism $\beta : Rf_*\mathcal{O}_Y \to \mathcal{O}_X$ such that $\beta \circ \alpha$ is a quasi-isomorphism in the derived category. Then X has only rational singularities.*

Proof. We construct the following commutative diagram:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\sigma"] \arrow[d, "\widetilde{f}"'] & Y \arrow[d, "f"] \\ \widetilde{X} \arrow[r, "\pi"'] & X \end{tikzcd}$$

such that σ and π are resolutions of singularities. Then we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\alpha"] \arrow[d, "a"'] & Rf_*\mathcal{O}_Y \arrow[d, "b"] \\ R\pi_*\mathcal{O}_{\widetilde{X}} \arrow[r, "c"'] & Rf_*R\sigma_*\mathcal{O}_{\widetilde{Y}}. \end{tikzcd}$$

Note that b is a quasi-isomorphism because Y has only rational singularities. Therefore,

$$(\beta \circ b^{-1} \circ c) \circ a : \mathcal{O}_X \to R\pi_*\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$$

is a quasi-isomorphism. Thus, we may assume that f is a resolution of singularities. We apply $R\mathcal{H}om(\underline{\quad}, \omega_X^\bullet)$ to

$$\mathcal{O}_X \xrightarrow{\alpha} Rf_*\mathcal{O}_Y \xrightarrow{\beta} \mathcal{O}_X.$$

By Grothendieck duality, we obtain

$$\omega_X^\bullet \xleftarrow{\alpha^*} Rf_*\omega_Y^\bullet \xleftarrow{\beta^*} \omega_X^\bullet$$

such that $\alpha^* \circ \beta^*$ is a quasi-isomorphism. Therefore, we obtain

$$h^i(\omega_X^\bullet) \subset R^if_*\omega_Y^\bullet \simeq R^{i+d}f_*\omega_Y$$

where $d = \dim X = \dim Y$. By the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7, we have $R^{i+d}\omega_Y = 0$ for every $i > -d$.

This implies that $h^i(\omega_X^\bullet) = 0$ for every $i > -d$. This means that X is Cohen–Macaulay. By the above argument, we obtain

$$\omega_X \xrightarrow{h^{-d}(\beta^*)} f_*\omega_Y \xrightarrow{h^{-d}(\alpha^*)} \omega_X$$

such that the composition is an isomorphism. This implies $f_*\omega_Y \simeq \omega_X$. Therefore, X has only rational singularities by Lemma 3.12.2. □

The arguments in the proof of Theorem 3.12.5 is very useful for various applications (see the proof of Theorem 3.13.6).

We close this section with a well-known vanishing theorem for varieties with only rational singularities. It is an easy application of the Kawamata–Viehweg vanishing theorem.

THEOREM 3.12.6 (Vanishing theorem for varieties with only rational singularities). *Let X be a normal complete variety with only rational singularities. Let D be a nef and big Cartier divisor on X. Then*

$$H^i(X, \omega_X \otimes \mathcal{O}_X(D)) = 0$$

for $i > 0$, equivalently, by Serre duality,

$$H^i(X, \mathcal{O}_X(-D)) = 0$$

for $i < \dim X$.

PROOF. Let $f : Y \to X$ be a resolution of singularities. Since X has only rational singularities,

$$H^i(X, \mathcal{O}_X(-D)) \simeq H^i(Y, \mathcal{O}_Y(-f^*D))$$

for every i. Since f is birational, f^*D is nef and big. Therefore, $H^i(Y, \mathcal{O}_Y(-f^*D)) = 0$ for every $i < \dim Y = \dim X$ by the Kawamata–Viehweg vanishing theorem: Theorem 3.2.1. Note that $H^i(X, \omega_X \otimes \mathcal{O}_X(D))$ is dual to $H^{\dim X - i}(X, \mathcal{O}_X(-D))$ by Serre duality because X is Cohen–Macaulay. □

3.13. Basic properties of dlt pairs

In this section, we prove some basic properties of dlt pairs. We note that the notion of dlt pairs plays very important roles in the recent developments of the minimal model program after [**KoMo**]. We also note that the notion of dlt pairs was introduced by Shokurov [**Sh2**].

First, let us prove the following well-known theorem.

THEOREM 3.13.1. *Let (X, D) be a dlt pair. Then X has only rational singularities.*

For the proof of Theorem 3.13.1, the following formulation of the Kawamata–Viehweg vanishing theorem is useful.

THEOREM 3.13.2 (Kawamata–Viehweg vanishing theorem). *Let $f : Y \to X$ be a projective surjective morphism onto a variety Y and let M be a Cartier divisor on Y. Let Δ be a boundary $\mathbb{R}$-divisor on Y such that* $\operatorname{Supp}\Delta$ *is a normal crossing divisor on Y. Assume that $M-(K_Y+\Delta)$ is f-ample. Then*

$$R^if_*\mathcal{O}_Y(M)=0$$

for every $i>0$.

It is obvious that Theorem 3.13.2 contains Norimatsu's vanishing theorem: Theorem 3.2.12.

PROOF OF THEOREM 3.13.2. We put $D=M-(K_Y+(1-\varepsilon)\Delta)$ for some small positive number ε. Then D is an f-ample $\mathbb{R}$-divisor on Y such that $\lceil D\rceil = M-K_Y$ and that $\operatorname{Supp}\{D\}$ is a normal crossing divisor on Y. By Theorem 3.2.9, we obtain $R^if_*\mathcal{O}_Y(K_Y+\lceil D\rceil)=0$ for every $i>0$. This means that $R^if_*\mathcal{O}_Y(M)=0$ for every $i>0$. □

Let us give a proof of Theorem 3.13.1 based on Theorem 3.12.5, which was first obtained in [**F17**, Theorem 4.9]. For a related result, see [**Nak2**, Chapter VII, 1.1.Theorem].

PROOF OF THEOREM 3.13.1. By the definition of dlt pairs, we can take a resolution $f : Y \to X$ such that $\operatorname{Exc}(f)$ and $\operatorname{Exc}(f)\cup \operatorname{Supp} f_*^{-1}D$ are both simple normal crossing divisors on Y and that

$$K_Y+f_*^{-1}D=f^*(K_X+D)+E$$

with $\lceil E\rceil \geq 0$. We can take an effective f-exceptional divisor A on Y such $-A$ is f-ample (see, for example, Remark 2.3.18 and [**F12**, Proposition 3.7.7]). Then

$$\lceil E\rceil-(K_Y+f_*^{-1}D+\{-E\}+\varepsilon A)=-f^*(K_X+D)-\varepsilon A$$

is f-ample for $\varepsilon>0$. If $0<\varepsilon \ll 1$, then $f_*^{-1}D+\{-E\}+\varepsilon A$ is a boundary $\mathbb{R}$-divisor whose support is a simple normal crossing divisor on Y. Therefore, $R^if_*\mathcal{O}_Y(\lceil E\rceil)=0$ for $i>0$ by Theorem 3.13.2 and $f_*\mathcal{O}_Y(\lceil E\rceil)\simeq \mathcal{O}_X$. Note that $\lceil E\rceil$ is effective and f-exceptional. Thus, the composition

$$\mathcal{O}_X\to Rf_*\mathcal{O}_Y\to Rf_*\mathcal{O}_Y(\lceil E\rceil)\simeq \mathcal{O}_X$$

is a quasi-isomorphism in the derived category. So, X has only rational singularities by Theorem 3.12.5. □

REMARK 3.13.3. It is curious that Theorem 3.13.1 is missing in [**Kv3**]. As we saw in the proof of Theorem 3.13.1, it easily follows from Kovács's characterization of rational singularities (see Theorem

3.12.5 and [**Kv3**, Theorem 1]). In [**Kv3**, Theorem 4], Kovács only proved the following statement. Let X be a variety with log terminal singularities. Then X has only rational singularities.

3.13.4 (Weak log-terminal singularities). The proof of Theorem 3.13.1 works for weak log-terminal singularities (see Definition 2.3.21). Thus, we can recover [**KMM**, Theorem 1-3-6], that is, we obtain the following statement.

THEOREM 3.13.5 (see [**KMM**, Theorem 1-3-6]). *All weak log-terminal singularities are rational.*

We do not need the difficult vanishing theorem due to Elkik and Fujita (see Theorem 3.14.1) to obtain the above theorem. In Theorem 3.13.1, if we assume that (X, D) is only weak log-terminal singularities, then we can not always make $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}D$ *simple* normal crossing divisors. We can only make them *normal crossing divisors*. However, Theorem 3.13.2 works in this setting. Thus, the proof of Theorem 3.13.1 works for weak log-terminal singularities. Anyway, the notion of weak log-terminal singularities is not useful in the recent minimal model program.

The following theorem generalizes [**Koetal**, 17.5 Corollary], where it was only proved that S is semi-normal and satisfies Serre's S_2 condition. Theorem 3.13.6 was first obtained in [**F17**] in order to understand [**Koetal**, 17.5 Corollary].

THEOREM 3.13.6 ([**F17**, Theorem 4.14]). *Let X be a normal variety and let $S + B$ be a boundary $\mathbb{R}$-divisor such that $(X, S + B)$ is dlt, S is reduced, and $\lfloor B \rfloor = 0$. Let $S = S_1 + \cdots + S_k$ be the irreducible decomposition. We put $T = S_1 + \cdots + S_l$ for some l with $1 \leq l \leq k$. Then T is semi-normal, Cohen–Macaulay, and has only Du Bois singularities.*

PROOF. Let $f : Y \to X$ be a resolution of singularities such that

$$K_Y + S' + B' = f^*(K_X + S + B) + E$$

with the following properties (see Remark 2.3.18):

(i) S' (resp. B') is the strict transform of S (resp. B).
(ii) $\mathrm{Supp}(S' + B') \cup \mathrm{Exc}(f)$ and $\mathrm{Exc}(f)$ are simple normal crossing divisors on Y.
(iii) f is an isomorphism over the generic point of any log canonical center of $(X, S + B)$.
(iv) $\lceil E \rceil \geq 0$.

We write $S = T + U$. Let T' (resp. U') be the strict transform of T (resp. U) on Y. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(-T' + \lceil E \rceil) \to \mathcal{O}_Y(\lceil E \rceil) \to \mathcal{O}_{T'}(\lceil E|_{T'} \rceil) \to 0.$$

Since $-T' + E \sim_{\mathbb{R},f} K_Y + U' + B'$ and $E \sim_{\mathbb{R},f} K_Y + S' + B'$, we have

$$-T' + \lceil E \rceil \sim_{\mathbb{R},f} K_Y + U' + B' + \{-E\}$$

and

$$\lceil E \rceil \sim_{\mathbb{R},f} K_Y + S' + B' + \{-E\}.$$

By Theorem 3.2.11, we obtain

$$R^i f_* \mathcal{O}_Y(-T' + \lceil E \rceil) = R^i f_* \mathcal{O}_Y(\lceil E \rceil) = 0$$

for every $i > 0$. Therefore, we have

$$0 \to f_* \mathcal{O}_Y(-T' + \lceil E \rceil) \to \mathcal{O}_X \to f_* \mathcal{O}_{T'}(\lceil E|_{T'} \rceil) \to 0$$

and $R^i f_* \mathcal{O}_{T'}(\lceil E|_{T'} \rceil) = 0$ for every $i > 0$. Note that $\lceil E \rceil$ is effective and f-exceptional. Thus we obtain

$$\mathcal{O}_T \simeq f_* \mathcal{O}_{T'} \simeq f_* \mathcal{O}_{T'}(\lceil E|_{T'} \rceil).$$

Since T' is a simple normal crossing divisor, T is semi-normal. By the above vanishing result, we obtain $Rf_* \mathcal{O}_{T'}(\lceil E|_{T'} \rceil) \simeq \mathcal{O}_T$ in the derived category. Therefore, the composition

$$\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to Rf_* \mathcal{O}_{T'}(\lceil E|_{T'} \rceil) \simeq \mathcal{O}_T$$

is a quasi-isomorphism. Apply

$$R\mathcal{H}om_T(\underline{\quad}, \omega_T^\bullet)$$

to

$$\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to \mathcal{O}_T.$$

Then the composition

$$\omega_T^\bullet \to Rf_* \omega_{T'}^\bullet \to \omega_T^\bullet$$

is a quasi-isomorphism by Grothendieck duality. Hence, we have

$$h^i(\omega_T^\bullet) \subseteq R^i f_* \omega_{T'}^\bullet \simeq R^{i+d} f_* \omega_{T'},$$

where $d = \dim T = \dim T'$.

Claim (see also Lemma 5.6.1). *$R^i f_* \omega_{T'} = 0$ for every $i > 0$.*

Proof of Claim. We use induction on the number of the irreducible components of T'. If T' is irreducible, then Claim follows from the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7. Let S_i' be the strict transform of S_i on Y for every i. Let W be any irreducible component of $S'_{i_1} \cap \cdots \cap S'_{i_m}$ for $\{i_1, \cdots, i_m\} \subset \{1, 2, \cdots, k\}$. Then $f : W \to f(W)$ is birational by the construction of f. Therefore, every Cartier divisor on W is f-big. We put $T' = S'_1 + T'_0$ and consider the short exact sequence

$$0 \to \mathcal{O}_{T'_0}(-S'_1) \to \mathcal{O}_{T'} \to \mathcal{O}_{S'_1} \to 0.$$

By taking $\otimes \omega_{T'}$, we obtain

$$0 \to \omega_{T'_0} \to \omega_{T'} \to \omega_{S'_1} \otimes \mathcal{O}_{S'_1}(T'|_{S'_1}) \to 0.$$

Then we have the following long exact sequence

$$\cdots \to R^i f_* \omega_{T'_0} \to R^i f_* \omega_{T'} \to R^i f_* \mathcal{O}_{S'_1}(K_{S'_1} + T'|_{S'_1}) \to \cdots .$$

By Theorem 3.2.11, $R^i f_* \mathcal{O}_{S'_1}(K_{S'_1} + T'|_{S'_1}) = 0$ for every $i > 0$. By induction on the number of the irreducible components, we obtain that $R^i f_* \omega_{T'_0} = 0$ for every $i > 0$. Therefore, we obtain the desired vanishing theorem. □

Therefore, by Claim, $h^i(\omega^\bullet_T) = 0$ for $i \neq -d$. Thus, T is Cohen–Macaulay. This argument is the same as the proof of Theorem 3.12.5. Since T' is a simple normal crossing divisor, T' has only Du Bois singularities (see, for example, Lemma 5.3.8). Note that the composition

$$\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to \mathcal{O}_T$$

is a quasi-isomorphism. It implies that T has only Du Bois singularities (see [**Kv1**, Corollary 2.4]). Since the composition

$$\omega_T \to f_* \omega_{T'} \to \omega_T$$

is an isomorphism, we obtain $f_* \omega_{T'} \simeq \omega_T$. By Grothendieck duality,

$$Rf_* \mathcal{O}_{T'} \simeq R\mathcal{H}om_T(Rf_* \omega^\bullet_{T'}, \omega^\bullet_T) \simeq R\mathcal{H}om_T(\omega^\bullet_T, \omega^\bullet_T) \simeq \mathcal{O}_T.$$

So, we have $R^i f_* \mathcal{O}_{T'} = 0$ for every $i > 0$. □

We obtained the following vanishing theorem in the proof of Theorem 3.13.6.

Corollary 3.13.7. *Under the notation in the proof of Theorem 3.13.6, $R^i f_* \mathcal{O}_{T'} = 0$ for every $i > 0$ and $f_* \mathcal{O}_{T'} \simeq \mathcal{O}_T$.*

As a special case, we have:

Corollary 3.13.8 ([**KoMo**, Corollary 5.52]). *Let $(X, S + B)$ be a dlt pair as in Theorem 3.13.6. Then S_i is normal for every i.*

Proof. We put $T = S_i$. Then S_i is normal since $f_* \mathcal{O}_{T'} \simeq \mathcal{O}_T$ (see Corollary 3.13.7). □

Let us discuss a nontrivial example. This example shows the subtleties of the notion of dlt pairs.

EXAMPLE 3.13.9 (cf. [KMM, Remark 0-2-11. (4)]). We consider the $\mathbb{P}^2$-bundle

$$\pi : V = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathbb{P}^2.$$

Let $F_1 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ and $F_2 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ be two hypersurfaces of V which correspond to projections

$$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$$

given by $(x, y, z) \mapsto (x, y)$ and $(x, y, z) \mapsto (x, z)$. Let $\Phi : V \to W$ be the flipping contraction that contracts the negative section of $\pi : V \to \mathbb{P}^2$, that is, the section corresponding to the projection

$$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

Let $C \subset \mathbb{P}^2$ be an elliptic curve. We put $Y = \pi^{-1}(C)$, $D_1 = F_1|_Y$, and $D_2 = F_2|_Y$. Let $f : Y \to X$ be the Stein factorization of $\Phi|_Y : Y \to \Phi(Y)$. Then the exceptional locus of f is $E = D_1 \cap D_2$. We note that Y is smooth, $D_1 + D_2$ is a simple normal crossing divisor, and $E \simeq C$ is an elliptic curve. Let $g : Z \to Y$ be the blow-up along E. Then

$$K_Z + D_1' + D_2' + D = g^*(K_Y + D_1 + D_2),$$

where D_1' (resp. D_2') is the strict transform of D_1 (resp. D_2) and D is the exceptional divisor of g. Note that $D \simeq C \times \mathbb{P}^1$. Since

$$-D + (K_Z + D_1' + D_2' + D) - (K_Z + D_1' + D_2') = 0,$$

we obtain that $R^i f_*(g_*\mathcal{O}_Z(-D + K_Z + D_1' + D_2' + D)) = 0$ for every $i > 0$ by Theorem 5.7.3 below. We note that $f \circ g$ is an isomorphism outside D. We consider the following short exact sequence

$$0 \to \mathcal{I}_E \to \mathcal{O}_Y \to \mathcal{O}_E \to 0,$$

where $\mathcal{I}_E$ is the defining ideal sheaf of E. Since $\mathcal{I}_E = g_*\mathcal{O}_Z(-D)$, we obtain that

$$\begin{aligned} 0 \to f_*(\mathcal{I}_E \otimes \mathcal{O}_Y(K_Y + D_1 + D_2)) &\to f_*\mathcal{O}_Y(K_Y + D_1 + D_2) \\ &\to f_*\mathcal{O}_E(K_Y + D_1 + D_2) \to 0 \end{aligned}$$

by $R^1 f_*(\mathcal{I}_E \otimes \mathcal{O}_Y(K_Y + D_1 + D_2)) = 0$. By adjunction,

$$\mathcal{O}_E(K_Y + D_1 + D_2) \simeq \mathcal{O}_E.$$

Therefore, $\mathcal{O}_Y(K_Y + D_1 + D_2)$ is f-free. In particular,

$$K_Y + D_1 + D_2 = f^*(K_X + B_1 + B_2),$$

where $B_1 = f_*D_1$ and $B_2 = f_*D_2$. Thus, $-D - (K_Z + D_1' + D_2') \sim_{f \circ g} 0$. So, we have

$$R^i f_*\mathcal{I}_E = R^i f_*(g_*\mathcal{O}_Z(-D)) = 0$$

for every $i > 0$ by Theorem 5.7.3 below. This implies that $R^i f_*\mathcal{O}_Y \simeq R^i f_*\mathcal{O}_E$ for every $i > 0$. Thus, $R^1 f_*\mathcal{O}_Y \simeq \mathbb{C}(P)$, where $P = f(E)$. We consider the following spectral sequence

$$E^{p,q} = H^p(X, R^q f_*\mathcal{O}_Y \otimes \mathcal{O}_X(-mA)) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y(-mA)),$$

where A is an ample Cartier divisor on X and m is any positive integer. Since $H^1(Y, \mathcal{O}_Y(-mf^*A)) = H^2(Y, \mathcal{O}_Y(-mf^*A)) = 0$ by the Kawamata–Viehweg vanishing theorem (see Theorem 3.2.1), we have

$$H^0(X, R^1 f_*\mathcal{O}_Y \otimes \mathcal{O}_X(-mA)) \simeq H^2(X, \mathcal{O}_X(-mA)).$$

If we assume that X is Cohen–Macaulay, then we have

$$H^2(X, \mathcal{O}_X(-mA)) = 0$$

for every $m \gg 0$ by Serre duality and Serre's vanishing theorem. On the other hand, $H^0(X, R^1 f_*\mathcal{O}_Y \otimes \mathcal{O}_X(-mA)) \simeq \mathbb{C}(P)$ because $R^1 f_*\mathcal{O}_Y \simeq \mathbb{C}(P)$. This is a contradiction. Thus, X is not Cohen–Macaulay. In particular, $(X, B_1 + B_2)$ is log canonical but not dlt. We note that $\mathrm{Exc}(f) = E$ is not a divisor on Y.

Let us recall that $\Phi : V \to W$ is a flipping contraction. Let $\Phi^+ : V^+ \to W$ be the flip of Φ. We can check that

$$V^+ = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$$

and the flipped curve $E^+ \simeq \mathbb{P}^1$ is the negative section of $\pi^+ : V^+ \to \mathbb{P}^1$, that is, the section corresponding to the projection

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Let Y^+ be the strict transform of Y on V^+. Then Y^+ is Gorenstein, log canonical along $E^+ \subset Y^+$, and smooth outside E^+. Let D_1^+ (resp. D_2^+) be the strict transform of D_1 (resp. D_2) on Y^+. If we take a Cartier divisor B on Y suitably, then

$$\begin{tikzcd}
(Y, D_1 + D_2) \arrow[rr, dashed] \arrow[dr, "f"'] & & (Y^+, D_1^+ + D_2^+) \arrow[dl] \\
& X &
\end{tikzcd}$$

is the B-flop of $f : Y \to X$. In this example, the flopping curve E is a smooth elliptic curve and the flopped curve E^+ is $\mathbb{P}^1$. We note that $(Y, D_1 + D_2)$ is dlt. However, $(Y^+, D_1^+ + D_2^+)$ is log canonical but not dlt.

We close this section with Kovács's vanishing theorem.

THEOREM 3.13.10 (cf. [Kv5, Theorem 1.2] and [F34, Theorem 1]). *Let (X, Δ) be a log canonical pair and let $f : Y \to X$ be a proper birational morphism from a smooth variety Y such that* $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta$ *is a simple normal crossing divisor on Y. In this situation, we can write*

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

We put $E = \sum_{a_i=-1} E_i$. Then we have

$$R^i f_* \mathcal{O}_Y(-E) = 0$$

for every $i > 0$.

The proof given in [F34] is essentially the same as the proof of Theorem 3.13.6 with the aid of the minimal model program. For a slightly simpler proof of Theorem 3.13.10, see [Ch2, Section 4]. The original proof of Theorem 3.13.10 in [Kv5] uses the notion of Du Bois pairs (see Definition 5.3.5) and the minimal model program. Anyway, we do not know any proof without using the minimal model program. So we omit the details here.

REMARK 3.13.11. If (X, Δ) is klt, then Theorem 3.13.10 says that X has only rational singularities.

REMARK 3.13.12. In [Kv5], Kovács proved Theorem 3.13.10 under the extra assumption that X is $\mathbb{Q}$-factorial. When we use Theorem 3.13.10 for the study of log canonical singularities, the assumption that X is $\mathbb{Q}$-factorial is very restrictive. See, for example, [Ch2].

3.14. Elkik–Fujita vanishing theorem

The Elkik–Fujita vanishing theorem (see [Elk] and [Ft3]) is a very difficult vanishing theorem in [KMM].

THEOREM 3.14.1 ([KMM, Theorem 1-3-1]). *Let $f : Y \to X$ be a projective birational morphism from a smooth variety Y onto a variety X. Let L and $\widetilde{L}$ be Cartier divisors on Y. Assume that there exist $\mathbb{R}$-divisors D and $\widetilde{D}$ on Y and an effective Cartier divisor E on Y such that the following conditions are satisfied:*

(i) $\mathrm{Supp}\, D$ *and* $\mathrm{Supp}\, \widetilde{D}$ *are simple normal crossing divisors, and* $\lfloor D \rfloor = \lfloor \widetilde{D} \rfloor = 0$,
(ii) *both $-L - D$ and $-\widetilde{L} - \widetilde{D}$ are f-nef,*
(iii) $K_Y \sim L + \widetilde{L} + E$, *and*
(iv) *E is f-exceptional.*

Then $R^i f_* \mathcal{O}_Y(L) = 0$ for every $i > 0$.

We give a very simple proof due to Chih-Chi Chou (see [Ch1]). The original proof in [KMM, §1-3] is much harder than the proof given below.

Proof. We consider

$$\alpha : f_*\mathcal{O}_Y(L) \simeq \tau_{\leq 0} Rf_*\mathcal{O}_Y(L) \to Rf_*\mathcal{O}_Y(L)$$

and

$$\beta : Rf_*\mathcal{O}_Y(L) \to Rf_*\mathcal{O}_Y(L+E)$$

in the derived category of coherent sheaves. Since

$$L + E - (K_Y + \widetilde{D}) \sim -\widetilde{L} - \widetilde{D}$$

is f-nef and f-big, we obtain

$$R^i f_*\mathcal{O}_Y(L+E) = 0$$

for every $i > 0$ by the Kawamata–Viehweg vanishing theorem: Theorem 3.2.9. By Lemma 3.14.2 below,

$$f_*\mathcal{O}_Y(L+E) = f_*\mathcal{O}_Y(L).$$

Therefore, the composition

$$\beta \circ \alpha : f_*\mathcal{O}_Y(L) \to Rf_*\mathcal{O}_Y(L) \to Rf_*\mathcal{O}_Y(L+E)$$

is a quasi-isomorphism. By taking $R\mathcal{H}om(\underline{\quad}, \omega^\bullet_X)$, we obtain

$$\begin{aligned} R\mathcal{H}om(f_*\mathcal{O}_Y(L), \omega^\bullet_X) &\xleftarrow{\alpha^*} R\mathcal{H}om(Rf_*\mathcal{O}_Y(L), \omega^\bullet_X) \\ &\xleftarrow{\beta^*} R\mathcal{H}om(Rf_*\mathcal{O}_Y(L+E), \omega^\bullet_X) \end{aligned}$$

such that $\alpha^* \circ \beta^*$ is a quasi-isomorphism. By Grothendieck duality,

$$R\mathcal{H}om(Rf_*\mathcal{O}_Y(L), \omega^\bullet_X) \simeq Rf_*\mathcal{O}_Y(K_Y - L)[n]$$

and

$$R\mathcal{H}om(Rf_*\mathcal{O}_Y(L+E), \omega^\bullet_X) \simeq Rf^*\mathcal{O}_Y(K_Y - L - E)[n]$$

where $n = \dim X = \dim Y$. Since

$$K_Y - L - (K_Y + D) = -L - D$$

is f-nef and f-big, we obtain

$$R^i f_*\mathcal{O}_Y(K_Y - L) = 0$$

for every $i > 0$ by the Kawamata–Viehweg vanishing theorem: Theorem 3.2.9. Since

$$\begin{aligned} \alpha^* \circ \beta^* : Rf_*\mathcal{O}_Y(K_Y - L - E)[n] &\to Rf_*\mathcal{O}_Y(K_Y - L)[n] \\ &\to R\mathcal{H}om(f_*\mathcal{O}_Y(L), \omega^\bullet_X) \end{aligned}$$

is a quasi-isomorphism, we have

$$R^i f_* \mathcal{O}_Y(K_Y - L - E) = 0$$

for every $i > 0$. This implies that $R^i f_* \mathcal{O}_Y(\widetilde{L}) = 0$ for every $i > 0$ by the condition (iii). By symmetry, we obtain $R^i f_* \mathcal{O}_Y(L) = 0$ for every $i > 0$. $\square$

We have already used the following Fujita's lemma in the proof of Theorem 3.14.1 (see also [**KMM**, Lemma 1-3-2]).

LEMMA 3.14.2 ([**Ft3**, (2.2) Lemma]). *Let $f : Y \to X$ be a projective birational morphism from a smooth variety Y onto a variety X, let L be a Cartier divisor on Y, let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y, and let E be a Cartier divisor on Y. Assume that* $\operatorname{Supp} D$ *is a simple normal crossing divisor, $\lfloor D \rfloor = 0$, $-L - D$ is f-nef, and that E is effective and f-exceptional. Then $f_* \mathcal{O}_E(L + E) = 0$. In particular, $f_* \mathcal{O}_Y(L) = f_* \mathcal{O}_Y(L + E)$.*

PROOF. For any reduced irreducible component E_j of E, we have the exact sequence

$$0 \to f_* \mathcal{O}_{E'}(L + E') \to f_* \mathcal{O}_E(L + E) \to f_* \mathcal{O}_{E_j}(L + E),$$

where $E' = E - E_j$. Thus, by induction on the number of components of E, it is sufficient to prove that there exists a reduced irreducible component E_0 of E such that $f_* \mathcal{O}_{E_0}(L + E) = 0$. We will prove this by induction on $n = \dim X$.

First, we assume $n = 2$. We write $E - D = A - B$, where A and B are effective $\mathbb{R}$-divisors without common components. Since $\lfloor D \rfloor = 0$, we have $A \neq 0$. Since $\operatorname{Supp} A \subset \operatorname{Supp} E$, A is f-exceptional. Therefore, by Lemma 2.3.24, we have $A \cdot E_0 < 0$ for some irreducible component E_0 of A. Sine $-L - D$ is f-nef, we have

$$(E + L) \cdot E_0 \leq (E - D) \cdot E_0 \leq A \cdot E_0 < 0,$$

which implies $f_* \mathcal{O}_{E_0}(L + E) = 0$.

Next, we assume $n \geq 3$. We will derive a contradiction assuming that $f_* \mathcal{O}_{E_j}(L + E_j) \neq 0$ for every irreducible component E_j of E. By replacing X with an arbitrary affine open set of X, we may assume that X is affine.

If $\dim f(E) = 0$, then

$$H^0(E_j, \mathcal{O}_{E_j}(L + E)) \neq 0$$

for every E_j. We take a general hyperplane section Y' of Y. Then

$$H^0(E_j \cap Y', \mathcal{O}_{E_j \cap Y'}(L + E)) \neq 0$$

for every E_j. This is a contradiction by induction hypothesis.

If $\dim f(E) \geq 1$, then we take a general hyperplane section X' of X and apply induction to $f : Y' := f^{-1}(X') \to X'$. Then, we have $f_*\mathcal{O}_{E_j \cap Y'}(L + E) \neq 0$ for every E_j with $\dim f(E_j) \geq 1$, which is a contradiction by induction hypothesis. $\square$

Remark 3.14.3. In [KMM], f in Theorem 3.14.1 and Lemma 3.14.2 is assumed to be *proper*. However, we assume that f is *projective* in Theorem 3.14.1 and Lemma 3.14.2 since we take a general hyperplane section Y' of Y in the proof of Lemma 3.14.2.

The following remark is obvious by the proof of Theorem 3.14.1.

Remark 3.14.4 (see [KMM, Remark 1-3-5]). It is easy to see that Theorem 3.14.1 holds under the following conditions (a) and (b) instead of (i) and (ii):

(a) $\operatorname{Supp} D$ and $\operatorname{Supp} \widetilde{D}$ are normal crossing divisors, $\lfloor D \rfloor = 0$, and $\widetilde{D}$ is a boundary $\mathbb{R}$-divisor,
(b) $-L - D$ is f-nef and $-\widetilde{L} - \widetilde{D}$ is f-ample.

We give a proof of [KMM, Theorem 1-3-6] using Theorem 3.14.1 for the reader's convenience.

Theorem 3.14.5 (see Theorem 3.13.5 and [KMM, Theorem 1-3-6]). *All weak log-terminal singularities are rational.*

Proof. Let (X, Δ) be a pair with only weak log-terminal singularities. Then we can take a resolution of singularities $f : Y \to X$ where

(1) there exists a divisor $\sum F_j$ with only normal crossings whose support is $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}\Delta$,
(2) $K_Y = f^*(K_X + \Delta) + \sum a_j F_j$ with the condition that $a_j > -1$ whenever F_j is f-exceptional, and
(3) there exists an f-ample Cartier divisor $A = \sum b_j F_j$ where $b_j = 0$ if F_j is not f-exceptional.

Note that $b_j \leq 0$ for every j, equivalently, $-A$ is effective, by the negativity lemma (see Lemma 2.3.26). We put

$$J' = \{j \mid F_j \text{ is } f\text{-exceptional}\}$$

and

$$J'' = \{j \mid F_j \text{ is not } f\text{-exceptional}\}.$$

We put

$$E' = \sum_{j \in J'} a_j F_j, \quad E = \lceil E' \rceil$$

and

$$\widetilde{D} = \sum_{j \in J''} (-a_j F_j) + E - E' - \delta A$$

for some sufficiently small number δ. Then

$$\widetilde{L} = K_Y - E, \quad L = 0, \quad \text{and} \quad D = 0$$

satisfy the conditions (a) and (b) in Remark 3.14.4 and (iii) and (iv) of Theorem 3.14.1. Therefore, we obtain

$$0 = R^i f_* \mathcal{O}_Y(L) = R^i f_* \mathcal{O}_Y$$

for every $i > 0$. □

3.15. Method of two spectral sequences

In this section, we give a proof of the following well-known theorem again (see Theorem 3.13.1).

THEOREM 3.15.1. *Let (X, D) be a dlt pair. Then X has only rational singularities.*

Our proof is a combination of the proofs in [KoMo, Theorem 5.22] and [Ko8, Section 11]. We need no difficult duality theorems.

Let us give a dual form of the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7.

LEMMA 3.15.2 (see also Lemma 7.1.2). *Let $f : Y \to X$ be a proper birational morphism from a smooth variety Y to a variety X. Let $x \in X$ be a closed point. We put $F = f^{-1}(x)$. Then we have*

$$H^i_F(Y, \mathcal{O}_Y) = 0$$

for every $i < n = \dim X$.

PROOF. We take a proper birational morphism $g : Z \to Y$ from a smooth variety Z such that $f \circ g$ is projective. We consider the following spectral sequence

$$E_2^{pq} = H^p_F(Y, R^q g_* \mathcal{O}_Z) \Rightarrow H^{p+q}_E(Z, \mathcal{O}_Z),$$

where $E = g^{-1}(F) = (f \circ g)^{-1}(x)$. Since $R^q g_* \mathcal{O}_Z = 0$ for $q > 0$ and $g_* \mathcal{O}_Z \simeq \mathcal{O}_Y$, we have $H^p_F(Y, \mathcal{O}_Y) \simeq H^p_E(Z, \mathcal{O}_Z)$ for every p. Therefore, we can replace Y with Z and assume that $f : Y \to X$ is projective. Without loss of generality, we may assume that X is affine. Then we compactify X and assume that X and Y are projective. It is well known that

$$H^i_F(Y, \mathcal{O}_Y) \simeq \varinjlim_m \operatorname{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y)$$

(see [Har2, Theorem 2.8]) and that

$$\operatorname{Hom}(\operatorname{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y), \mathbb{C}) \simeq H^{n-i}(Y, \mathcal{O}_{mF} \otimes \omega_Y)$$

by duality on a smooth projective variety Y (see [Har4, Theorem 7.6 (a)]). Therefore,

$$\begin{aligned}\operatorname{Hom}(H^i_F(Y, \mathcal{O}_Y), \mathbb{C}) &\simeq \operatorname{Hom}(\varinjlim_m \operatorname{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y), \mathbb{C}) \\ &\simeq \varprojlim_m H^{n-i}(Y, \mathcal{O}_{mF} \otimes \omega_Y) \\ &\simeq (R^{n-i}f_*\omega_Y)^\wedge_x\end{aligned}$$

by the theorem on formal functions (see [Har4, Theorem 11.1]), where $(R^{n-i}f_*\omega_Y)^\wedge_x$ is the completion of $R^{n-i}f_*\omega_Y$ at $x \in X$. On the other hand, $R^{n-i}f_*\omega_Y = 0$ for $i < n$ by the Grauert–Riemenschneider vanishing theorem: Theorem 3.2.7. Thus, $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i < n$. $\square$

Remark 3.15.3. Lemma 3.15.2 holds true even when Y has rational singularities. This is because $R^q g_*\mathcal{O}_Z = 0$ for $q > 0$ and $g_*\mathcal{O}_Z \simeq \mathcal{O}_Y$ holds in the proof of Lemma 3.15.2.

Let us go to the proof of Theorem 3.15.1.

Proof of Theorem 3.15.1. Without loss of generality, we may assume that X is affine. Moreover, by taking general hyperplane sections of X, we may also assume that X has only rational singularities outside a closed point $x \in X$. By the definition of dlt, we can take a resolution $f : Y \to X$ such that $\operatorname{Exc}(f)$ and $\operatorname{Exc}(f) \cup \operatorname{Supp} f^{-1}_*D$ are both simple normal crossing divisors on Y,

$$K_Y + f^{-1}_*D = f^*(K_X + D) + E$$

with $\lceil E \rceil \geq 0$, and that f is projective. Moreover, we can make f an isomorphism over the generic point of any log canonical center of (X, D) (see Remark 2.3.18). Therefore, by Lemma 3.2.11, we can check that $R^if_*\mathcal{O}_Y(\lceil E \rceil) = 0$ for every $i > 0$. We note that $f_*\mathcal{O}_Y(\lceil E \rceil) \simeq \mathcal{O}_X$ since $\lceil E \rceil$ is effective and f-exceptional. For every $i > 0$, by the above assumption, $R^if_*\mathcal{O}_Y$ is supported at a point $x \in X$ if it ever has a non-empty support at all. We put $F = f^{-1}(x)$. Then we have a spectral sequence

$$E^{i,j}_2 = H^i_x(X, R^jf_*\mathcal{O}_Y(\lceil E \rceil)) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y(\lceil E \rceil)).$$

By the above vanishing result, we have

$$H^i_x(X, \mathcal{O}_X) \simeq H^i_F(Y, \mathcal{O}_Y(\lceil E \rceil))$$

for every $i \geq 0$. We obtain a commutative diagram

$$\begin{tikzcd} H^i_F(Y, \mathcal{O}_Y) \arrow[r] & H^i_F(Y, \mathcal{O}_Y(\lceil E \rceil)) \\ H^i_x(X, \mathcal{O}_X) \arrow[u, "\alpha"] \arrow[r, equal] & H^i_x(X, \mathcal{O}_X). \arrow[u, "\beta"'] \end{tikzcd}$$

We have already checked that β is an isomorphism for every i and that $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i < n$ (see Lemma 3.15.2). Therefore, $H^i_x(X, \mathcal{O}_X) = 0$ for every $i < n = \dim X$. Thus, X is Cohen–Macaulay. For $i = n$, we obtain that

$$\alpha : H^n_x(X, \mathcal{O}_X) \to H^n_F(Y, \mathcal{O}_Y)$$

is injective. We consider the following spectral sequence

$$E_2^{i,j} = H^i_x(X, R^j f_* \mathcal{O}_Y) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y).$$

We note that $H^i_x(X, R^j f_* \mathcal{O}_Y) = 0$ for every $i > 0$ and $j > 0$ since $\operatorname{Supp} R^j f_* \mathcal{O}_Y \subset \{x\}$ for $j > 0$. On the other hand,

$$E_2^{i,0} = H^i_x(X, \mathcal{O}_X) = 0$$

for every $i < n$. Therefore,

$$H^0_x(X, R^j f_* \mathcal{O}_Y) \simeq H^j_x(X, \mathcal{O}_X) = 0$$

for all $j \leq n - 2$. Thus, $R^j f_* \mathcal{O}_Y = 0$ for $1 \leq j \leq n - 2$. Since $H^{n-1}_x(X, \mathcal{O}_X) = 0$, we obtain that

$$0 \to H^0_x(X, R^{n-1} f_* \mathcal{O}_Y) \to H^n_x(X, \mathcal{O}_X) \xrightarrow{\alpha} H^n_F(Y, \mathcal{O}_Y) \to 0$$

is exact. We have already checked that α is injective. So, we obtain that $H^0_x(X, R^{n-1} f_* \mathcal{O}_Y) = 0$. This means that $R^{n-1} f_* \mathcal{O}_Y = 0$. Thus, we have $R^i f_* \mathcal{O}_Y = 0$ for every $i > 0$. We complete the proof. $\square$

REMARK 3.15.4. The method of two spectral sequences was introduced in the proof of [**KoMo**, Theorem 5.22]. In [**Ale3**], Alexeev used this method in order to establish his criterion for Serre's S_3 condition. The method of two spectral sequences of local cohomology groups discussed in this section was first used in [**F17**, Section 4.3] in order to generalize Alexeev's criterion for S_3 condition (see Section 7.1). The proof of Theorem 3.15.1 in this section first appeared in [**F17**, Subsection 4.2.1].

3.16. Toward new vanishing theorems

In [**F28**], the following results played crucial roles. For the proof and the details, see [**F28**].

Proposition 3.16.1 (see, for example, [**F28**, Proposition 5.1]) can be proved by the theory of mixed Hodge structures. Note that Theorem 5.4.1 below is a complete generalization of Proposition 3.16.1.

PROPOSITION 3.16.1 (Fundamental injectivity theorem). *Let X be a smooth projective variety and let $S+B$ be a boundary $\mathbb{R}$-divisor on X such that the support of $S+B$ is simple normal crossing and $\lfloor S+B \rfloor = S$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in* $\operatorname{Supp} B$. *Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ are injective for all q.

Proposition 3.16.1 is one of the correct generalizations of Kollár's injectivity theorem (see Theorem 3.6.2) from the Hodge theoretic viewpoint. By Proposition 3.16.1, we can prove Theorem 3.16.2 (see, for example, [**F28**, Theorem 6.1]), which is a generalization of Kollár's injectivity theorem. Theorem 5.6.2 below is a generalization of Theorem 3.16.2 for simple normal crossing pairs.

THEOREM 3.16.2 (Injectivity theorem). *Let X be a smooth projective variety and let Δ be a boundary $\mathbb{R}$-divisor such that* $\operatorname{Supp}\Delta$ *is simple normal crossing. Let L be a Cartier divisor on X and let D be an effective Cartier divisor that contains no log canonical centers of (X, Δ). Assume the following conditions.*

(i) *$L \sim_{\mathbb{R}} K_X + \Delta + H$,*
(ii) *H is a semi-ample $\mathbb{R}$-divisor, and*
(iii) *$tH \sim_{\mathbb{R}} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support contains no log canonical centers of (X, Δ).*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ are injective for all q.

There are no difficulties to prove Theorem 3.16.3 as an application of Theorem 3.16.2 (see, for example, [**F28**, Theorem 6.3]). Theorem 3.16.3 contains Kollár's torsion-free theorem and Kollár's vanishing theorem (see Theorem 3.6.3). We will prove a generalization of Theorem 3.16.3 for simple normal crossing pairs (see Theorem 5.6.3 below). Theorem 5.6.3 is a key ingredient of the theory of quasi-log schemes discussed in Chapter 6.

THEOREM 3.16.3 (Torsion-freeness and vanishing theorem). *Let Y be a smooth variety and let Δ be a boundary $\mathbb{R}$-divisor such that* $\operatorname{Supp}\Delta$ *is simple normal crossing. Let $f : Y \to X$ be a projective morphism and let L be a Cartier divisor on Y such that $L-(K_Y+\Delta)$ is f-semi-ample.*

(i) *Let q be an arbitrary non-negative integer. Then every associated prime of $R^qf_*\mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y,Δ).*

(ii) *Let $\pi : X \to S$ be a projective morphism. Assume that*
$$L-(K_Y+\Delta)\sim_{\mathbb{R}} f^*H$$
for some π-ample $\mathbb{R}$-divisor H on X. Then
$$R^p\pi_*R^qf_*\mathcal{O}_Y(L)=0$$
for every $p>0$ and $q\geq 0$.

As an easy consequence of Theorem 3.16.3, we obtain Theorem 3.16.4 in [F28].

THEOREM 3.16.4 (see [F28, Theorem 8.1]). *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let D be a Cartier divisor on X. Assume that $D-(K_X+\Delta)$ is π-ample, where $\pi : X \to S$ is a projective morphism onto a variety S. Let $\{C_i\}$ be* any *set of log canonical centers of the pair (X,Δ). We put $W=\bigcup C_i$ with the reduced scheme structure. Assume that W is disjoint from the non-lc locus* $\operatorname{Nlc}(X,\Delta)$ *of (X,Δ). Then we have*
$$R^i\pi_*(\mathcal{J}\otimes\mathcal{O}_X(D))=0$$
for every $i>0$, where $\mathcal{J}=\mathcal{I}_W\cdot\mathcal{J}_{NLC}(X,\Delta)\subset\mathcal{O}_X$ and $\mathcal{I}_W$ is the defining ideal sheaf of W on X. Therefore, the restriction map
$$\pi_*\mathcal{O}_X(D)\to\pi_*\mathcal{O}_W(D)\oplus\pi_*\mathcal{O}_{\operatorname{Nlc}(X,\Delta)}(D)$$
is surjective and
$$R^i\pi_*\mathcal{O}_W(D)=0$$
for every $i>0$. In particular, the restriction maps
$$\pi_*\mathcal{O}_X(D)\to\pi_*\mathcal{O}_W(D)$$
and
$$\pi_*\mathcal{O}_X(D)\to\pi_*\mathcal{O}_{\operatorname{Nlc}(X,\Delta)}(D)$$

are surjective.

In [**F28**], Theorem 3.16.5 (see [**F28**, Theorem 11.1]) plays crucial roles for the proof of the non-vanishing theorem in [**F28**, Theorem 12.2].

THEOREM 3.16.5 (Vanishing theorem for minimal log canonical centers). *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let W be a minimal log canonical center of (X,Δ) such that W is disjoint from the non-lc locus* $\mathrm{Nlc}(X,\Delta)$ *of (X,Δ). Let $\pi : X \to S$ be a projective morphism onto a variety S. Let D be a Cartier divisor on W such that $D-(K_X+\Delta)|_W$ is π-ample. Then $R^i\pi_*\mathcal{O}_W(D)=0$ for every $i>0$.*

In [**F28**], we proved Theorem 3.16.5 by using dlt blow-ups (see Theorem 4.4.21), which depend on the recent developments of the minimal model program, and Theorem 3.16.3. Therefore, Theorem 3.16.5 is much harder than Theorem 3.16.4. Note that Theorem 3.16.4 and Theorem 3.16.5 are special cases of Theorem 6.3.4 below. The proof of Theorem 6.3.4 in Chapter 6 does not need the minimal model program but uses the theory of mixed Hodge structures for reducible varieties.

In [**F28**], we obtained the fundamental theorems, that is, various Kodaira type vanishing theorem, the cone and contraction theorem, and so on, for normal pairs by using Theorem 3.16.3 and Theorem 3.16.5 (see Section 4.5). Our formulation in [**F28**] is different from the traditional X-method and is similar to the theory of (algebraic) multiplier ideal sheaves based on the Nadel vanishing theorem (see, for example, [**La2**, Part Three]). In [**F28**], we need no vanishing theorems for *reducible* varieties.

REMARK 3.16.6. Theorem 3.16.2 (resp. Theorem 3.16.3) is a special case of [**Am1**, Theorem 3.1] (resp. [**Am1**, Theorem 3.2]). The proof of [**Am1**, Theorem 3.1] contains several difficulties (see, for example, Example 5.1.4). Moreover, Ambro's original proof of [**Am1**, Theorem 3.2 (ii)] used [**Am1**, Theorem 3.2 (i)] for *embedded normal crossing pairs* even when Y is smooth in [**Am1**, Theorem 3.2 (ii)]. On the other hand, the proof of Theorem 3.16.3 in [**F28**] does not need *reducible* varieties.

In Chapter 5 and Chapter 6, we will prove more general results than the theorems in this section. Note that [**KMM**, Theorem 1-2-5 and Remark 1-2-6] will be generalized as follows.

THEOREM 3.16.7 (Theorem 5.7.6). *Let (X,Δ) be a log canonical pair such that Δ is a boundary $\mathbb{R}$-divisor and let L be a $\mathbb{Q}$-Cartier Weil* *divisor on X. Assume that $L-(K_X+\Delta)$ is nef and log big over V with respect to (X,Δ), where $\pi : X \to V$ is a proper morphism. Then $R^q\pi_*\mathcal{O}_X(L)=0$ for every $q>0$.*

The proof of Theorem 3.16.7 (see Theorem 5.7.6) needs the vanishing theorem for reducible varieties. Therefore, it is much harder than the arguments in this chapter.

We strongly recommend the reader to see [**F28**, Section 3], where we discussed the conceptual difference between the traditional arguments based on the Kawamata–Viehweg–Nadel vanishing theorem and our new approach depending on the theory of mixed Hodge structures. It will help the reader to understand the results and the framework discussed in Chapter 5 and Chapter 6.

CHAPTER 4

Minimal model program

In this chapter, we discuss the minimal model program. Although we explain the recent developments of the minimal model program mainly due to Birkar–Cascini–Hacon–M$^{\mathrm{c}}$Kernan in Section 4.4, we do not discuss the proof of the main results of [**BCHM**]. For the details of [**BCHM**], see [**BCHM**], [**HaKo**, Part II], [**HaMc1**], [**HaMc2**], and so on. The papers [**Dr**], [**F25**], and [**Ka4**] are survey articles on [**BCHM**]. For slightly different approaches, see [**BirPa**], [**CoLa**], [**CaL**], [**P**], and so on. In this book, we mainly discuss the topics of the minimal model program which are not directly related to [**BCHM**].

In Sections 4.1, 4.2, and 4.3, we quickly review the basic results on the minimal model program, X-method, and so on. Section 4.4 is devoted to the explanation on [**BCHM**] and some related results and examples. In Section 4.5, we discuss the fundamental theorems for normal pairs (see [**F28**]) and various examples of the Kleiman–Mori cone. The results in Section 4.5 are sufficient for the minimal model program for log canonical pairs. In Section 4.6 and Section 4.7, we prove that Shokurov polytope is a polytope. In Section 4.8, we discuss the minimal model program for log canonical pairs and various conjectures. In Section 4.9, we explain the minimal model program for (not necessarily $\mathbb{Q}$-factorial) log canonical pairs. It is the most general minimal model program in the usual sense. In Section 4.10, we review the minimal model theory for singular surfaces following [**F29**]. In Section 4.11, we quickly explain the author's recent result on semi log canonical pairs without proof.

4.1. Fundamental theorems for klt pairs

In this section, we assume that X is a projective irreducible variety and Δ is an effective $\mathbb{Q}$-divisor for simplicity. Let us recall the fundamental theorems for klt pairs. For the details, see, for example, [**KoMo**, Chapter 3]. A starting point is the following vanishing theorem (see Theorem 3.1.7).

THEOREM 4.1.1 (Kawamata–Viehweg vanishing theorem). *Let X be a smooth projective variety and let D be a $\mathbb{Q}$-divisor such that*

Supp$\{D\}$ *is a simple normal crossing divisor on* X*. Assume that* D *is ample. Then*

$$H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$$

for every $i > 0$*.*

The next theorem is Shokurov's non-vanishing theorem (see [Sh1]).

THEOREM 4.1.2 (Non-vanishing theorem). *Let* X *be a projective variety, let* D *be a nef Cartier divisor, and let* G *be a* $\mathbb{Q}$*-divisor. Suppose*

(i) $aD + G - K_X$ *is an ample* $\mathbb{Q}$*-divisor for some* $a > 0$*, and*
(ii) $(X, -G)$ *is sub klt.*

Then there is a positive integer m_0 *such that*

$$H^0(X, \mathcal{O}_X(mD + \lceil G \rceil)) \neq 0$$

for every $m \geq m_0$*.*

It plays important roles in the proof of the basepoint-free and rationality theorems below.

THEOREM 4.1.3 (Basepoint-free theorem). *Let* (X, Δ) *be a projective klt pair. Let* D *be a nef Cartier divisor such that* $aD - (K_X + \Delta)$ *is ample for some* $a > 0$*. Then there is a positive integer* b_0 *such that* $|bD|$ *has no base points for every* $b \geq b_0$*.*

THEOREM 4.1.4 (Rationality theorem). *Let* (X, Δ) *be a projective klt pair such that* $K_X + \Delta$ *is not nef. Let* $a > 0$ *be an integer such that* $a(K_X + \Delta)$ *is Cartier. Let* H *be an ample Cartier divisor. We define*

$$r = \max\{\, t \in \mathbb{R} \mid H + t(K_X + \Delta) \text{ is nef}\}.$$

Then r *is a rational number of the form* u/v*, where* u *and* v *are integers with*

$$0 < v \leq a(\dim X + 1).$$

The final theorem is the cone and contraction theorem. It easily follows from the basepoint-free and rationality theorems: Theorems 4.1.3 and 4.1.4.

THEOREM 4.1.5 (Cone and contraction theorem). *Let* (X, Δ) *be a projective klt pair. Then we have the following properties.*

(i) *There are (countably many possibly singular) rational curves* $C_j \subset X$ *such that*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

(ii) *Let $R \subset \overline{NE}(X)$ be a $(K_X+\Delta)$-negative extremal ray. Then there is a unique morphism $\varphi_R : X \to Z$ to a projective variety Z such that $(\varphi_R)_*\mathcal{O}_X \simeq \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point by φ_R if and only if $[C] \in R$.*

We note that the cone and contraction theorem can be proved for dlt pairs in the relative setting (see, for example, [KMM]). We omit it here because we will give a complete generalization of the cone and contraction theorem for quasi-log schemes in Chapter 6. See also Theorem 4.5.2 below.

The main purpose of this book is to establish the cone and contraction theorem for quasi-log schemes (see Chapter 6). Note that a log canonical pair has a natural quasi-log structure which is compatible with the original log canonical structure.

4.2. X-method

In this section, we give a proof of the basepoint-free theorem for klt pairs (see Theorem 4.1.3) by assuming the non-vanishing theorem (see Theorem 4.1.2). The following proof is taken almost verbatim from [KoMo, 3.2 Basepoint-free Theorem]. This type of argument is usually called *X-method*. It has various applications in many different contexts.

Proof of the basepoint-free theorem: Theorem 4.1.3. We prove the basepoint-free theorem.

Step 1. In this step, we establish that $|mD| \neq \emptyset$ for every $m \gg 0$. We can construct a resolution of singularities $f : Y \to X$ such that

(i) $K_Y = f^*(K_X+\Delta) + \sum a_j F_j$ with all $a_j > -1$,
(ii) $f^*(aD - (K_X+\Delta)) - \sum p_j F_j$ is ample for some $a > 0$ and for suitable $0 < p_j \ll 1$, and
(iii) $\sum F_j (\supset \mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta)$ is a simple normal crossing divisor on Y.

We note that the F_j is not necessarily f-exceptional. On Y, we write

$$\begin{aligned}
&f^*(aD - (K_X+\Delta)) - \sum p_j F_j \\
&= af^*D + \sum (a_j - p_j)F_j - (f^*(K_X+\Delta) + \sum a_j F_j) \\
&= af^*D + G - K_Y,
\end{aligned}$$

where $G = \sum (a_j - p_j)F_j$. By assumption, $\lceil G \rceil$ is an effective f-exceptional divisor, $af^*D + G - K_Y$ is ample, and

$$H^0(Y, \mathcal{O}_Y(mf^*D + \lceil G \rceil)) \simeq H^0(X, \mathcal{O}_X(mD)).$$

We can now apply the non-vanishing theorem (see Theorem 4.1.2) to get that $H^0(X, \mathcal{O}_X(mD)) \neq 0$ for every $m \gg 0$.

STEP 2. For a positive integer s, let $B(s)$ denote the reduced base locus of $|sD|$. Clearly, we have $B(s^u) \subset B(s^v)$ for every positive integers $u > v$. The noetherian induction implies that the sequence $B(s^u)$ stabilizes, and we call the limit B_s. So either B_s is non-empty for some s or B_s and $B_{s'}$ are empty for two relatively prime integers s and s'. In the latter case, take u and v such that $B(s^u)$ and $B(s'^v)$ are empty, and use the fact that every sufficiently large integer is a linear combination of s^u and s'^v with non-negative coefficients to conclude that $|mD|$ is basepoint-free for every $m \gg 0$. So, we must show that the assumption that some B_s is non-empty leads to a contradiction. We let $m = s^u$ such that $B_s = B(m)$ and assume that this set is non-empty.

Starting with the linear system obtained from Step 1, we can blow up further to obtain a new $f : Y \to X$ for which the conditions of Step 1 hold, and, for some $m > 0$,

$$f^*|mD| = |L| \text{ (moving part)} + \sum r_j F_j \text{ (fixed part)}$$

such that $|L|$ is basepoint-free. Therefore, $\bigcup\{f(F_j) | r_j > 0\}$ is the base locus of $|mD|$. Note that $f^{-1} \operatorname{Bs}|mD| = \operatorname{Bs}|mf^*D|$. We obtain the desired contradiction by finding some F_j with $r_j > 0$ such that, for every $b \gg 0$, F_j is not contained in the base locus of $|bf^*D|$.

STEP 3. For an integer $b > 0$ and a rational number $c > 0$ such that $b \geq cm + a$, we define divisors:

$$\begin{aligned}
N(b, c) &= bf^*D - K_Y + \sum(-cr_j + a_j - p_j)F_j \\
&= (b - cm - a)f^*D \qquad \text{(nef)} \\
&\quad +c(mf^*D - \sum r_j F_j) \qquad \text{(basepoint-free)} \\
&\quad +f^*(aD - (K_X + \Delta)) - \sum p_j F_j \qquad \text{(ample).}
\end{aligned}$$

Thus, $N(b, c)$ is ample for $b \geq cm + a$. If that is the case, then, by Theorem 4.1.1, $H^1(Y, \mathcal{O}_Y(\lceil N(b, c)\rceil + K_Y)) = 0$, and

$$\lceil N(b, c)\rceil = bf^*D + \sum\lceil -cr_j + a_j - p_j\rceil F_j - K_Y.$$

STEP 4. c and p_j can be chosen so that

$$\sum(-cr_j + a_j - p_j)F_j = A - F$$

for some $F = F_{j_0}$, where $\lceil A \rceil$ is effective and A does not have F as a component. In fact, we choose $c > 0$ so that

$$\min_j(-cr_j + a_j - p_j) = -1.$$

If this last condition does not single out a unique j, we wiggle the p_j slightly to achieve the desired uniqueness. This j satisfies $r_j > 0$ and $\lceil N(b,c) \rceil + K_Y = bf^*D + \lceil A \rceil - F$. Now Step 3 implies that

$$H^0(Y, \mathcal{O}_Y(bf^*D + \lceil A \rceil)) \to H^0(F, \mathcal{O}_F(bf^*D + \lceil A \rceil))$$

is surjective for $b \geq cm + a$. If F_j appears in $\lceil A \rceil$, then $a_j > 0$, so F_j is f-exceptional. Thus, $\lceil A \rceil$ is f-exceptional.

Step 5. Notice that

$$N(b,c)|_F = (bf^*D + A - F - K_Y)|_F = (bf^*D + A)|_F - K_F.$$

So we can apply the non-vanishing theorem (see Theorem 4.1.2) on F to get

$$H^0(F, \mathcal{O}_F(bf^*D + \lceil A \rceil)) \neq 0.$$

Thus, $H^0(Y, \mathcal{O}_Y(bf^*D + \lceil A \rceil))$ has a section not vanishing on F. Since $\lceil A \rceil$ is f-exceptional and effective,

$$H^0(Y, \mathcal{O}_Y(bf^*D + \lceil A \rceil)) \simeq H^0(X, \mathcal{O}_X(bD)).$$

Therefore, $f(F)$ is not contained in the base locus of $|bD|$ for every $b \gg 0$.

This completes the proof of the basepoint-free theorem. □

The X-method is very powerful and very useful for klt pairs. Unfortunately, it can not be applied for log canonical pairs. So we need the framework discussed in [**F28**] or the theory of quasi-log schemes (see Chapter 6) in order to treat log canonical pairs. For the details of X-method, see [**KMM**] and [**KoMo**].

We note that the X-method, the technique which was used for the proofs of Theorems 4.1.2, 4.1.3, 4.1.4, and 4.1.5, was developed by several authors. The main contributions are [**Ka2**], [**Ko1**], [**R1**], and [**Sh1**].

4.3. MMP for ℚ-factorial dlt pairs

In this section, we quickly explain the minimal model program for ℚ-factorial dlt pairs. First, let us recall the definition of the (log) minimal models. Definition 4.3.1 is a traditional definition of minimal models. For slightly different other definitions of minimal models, see Definition 4.4.4 and Definition 4.8.5.

DEFINITION 4.3.1 ((Log) minimal model). Let (X, Δ) be a log canonical pair and let $f : X \to S$ be a proper morphism. A pair (X', Δ') sitting in a diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

is called a *(log) minimal model of* (X, Δ) *over* S if

(i) f' is proper,
(ii) ϕ^{-1} has no exceptional divisors,
(iii) $\Delta' = \phi_*\Delta$,
(iv) $K_{X'} + \Delta'$ is f'-nef, and
(v) $a(E, X, \Delta) < a(E, X', \Delta')$ for every ϕ-exceptional divisor $E \subset X$.

Furthermore, if $K_{X'} + \Delta'$ is f'-semi-ample, then (X', Δ') is called a *good minimal model of* (X, Δ) *over* S.

We note the following easy lemma.

LEMMA 4.3.2. *Let* (X, Δ) *be a log canonical pair and let* $f : X \to S$ *be a proper morphism. Let* (X', Δ') *be a minimal model of* (X, Δ) *over* S. *Then* $a(E, X, \Delta) \leq a(E, X', \Delta')$ *for every divisor* E *over* X.

PROOF. We take any common resolution

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X' \end{tikzcd}$$

of X and X'. Then we can write

$$K_W = p^*(K_X + \Delta) + F$$

and

$$K_W = q^*(K_{X'} + \Delta') + G.$$

It is sufficient to prove $G \geq F$. Note that

$$p^*(K_X + \Delta) = q^*(K_{X'} + \Delta') + G - F.$$

Then $-(G - F)$ is p-nef since $K_{X'} + \Delta'$ is nef over S. Note that $p_*(G - F)$ is effective by (v). Therefore, by the negativity lemma (see Lemma 2.3.26), $G - F$ is effective. $\square$

Next, we recall the flip theorem for dlt pairs in [**BCHM**] and [**HaMc1**] (see also [**HaMc2**]). We need the notion of small morphisms to treat flips.

Definition 4.3.3 (Small morphism). Let $f : X \to Y$ be a proper birational morphism between normal varieties. If $\mathrm{Exc}(f)$ has codimension ≥ 2, then f is called *small*.

Theorem 4.3.4 ((Log) flip for dlt pairs). *Let $\varphi : (X, \Delta) \to W$ be an extremal flipping contraction, that is,*

(i) *(X, Δ) is dlt,*
(ii) *φ is small projective and φ has connected fibers,*
(iii) *$-(K_X + \Delta)$ is φ-ample,*
(iv) *$\rho(X/W) = 1$, and*
(v) *X is $\mathbb{Q}$-factorial.*

Then we have the following diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

(1) *X^+ is a normal variety,*
(2) *$\varphi^+ : X^+ \to W$ is small projective, and*
(3) *$K_{X^+} + \Delta^+$ is φ^+-ample, where Δ^+ is the strict transform of Δ.*

We call $\varphi^+ : (X^+, \Delta^+) \to W$ a $(K_X+\Delta)$-flip of φ. In this situation, we can check that (X^+, Δ^+) is a $\mathbb{Q}$-factorial dlt pair with $\rho(X^+/W) = 1$ (see, for example, Lemma 4.8.13 and Proposition 4.8.16 below).

Let us explain the relative minimal model program (MMP, for short) for $\mathbb{Q}$-factorial dlt pairs.

4.3.5 (MMP for $\mathbb{Q}$-factorial dlt pairs). We start with a pair $(X, \Delta) = (X_0, \Delta_0)$. Let $f_0 : X_0 \to S$ be a projective morphism. The aim is to set up a recursive procedure which creates intermediate pairs (X_i, Δ_i) and projective morphisms $f_i : X_i \to S$. After some steps, it should stop with a final pair (X', Δ') and $f' : X' \to S$.

Step 0 (Initial datum). Assume that we have already constructed (X_i, Δ_i) and $f_i : X_i \to S$ with the following properties:

(i) X_i is $\mathbb{Q}$-factorial,
(ii) (X_i, Δ_i) is dlt, and
(iii) f_i is projective.

Step 1 (Preparation). If $K_{X_i} + \Delta_i$ is f_i-nef, then we go directly to Step 3 (ii). If $K_{X_i} + \Delta_i$ is not f_i-nef, then we have established the following two results:

(i) (Cone theorem) We have the following equality.
$$\overline{NE}(X_i/S) = \overline{NE}(X_i/S)_{(K_{X_i}+\Delta_i)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

(ii) (Contraction theorem) Any $(K_{X_i} + \Delta_i)$-negative extremal ray $R_i \subset \overline{NE}(X_i/S)$ can be contracted. Let $\varphi_{R_i} : X_i \to Y_i$ denote the corresponding contraction. It sits in a commutative diagram.
$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{R_i}"] \arrow[dr, "f_i"'] & & Y_i \arrow[dl, "g_i"] \\ & S & \end{tikzcd}$$

Step 2 (Birational transformations). If $\varphi_{R_i} : X_i \to Y_i$ is birational, then we produce a new pair (X_{i+1}, Δ_{i+1}) as follows.

(i) (Divisorial contraction). If φ_{R_i} is a divisorial contraction, that is, φ_{R_i} contracts a divisor, then we set $X_{i+1} = Y_i$, $f_{i+1} = g_i$, and $\Delta_{i+1} = (\varphi_{R_i})_*\Delta_i$.

(ii) (Flipping contraction). If φ_{R_i} is a flipping contraction, that is, φ_{R_i} is small, then we set $(X_{i+1}, \Delta_{i+1}) = (X_i^+, \Delta_i^+)$, where (X_i^+, Δ_i^+) is the flip of φ_{R_i}
$$\begin{tikzcd} (X_i, \Delta_i) \arrow[rr, dashed] \arrow[dr, "\varphi_{R_i}"'] & & (X_i^+, \Delta_i^+) \arrow[dl, "\varphi^+_{R_i}"] \\ & Y_i & \end{tikzcd}$$
and $f_{i+1} = g_i \circ \varphi^+_{R_i}$ (see Theorem 4.3.4).

In both cases, we can prove that X_{i+1} is $\mathbb{Q}$-factorial, f_{i+1} is projective and (X_{i+1}, Δ_{i+1}) is dlt (see, for example, Lemma 4.8.13, Proposition 4.8.14, and Proposition 4.8.16). Then we go back to Step 0 with (X_{i+1}, Δ_{i+1}) and start anew.

Step 3 (Final outcome). We expect that eventually the procedure stops, and we get one of the following two possibilities:

(i) (Mori fiber space). If φ_{R_i} is a Fano contraction, that is, $\dim Y_i < \dim X_i$, then we set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$. In this case, we usually call $f' : (X', \Delta') \to Y_i$ a *Mori fiber space of* (X, Δ) *over* S.

(ii) (Minimal model). If $K_{X_i} + \Delta_i$ is f_i-nef, then we again set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$. We can easily check that (X', Δ') is a minimal model of (X, Δ) over S in the sense of Definition 4.3.1.

By the results in [**BCHM**] and [**HaMc1**] (see also [**HaMc2**]), all we have to do is to prove that there are no infinite sequence of flips in the above process.

Conjecture 4.3.6 (Flip conjecture II). *A sequence of (log) flips*

$$(X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i) \dashrightarrow \cdots$$

terminates after finitely many steps. Namely there does not exist an infinite sequence of (log) flips.

Remark 4.3.7. In Conjecture 4.3.6, each flip

$$(X_i, \Delta_i) \dashrightarrow (X_{i+1}, \Delta_{i+1})$$

is a flip as in Theorem 4.3.4.

Lemma 4.3.8. *We assume that Conjecture 4.3.6 holds in the following two cases:*

(i) (X_0, Δ_0) *is klt with* $\dim X_0 = n$*, and*
(ii) (X_0, Δ_0) *is dlt with* $\dim X_0 \le n - 1$*.*

Then Conjecture 4.3.6 holds for n-dimensional dlt pair (X_0, Δ_0). Therefore, by induction on the dimension, it is sufficient to prove Conjecture 4.3.6 under the extra assumption that (X_0, Δ_0) is klt.

Proof. Let

$$(X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i) \dashrightarrow \cdots$$

be a sequence of flips as in Conjecture 4.3.6 with $\dim X_0 = n$. By the case (ii), the special termination theorem holds in dimension n (see, for example, [**F13**, Theorem 4.2.1]). Therefore, after finitely many steps, the flipping locus (and thus the flipped locus) is disjoint from $\lfloor \Delta_i \rfloor$. Thus, we may assume that $\lfloor \Delta_i \rfloor = 0$ by replacing Δ_i with $\{\Delta_i\}$. In this case, the above sequence terminates by the case (i). $\square$

Conjecture 4.3.6 was completely solved in dimension ≤ 3 (see, for example, [**Koetal**, Chapter 6] and [**Sh3**, 5.1.3]). Conjecture 4.3.6 is still open even when $\dim X_0 = 4$. For the details of Conjecture 4.3.6 in dimension 4, see [**KMM**, Theorem 5-1-15], [**F5**], [**F7**], [**F8**], [**AHK**], and [**Bir1**].

4.4. BCHM and some related results

In this section, we quickly review the main results of [**HaMc1**], [**HaMc2**], and [**BCHM**] for the reader's convenience. We also discuss some related results. We closely follow the presentation of [**HaKo**, 5.D].

Roughly speaking, [**BCHM**] established:

THEOREM 4.4.1 (Minimal model program). *Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and let (X, Δ) be a $\mathbb{Q}$-factorial klt pair such that Δ is π-big. Then there exists a finite sequence of flips and divisorial contractions for the $(K_X + \Delta)$-minimal model program over S:*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N$$

such that either $K_{X_N} + \Delta_N$ is nef over S or there exists a morphism $X_N \to Z$ which is a $(K_{X_N} + \Delta_N)$-Mori fiber space over S.

From now on, let us explain the results in [**BCHM**] more details.

DEFINITION 4.4.2 (Pl-flipping contraction). Let (X, Δ) be a plt pair with $S = \lfloor \Delta \rfloor$. A *pl-flipping contraction* is a flipping contraction $\varphi : X \to W$, that is, φ is small, $\varphi_* \mathcal{O}_X \simeq \mathcal{O}_W$, $\rho(X/W) = 1$, and $-(K_X + \Delta)$ is φ-ample, such that Δ is a $\mathbb{Q}$-divisor, S is irreducible, and $-S$ is φ-ample.

For the definition of pl-flipping contractions, see also [**F13**, Definition 4.3.1 and Caution 4.3.2]. Note that the notion of pl-flipping contractions and pl-flips is due to Shokurov (see [**Sh2**]).

THEOREM 4.4.3 (Existence of pl-flips). *Let (X, Δ) be a plt pair and let $\varphi : X \to W$ be a pl-flipping contraction. Then the flip*

$$\varphi^+ : X^+ \to W$$

of φ exists.

In this section, we adopt the following definition of minimal models, which is slightly different from Definition 4.3.1.

DEFINITION 4.4.4 (Minimal models). Let (X, Δ) be a dlt pair and let $\pi : X \to S$ be a projective morphism onto a variety S. Let $\phi : X \dashrightarrow Y$ be a rational map over S such that

(i) ϕ^{-1} contracts no divisors,
(ii) Y is $\mathbb{Q}$-factorial,
(iii) $K_Y + \phi_* \Delta$ is nef over S, and
(iv) $a(E, X, \Delta) < a(E, Y, \phi_* \Delta)$ for every ϕ-exceptional divisor E on X.

Then $(Y, \phi_*\Delta)$ is called a *minimal model of* (X, Δ) *over* S. Furthermore, if $K_Y + \phi_*\Delta$ is semi-ample over S, then $(Y, \phi_*\Delta)$ is called a *good minimal model of* (X, Δ) *over* S.

It is obvious that a minimal model in the sense of Definition 4.4.4 is a minimal model in the sense of Definition 4.3.1.

THEOREM 4.4.5 (Existence of minimal models). *Let* $\pi : X \to S$ *be a projective morphism of normal quasi-projective varieties. Let* (X, Δ) *be a klt pair and let* D *be an effective* $\mathbb{R}$*-divisor on* X *such that* Δ *is* π*-big and* $K_X + \Delta \sim_{\mathbb{R},\pi} D$. *Then there exists a minimal model of* (X, Δ) *over* S.

THEOREM 4.4.6 (Non-vanishing theorem). *Let* $\pi : X \to S$ *be a projective morphism of normal quasi-projective varieties. Let* (X, Δ) *be a klt pair such that* Δ *is* π*-big. If* $K_X + \Delta$ *is pseudo-effective over* S*, then there exists an effective* $\mathbb{R}$*-divisor* D *on* X *such that* $K_X + \Delta \sim_{\mathbb{R},\pi} D$.

DEFINITION 4.4.7. On a normal variety X, the group of Weil divisors with rational coefficients $\mathrm{Weil}(X)_{\mathbb{Q}}$, or with real coefficients $\mathrm{Weil}(X)_{\mathbb{R}}$, forms a vector space, with a canonical basis given by the prime divisors. Let D be an $\mathbb{R}$-divisor on X. Then $\|D\|$ denotes the sup norm with respect to this basis.

THEOREM 4.4.8 (Finiteness of marked minimal models). *Let* $\pi : X \to S$ *be a projective morphism of normal quasi-projective varieties. Let* $\mathcal{C} \subset \mathrm{Weil}(X)_{\mathbb{R}}$ *be a rational polytope such that for every* $K_X + \Delta \in \mathcal{C}$, Δ *is* π*-big, and* (X, Δ) *is klt. Then there exist finitely many birational maps* $\phi_i : X \dashrightarrow Y_i$ *over* S *with* $1 \le i \le k$ *such that if* $K_X + \Delta \in \mathcal{C}$ *and* $K_X + \Delta$ *is pseudo-effective over* S*, then*

(i) *There exists an index* $1 \le j \le k$ *such that* $\phi_j : X \dashrightarrow Y_j$ *is a minimal model of* (X, Δ) *over* S.
(ii) *If* $\phi : X \dashrightarrow Y$ *is a minimal model of* (X, Δ) *over* S*, then there exists an index* $1 \le j \le k$ *such that the rational map* $\phi_j \circ \phi^{-1} : Y \dashrightarrow Y_j$ *is an isomorphism.*

We need the notion of stable base locus and stable augmented base locus.

DEFINITION 4.4.9 (Stable base locus and stable augmented base locus). Let $\pi : X \to S$ be a morphism from a normal variety X onto a variety S. The *real linear system* over S associated to an $\mathbb{R}$-divisor D on X is

$$|D/S|_{\mathbb{R}} = \{D' \ge 0 \,|\, D' \sim_{\mathbb{R},\pi} D\}.$$

We can define

$$|D/S|_{\mathbb{Q}} = \{D' \geq 0 \,|\, D' \sim_{\mathbb{Q},\pi} D\}$$

similarly. The *stable base locus* of D over S is the Zariski closed subset

$$\mathbf{B}(D/S) = \bigcap_{D' \in |D/S|_{\mathbb{R}}} \operatorname{Supp} D'.$$

If $|D/S|_{\mathbb{R}} = \emptyset$, then we put $\mathbf{B}(D/S) = X$. When D is $\mathbb{Q}$-Cartier, $\mathbf{B}(D/S)$ is the usual stable base locus (see [BCHM, Lemma 3.5.3]). When S is affine, we sometimes simply use $\mathbf{B}(D)$ to denote $\mathbf{B}(D/S)$.

The *stable augmented base locus* of D over S is the Zariski closed set

$$\mathbf{B}_{+}(D/S) = \mathbf{B}((D - \varepsilon A)/S)$$

for any π-ample $\mathbb{R}$-divisor A and any sufficiently small rational number $\varepsilon > 0$.

Let Λ be a non-empty linear system on X. Then the *fixed divisor* $\operatorname{Fix} \Lambda$ is the largest effective divisor F on X such that $D \geq F$ for all $D \in \Lambda$.

THEOREM 4.4.10 (Zariski decomposition). *Let $\pi : X \to S$ be a projective morphisim to a normal affine variety S. Let (X, Δ) be a klt pair where $K_X + \Delta$ is pseudo-effective over S, $\Delta = A + B$, A is an ample effective $\mathbb{Q}$-divisor, and B is an effective $\mathbb{R}$-divisor. Then we have the following properties.*

(i) *(X, Δ) has a minimal model $\phi : X \dashrightarrow Y$ over S. In particular, if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely generated.

(ii) *Let $V \subset \operatorname{Weil}(X)_{\mathbb{R}}$ be a finite dimensional affine subspace of $\operatorname{Weil}(X)_{\mathbb{R}}$ containing Δ which is defined over $\mathbb{Q}$. Then there exists a constant $\delta > 0$ such that if P is a prime divisor contained in $\mathbf{B}(K_X + \Delta)$, then P is contained in $\mathbf{B}(K_X + \Delta')$ for any $\mathbb{R}$-divisor $\Delta' \in V$ with $\|\Delta - \Delta\| \leq \delta$.*

(iii) *Let $W \subset \operatorname{Weil}(X)_{\mathbb{R}}$ be the smallest affine subspace containing Δ which is defined over $\mathbb{Q}$. Then there exist a real number $\eta > 0$ and a positive integer r such that if $\Delta' \in W$, $\|\Delta - \Delta'\| \leq \eta$ and k is a positive integer such that $k(K_X + \Delta')/r$ is Cartier, then $|k(K_X+\Delta')| \neq \emptyset$ and every component of $\operatorname{Fix} |k(K_X+\Delta')|$ is a component of $\mathbf{B}(K_X + \Delta)$.*

Let us explain the minimal model program with scaling.

4.4.11 (Minimal model program with scaling). Let $(X, \Delta + C)$ be a log canonical pair and let $\pi : X \to S$ be a projective morphism onto a variety S such that $K_X + \Delta + C$ is π-nef, Δ is an effective $\mathbb{R}$-divisor, and C is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We put

$$(X_0, \Delta_0 + C_0) = (X, \Delta + C).$$

Assume that $K_{X_0} + \Delta_0$ is nef over S or there exists a $(K_{X_0} + \Delta_0)$-negative extremal ray R_0 over S such that $(K_{X_0} + \Delta_0 + \lambda_0 C_0) \cdot R_0 = 0$ where

$$\lambda_0 = \inf\{t \geq 0 \,|\, K_{X_0} + \Delta_0 + tC_0 \text{ is nef over } S\}.$$

If $K_{X_0} + \Delta_0$ is nef over S or if R_0 defines a Mori fiber space structure over S, then we stop. Otherwise, we assume that R_0 gives a divisorial contraction $X_0 \to X_1$ over S or a flip $X_0 \dashrightarrow X_1$ over S. We can consider $(X_1, \Delta_1 + \lambda_0 C_1)$ where $\Delta_1 + \lambda_0 C_1$ is the strict transform of $\Delta_0 + \lambda_0 C_0$. Assume that $K_{X_1} + \Delta_1$ is nef over S or there exists a $(K_{X_1} + \Delta_1)$-negative extremal ray R_1 such that $(K_{X_1} + \Delta_1 + \lambda_1 C_1) \cdot R_1 = 0$ where

$$\lambda_1 = \inf\{t \geq 0 \,|\, K_{X_1} + \Delta_1 + tC_1 \text{ is nef over } S\}.$$

By repeating this process, we obtain a sequence of positive real numbers λ_i and a special kind of the minimal model program over S:

$$(X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i) \dashrightarrow \cdots,$$

which is called the *minimal model program over S on $K_X + \Delta$ with scaling of C*. We note that $\lambda_i \geq \lambda_{i+1}$ for every i.

In [**KoMo**, Section 7.4], it was called a minimal model program over S *guided with C*.

Theorem 4.4.12 (Termination of flips with scaling). *We use the same notation as in 4.4.11. We assume that $(X, \Delta + C)$ is a $\mathbb{Q}$-factorial klt pair, S is quasi-projective, and Δ is π-big. Then we can run the minimal model program with respect to $K_X + \Delta$ over S with scaling of C. Moreover, any sequence of flips and divisorial contractions for the $(K_X + \Delta)$-minimal model program over S with scaling of C is finite.*

Remark 4.4.13 follows from the argument in [**BCHM**, Remark 3.10.9].

Remark 4.4.13. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair and let $\pi : X \to S$ be a projective morphism between quasi-projective varieties. Let C be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(X, \Delta + C)$ is log canonical, $\mathbf{B}_+(C/S)$ contains no log canonical centers of (X, Δ), and $K_X + \Delta + C$ is nef over S. Then we can run the minimal model program with respect to $K_X + \Delta$ over S with scaling of C. Note that the termination of this minimal model program is still an open problem. However, it is useful for some applications.

4.4.14 (Finite generation of log canonical rings). By combining [FM, Theorem 5.2] with Theorem 4.4.5, we have:

THEOREM 4.4.15. *Let (X, Δ) be a projective klt pair such that Δ is a $\mathbb{Q}$-divisor on X. Then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

As a corollary of Theorem 4.4.15, we obtain:

COROLLARY 4.4.16. *Let X be a smooth projective variety. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

In [F38], the author obtained the following generalizations of Theorem 4.4.15 and Corollary 4.4.16.

THEOREM 4.4.17. *Let X be a complex analytic variety in Fujiki's class $\mathcal{C}$. Let (X, Δ) be a klt pair such that Δ is a $\mathbb{Q}$-divisor on X. Then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

As a special case of Theorem 4.4.17, we obtain:

COROLLARY 4.4.18 ([F38, Theorem 5.1]). *Let X be a compact Kähler manifold, or more generally, let X be a complex manifold in Fujiki's class $\mathcal{C}$. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \omega_X^{\otimes m})$$

is a finitely generated $\mathbb{C}$-algebra.

REMARK 4.4.19 ([F38, Corollary 5.2]). In [W], Wilson constructed a compact complex manifold which is not Kähler whose canonical ring is not a finitely generated $\mathbb{C}$-algebra. For the details, see [F38, Section 6].

4.4.20 (Dlt blow-ups). Let us recall a very important application of the minimal model program with scaling. Theorem 4.4.21 is originally due to Hacon.

THEOREM 4.4.21 (Dlt blow-ups). *Let X be a normal quasi-projective variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. In this case, we can construct a projective birational morphism $f: Y \to X$ from a normal quasi-projective variety Y with the following properties.*

(i) *Y is $\mathbb{Q}$-factorial.*
(ii) *$a(E, X, \Delta) \le -1$ for every f-exceptional divisor E on Y.*
(iii) *We put*
$$\Delta_Y = f_*^{-1}\Delta + \sum_{E: f\text{-exceptional}} E.$$
Then (Y, Δ_Y) is dlt and
$$K_Y + \Delta_Y = f^*(K_X+\Delta) + \sum_{a(E,X,\Delta)<-1} (a(E,X,\Delta)+1)E.$$
In particular, if (X, Δ) is log canonical, then
$$K_Y+\Delta_Y = f^*(K_X+\Delta).$$
Moreover, if (X,Δ) is dlt, then we can make f small, that is, f is an isomorphism in codimension one.

We closely follow the argument in [**F26**]. For the proof of Theorem 4.4.21, see also [**F28**, Section 10].

PROOF. Let $g: Z \to X$ be a resolution such that $\mathrm{Exc}(g) \cup \mathrm{Supp}\, g_*^{-1}\Delta$ is a simple normal crossing divisor on X and g is projective. We write
$$K_Z + \Delta_Z = g^*(K_X+\Delta) + F$$
where
$$\Delta_Z = g_*^{-1}\Delta + \sum_{E:\ g\text{-exceptional}} E.$$
Let C be a g-ample effective $\mathbb{Q}$-divisor on Z such that $(Z, \Delta_Z + C)$ is dlt and that $K_Z+\Delta_Z+C$ is g-nef. We run the minimal model program with respect to $K_Z+\Delta_Z$ over X with scaling of C (see Remark 4.4.13). We obtain a sequence of divisorial contractions and flips
$$(Z, \Delta_Z) = (Z_0, \Delta_{Z_0}) \dashrightarrow (Z_1, \Delta_{Z_1}) \dashrightarrow \cdots \dashrightarrow (Z_k, \Delta_{Z_k}) \dashrightarrow \cdots$$
over X. We note that
$$\lambda_i = \inf\{t \in \mathbb{R} \,|\, K_{Z_i} + \Delta_{Z_i} + tC_i \text{ is nef over } X\},$$

where C_i (resp. Δ_{Z_i}) is the pushforward of C (resp. Δ_Z) on Z_i for every i. By definition, $0 \leq \lambda_i \leq 1$, $\lambda_i \in \mathbb{R}$ for every i and

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_k \geq \cdots .$$

Let F_i be the pushforward of F on Z_i for every i. It is sufficient to prove:

Claim. *There is i_0 such that $-F_{i_0}$ is effective.*

Proof of Claim. If we prove that the above minimal model program terminates after finitely many steps, then there is i_0 such that F_{i_0} is nef over X. Since F_{i_0} is exceptional over X, $-F_{i_0}$ is effective by the negativity lemma (see Lemma 2.3.26). Therefore, we may assume that the above minimal model program does not terminate. We put

$$\lambda = \lim_{i \to \infty} \lambda_i.$$

Case 1 ($\lambda > 0$). *In this case, we can see that the above minimal model program is a minimal model program with respect to $(K_Z + \Delta_Z + \frac{1}{2}\lambda C)$ over X with scaling of $(1 - \frac{1}{2}\lambda)C$. By assumption, we can write*

$$\Delta_Z + \frac{1}{2}\lambda C \sim_{\mathbb{R},\pi} B$$

such that (Z, B) and $(Z, B + (1 - \frac{1}{2}\lambda)C)$ are klt. Therefore, it is a minimal model program with respect to $K_Z + B$ over X with scaling of $(1 - \frac{1}{2}\lambda)C$. This contradicts Theorem 4.4.12.

Case 2 ($\lambda = 0$). *After finitely many steps, every step of the above minimal model program is flip. Therefore, without loss of generality, we may assume that all the steps are flips. Let G_i be a relative ample $\mathbb{Q}$-divisor on Z_i such that $G_{iZ} \to 0$ in $N^1(Z/X)$ for $i \to \infty$ where G_{iZ} is the strict transform of G_i on Z. We note that*

$$K_{Z_i} + \Delta_{Z_i} + \lambda_i C_i + G_i$$

is ample over X for every i. Therefore, the strict transform

$$K_Z + \Delta_Z + \lambda_i C + G_{iZ}$$

is movable on Z for every i. Thus $K_Z + \Delta_Z$ is a limit of movable $\mathbb{R}$-divisors in $N^1(Z/X)$. So $K_Z + \Delta_Z \in \overline{\mathrm{Mov}}(Z/X)$. Note that $K_Z + \Delta_Z \sim_{\mathbb{R},g} F$ and F is g-exceptional. By Lemma 2.4.4, $-F$ is effective.

Anyway, there is i_0 such that $-F_{i_0}$ is effective. □

We put $(Y, \Delta_Y) = (Z_{i_0}, \Delta_{Z_{i_0}})$. Then this is a desired model. When (X, Δ) is dlt, we can make $a(E, X, \Delta) > -1$ for every g-exceptional divisor by the definition of dlt pairs. In this case, $f : Y \to X$ is automatically small by the above construction. □

Remark 4.4.22. It is conjectured that every minimal model program terminates. We can easily see that the minimal model program in the proof of Theorem 4.4.21 always terminates when (X, Δ) is log canonical. Note that $F_{i_0} = 0$ since F_i is always effective for every i. Therefore, $K_Y + \Delta_Y = f^*(K_X + \Delta)$ holds and is obviously f-nef when (X, Δ) is log canonical.

4.4.23 (Infinitely many marked minimal models). The following example is due to Gongyo (see [G1]). For related examples, see Example 4.5.12 and Example 4.5.9 below.

Example 4.4.24 (Infinitely many marked minimal models). There exists a three-dimensional projective plt pair (X, Δ) with the following properties:

(i) $K_X + \Delta$ is nef and big, and
(ii) there are infinitely many $(K_X + \Delta)$-flops.

Here we construct an example explicitly. We take a $K3$ surface S which contains infinitely many (-2)-curves. We take a projectively normal embedding $S \subset \mathbb{P}^N$. Let $Z \subset \mathbb{P}^{N+1}$ be a cone over $S \subset \mathbb{P}^N$ and let $\varphi : X \to Z$ be the blow-up at the vertex P of the cone Z. Then the projection $Z \dashrightarrow S$ from the vertex P induces a natural $\mathbb{P}^1$-bundle structure $p : X \to S$. Let E be the φ-exceptional divisor on X. Then E is a section of p. In particular, $E \simeq S$. Note that

$$K_X + E = \varphi^* K_Z.$$

We take a sufficiently ample smooth Cartier divisor H on Z which does not pass through P. We further assume that $K_Z + H$ is ample. We put $\Delta = E + \varphi^* H$ and consider the pair (X, Δ). By construction, (X, Δ) is a plt threefold such that X is smooth and that $K_X + \Delta$ is big and semi-ample. Since $p : X \to S$ is a $\mathbb{P}^1$-bundle and E is a section of p, we have

$$N_1(X) = N_1(E) \oplus \mathbb{R}[l]$$

where $l \simeq \mathbb{P}^1$ is a fiber of p. Therefore, it is easy to see that

$$\overline{NE}(E) \subset \overline{NE}(X) \cap (\varphi^* H = 0).$$

Claim. *Let C be a (-2)-curve on E. Then $\mathbb{R}_{\geq 0}[C]$ is an extremal ray of $\overline{NE}(X)$ such that $C \cdot (K_X + \Delta) = 0$.*

Proof of Claim. Since $C^2 = -2 < 0$, $\mathbb{R}_{\geq 0}[C]$ is an extremal ray of $\overline{NE}(E)$. Let L be a supporting Cartier divisor of the extremal ray $\mathbb{R}_{\geq 0}[C] \subset \overline{NE}(E)$, that is,

$$\overline{NE}(E) \cap (L = 0) = \mathbb{R}_{\geq 0}[C].$$

We can see that L is a Cartier divisor on S since $S \simeq E$. Then

$$(\varphi^* H + p^* L = 0) \cap \overline{NE}(X) = \mathbb{R}_{\geq 0}[C].$$

Note that

$$(K_X + \Delta) \cdot C = K_E \cdot C = 0.$$

Thus $\mathbb{R}_{\geq 0}[C]$ is an extremal ray of $\overline{NE}(X)$ with the desired intersection number. $\square$

We put $D = p^*(p(C))$. Note that $(X, \Delta + \delta D)$ is plt for a small positive rational number δ. Then $\mathbb{R}_{\geq 0}[C]$ is a $(K_X + \Delta + \delta D)$-negative extremal ray. Therefore, we obtain a $(K_X + \Delta + \delta D)$-flip

$$\begin{tikzcd} (X, \Delta + \delta D) \arrow[rr, dashed] \arrow[dr] & & (X^+, \Delta^+ + \delta D^+) \arrow[dl] \\ & W & \end{tikzcd}$$

which is a $(K_X + \Delta)$-flop associated to the extremal ray $\mathbb{R}_{\geq 0}[C]$. Since there are infinitely many (-2)-curves on S, we obtain infinitely many $(K_X + \Delta)$-flops.

4.5. Fundamental theorems for normal pairs

In this section, we explain the main result of [**F28**] and some related results and examples.

First, let us introduce the notion of normal pairs.

Definition 4.5.1 (Normal pairs). Let X be a normal algebraic variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We call the pair (X, Δ) a *normal pair*.

Next, we recall the main result of [**F28**], which covers the main result of [**Am1**] (see [**Am1**, Theorem 2]).

Theorem 4.5.2. *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier, and let $\pi : X \to S$ be a projective morphism onto a variety S. Then we have*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+\Delta \geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} + \sum R_j$$

with the following properties.

(1) $\mathrm{Nlc}(X, \Delta)$ *is the non-lc locus of* (X, Δ) *and*

$$\overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \mathrm{Im}(\overline{NE}(\mathrm{Nlc}(X, \Delta)/S) \to \overline{NE}(X/S)).$$

(2) R_j *is a* $(K_X + \Delta)$*-negative extremal ray of* $\overline{NE}(X/S)$ *such that* $R_j \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \{0\}$ *for every* j.

(3) *Let A be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(K_X + \Delta + A)_{<0}$. In particular, the R_j's are discrete in the half-space $(K_X + \Delta)_{<0}$.*

(4) *Let F be a face of $\overline{NE}(X/S)$ such that*

$$F \cap (\overline{NE}(X/S)_{K_X+\Delta\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)}) = \{0\}.$$

Then there exists a contraction morphism $\varphi_F : X \to Y$ over S.

(i) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*

(ii) *$\mathcal{O}_Y \simeq (\varphi_F)_*\mathcal{O}_X$.*

(iii) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Y on Y such that $L \simeq \varphi_F^* L_Y$.*

(5) *Every $(K_X + \Delta)$-negative extremal ray R with*

$$R \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \{0\}$$

is spanned by a rational curve C with $0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$.

From now on, we further assume that (X, Δ) is log canonical, that is, $\mathrm{Nlc}(X, \Delta) = \emptyset$. Then we have the following properties.

(6) *Let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $K_X + \Delta + H$ is π-nef and $(X, \Delta + H)$ is log canonical. Then, either $K_X + \Delta$ is also π-nef or there is a $(K_X + \Delta)$-negative extremal ray R such that $(K_X + \Delta + \lambda H) \cdot R = 0$ where*

$$\lambda := \inf\{t \geq 0 \mid K_X + \Delta + tH \textit{ is } \pi\textit{-nef}\}.$$

Of course, $K_X + \Delta + \lambda H$ is π-nef.

In [**Am1**], Ambro proved the properties (1), (2), (3), and (4) in Theorem 4.5.2 by using the theory of quasi-log schemes. More precisely, they are the main results of [**Am1**]. In [**F28**], the author obtained Theorem 4.5.2 without using the theory of quasi-log schemes. Our approach in [**F28**] is much simpler than Ambro's in [**Am1**]. For (5), see Theorem 4.6.7. For (6), see Theorem 4.7.3.

Let us include the following easy corollaries for the reader's convenience.

Corollary 4.5.3 (cf. [**KoMo**, Corollary 3.17]). *Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a projective morphism. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/S)$ with contraction* *morphism $\varphi_R : X \to Y$. Let C be a curve on X which generates R. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(Y) \xrightarrow{L\mapsto \varphi_R^*L} \mathrm{Pic}(X) \xrightarrow{M\mapsto (M\cdot C)} \mathbb{Z}.$$

In particular, we have $\rho(Y/S) = \rho(X/S) - 1$.

PROOF. Let L be a line bundle on Y. Then $(\varphi_R)_*(\varphi_R^*L) = L$. Therefore, $L \mapsto \varphi_R^*L$ is an injection. Note that M is a line bundle on X with $(M \cdot C) = 0$ if and only if $M = \varphi_R^*L$ for some L by Theorem 4.5.2 (4). $\square$

COROLLARY 4.5.4 (cf. [**KoMo**, Corollary 3.18]). *Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a projective morphism. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/S)$ with contraction morphism $\varphi_R : X \to Y$. Assume that X is $\mathbb{Q}$-factorial and that φ_R is either a divisorial or a Fano contraction. Then Y is also $\mathbb{Q}$-factorial.*

PROOF. First, we assume that φ_R is divisorial. Let E be the exceptional divisor on X. Then it is easy to see that $(E \cdot R) < 0$ and that E is irreducible. Let D be a Weil divisor on Y. Then there is a rational number s such that

$$((\varphi_R)_*^{-1}D + sE \cdot R) = 0.$$

We take a positive integer m such that $m((\varphi_R)_*^{-1}D + sE)$ is a Cartier divisor on X. Then, by Theorem 4.5.2 (4), it is the pull-back of a Cartier divisor D_Y on Y. Thus, $mD \sim D_Y$. This implies that D is $\mathbb{Q}$-Cartier.

Next, we assume that φ_R is a Fano contraction. Let D be a Weil divisor on Y. Let Y^0 be the smooth locus of Y. Let D_X be the closure of $(\varphi_R|_{\varphi_R^{-1}(Y^0)})^*(D|_{Y^0})$. Then D_X is disjoint from the general fiber of φ_R. Thus $(D_X \cdot R) = 0$. We take a positive integer m such that mD_X is a Cartier divisor on X. Thus, by Theorem 4.5.2 (4), $mD_X \sim \varphi_R^*D_Y$ for some Cartier divisor D_Y on Y. Thus, $mD \sim D_Y$. This implies that D is $\mathbb{Q}$-Cartier. $\square$

Let us include the basepoint-free theorem for normal pairs in [**F28**] without proof for the reader's convenience. Note that Theorem 4.5.5 is a special case of Theorem 6.5.1 below.

THEOREM 4.5.5 (see [**F28**, Theorem 13.1]). *Let (X, Δ) be a normal pair and let $\pi : X \to S$ be a projective morphism onto a variety S, and let L be a π-nef Cartier divisor on X. Assume that*

(i) *$aL - (K_X + \Delta)$ is π-ample for some real number $a > 0$, and*
(ii) *$\mathcal{O}_{\mathrm{Nlc}(X,\Delta)}(mL)$ is $\pi|_{\mathrm{Nlc}(X,\Delta)}$-generated for every $m \gg 0$.*

Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.

As an easy consequence of Theorem 4.5.5, we have:

COROLLARY 4.5.6. *Let (X, Δ) be a projective normal pair and let L be a nef Cartier divisor on X such that $aL-(K_X+\Delta)$ is nef and big for some real number $a > 0$. Assume that $\mathcal{O}_{\mathrm{Nklt}(X,\Delta)}(mL)$ is generated by global sections for every $m \gg 0$. Then $\mathcal{O}_X(mL)$ is generated by global sections for every $m \gg 0$.*

PROOF. By Kodaira's lemma (see Lemma 2.1.18), we can write

$$aL-(K_X+\Delta) \sim_{\mathbb{R}} A+E$$

where A is an ample $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let ε be a small positive number. Then

$$\mathrm{Nklt}(X,\Delta) = \mathrm{Nklt}(X,\Delta+\varepsilon E)$$

scheme theoretically and $aL-(K_X+\Delta+\varepsilon E)$ is ample. By replacing Δ with $\Delta+\varepsilon E$, we may assume that $aL-(K_X+\Delta)$ is ample. Since there is a natural surjective morphism $\mathcal{O}_{\mathrm{Nklt}(X,\Delta)} \to \mathcal{O}_{\mathrm{Nqlc}(X,\Delta)}$, $\mathcal{O}_{\mathrm{Nqlc}(X,\Delta)}(mL)$ is generated by global sections for every $m \gg 0$. Therefore, by Theorem 4.5.5, $\mathcal{O}_X(mL)$ is generated by global sections for every $m \gg 0$. $\square$

Note that it is well known that Corollary 4.5.6 can be proved by the usual X-method (see Section 4.2) with the aid of the Nadel vanishing theorem (see Theorem 3.4.2).

4.5.7 (Examples of the Kleiman–Mori cone). From now on, we discuss various examples of the Kleiman–Mori cone. The following example is well known (see, for example, [KMM, Example 4-2-4]).

EXAMPLE 4.5.8. We take two smooth elliptic curves E_1 and E_2 on $\mathbb{P}^2$ such that $P_1 - P_2$ is not of finite order on the abelian group E_1, where P_1 and P_2 are two of the nine intersection points of E_1 and E_2. Let f_1 and f_2 be the defining equations of E_1 and E_2 respectively. The rational map which maps $x \in \mathbb{P}^2 \setminus (E_1 \cap E_2)$ to $(f_1(x) : f_2(x)) \in \mathbb{P}^1$ becomes a morphism from S which is obtained by taking blow-ups of $\mathbb{P}^2$ at the nine intersection points of E_1 and E_2. Then it is easy to see that the inverse images of P_1 and P_2 on S are sections of $\pi : S \to \mathbb{P}^1$. By the choice of P_1 and P_2, there are infinitely many sections of π, which are (-1)-curves. Therefore, $\overline{NE}(S)$ has infinitely many K_S-negative extremal rays.

Example 4.5.9, which is essentially the same as [G2, Example 5.6], is an answer to [KMM, Problem 4-2-5]. Although the construction is essentially the same as that of Example 4.4.24, we explain the details of the construction for the reader's convenience.

EXAMPLE 4.5.9 (Infinitely many flipping contractions). There exists a three-dimensional projective plt pair (X, Δ) with the following properties:

(i) $K_X + \Delta$ is big, and
(ii) there are infinitely many $(K_X + \Delta)$-negative extremal rays.

Here we construct an example explicitly. Let S be a rational elliptic surface with infinitely many (-1)-curves constructed in Example 4.5.8. We take a projectively normal embedding $S \subset \mathbb{P}^N$. Let $Z \subset \mathbb{P}^{N+1}$ be a cone over $S \subset \mathbb{P}^N$ and let $\varphi : X \to Z$ be the blow-up at the vertex P of the cone Z. Then the projection $Z \dashrightarrow S$ from the vertex P induces a natural $\mathbb{P}^1$-bundle structure $p : X \to S$. Let E be the φ-exceptional divisor on X. Then E is a section of p. In particular, $E \simeq S$. We take a sufficiently ample smooth divisor H on Z which does not pass through P. We put $\Delta = E + \varphi^* H$ and consider the pair (X, Δ). By the construction, (X, Δ) is a plt threefold such that X is smooth and that $K_X + \Delta$ is big. Since $p : X \to S$ is a $\mathbb{P}^1$-bundle and E is a section of p, we have

$$N_1(X) = N_1(E) \oplus \mathbb{R}[l]$$

where $l \simeq \mathbb{P}^1$ is a fiber of p. Therefore, it is easy to see that

$$\overline{NE}(E) \subset \overline{NE}(X) \cap (\varphi^* H = 0).$$

CLAIM. *Let C be a (-1)-curve on E. Then $\mathbb{R}_{\geq 0}[C]$ is a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X)$.*

PROOF OF CLAIM. Note that $\mathbb{R}_{\geq 0}[C]$ is a K_E-negative extremal ray of $\overline{NE}(E)$. Let L be a supporting Cartier divisor of $\mathbb{R}_{\geq 0}[C] \subset \overline{NE}(E)$. We can see that L is a Cartier divisor on S since $S \simeq E$. Then

$$(\varphi^* H + p^* L = 0) \cap \overline{NE}(X) = \mathbb{R}_{\geq 0}[C].$$

Note that

$$(K_X + \Delta) \cdot C = K_E \cdot C = -1.$$

Thus $\mathbb{R}_{\geq 0}[C]$ is a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X)$. $\square$

Therefore, there are infinitely many $(K_X + \Delta)$-negative extremal rays of $\overline{NE}(X)$. Note that every extremal ray corresponds to a flipping contraction with respect to $K_X + \Delta$.

REMARK 4.5.10. Let (X, Δ) be a projective klt pair. Assume that $K_X + \Delta$ is big. Then there are only finitely many $(K_X + \Delta)$-negative extremal rays. We can check this well-known result as follows. By Kodaira's lemma (see Lemma 2.1.18), we can write

$$K_X + \Delta \sim_{\mathbb{R}} A + E$$

where A is an ample $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let ε be a small rational number such that $(X, \Delta + \varepsilon E)$ is klt. In this case, $(K_X + \Delta)$-negative extremal ray is nothing but $(K_X + \Delta + \varepsilon E + \varepsilon A)$-negative extremal ray. By Theorem 4.5.2 (3), there are only finitely many $(K_X + \Delta)$-negative extremal rays.

LEMMA 4.5.11. *Let (X, Δ) be a $\mathbb{Q}$-factorial projective log canonical pair such that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. Then there are only finitely many $(K_X + \Delta)$-negative extremal rays inducing divisorial contractions. In particular, if X is a smooth projective threefold with $\kappa(X, K_X) \geq 0$, then there are only finitely many K_X-negative extremal rays.*

PROOF. Let R be a $(K_X + \Delta)$-negative extremal ray such that the associated contraction $\varphi_R : X \to Y$ is divisorial. Then the exceptional locus of φ_R is a prime divisor E on X which is an irreducible component of $\operatorname{Supp} D$. Therefore, there are only finitely many $(K_X + \Delta)$-negative divisorial contractions. When X is a smooth projective threefold with $\kappa(X, K_X) \geq 0$, the contraction morphism $\varphi_R : X \to Y$ is divisorial for every K_X-negative extremal ray R by [Mo2] (see Theorem 1.1.4). Therefore, there are only finitely many K_X-negative extremal rays for a smooth projective threefold X with $\kappa(X, K_X) \geq 0$. $\square$

Yoshinori Gongyo and Yoshinori Namikawa informed the author of the following example. It is well known as Schoen's Calabi–Yau threefold and is an answer to [KMM, Problem 4-2-5].

EXAMPLE 4.5.12. Let $\pi_1 : S_1 \to \mathbb{P}^1$ and $\pi_2 : S_2 \to \mathbb{P}^1$ be rational elliptic surfaces with infinitely many (-1)-curves constructed in Example 4.5.8. We put $X = S_1 \times_{\mathbb{P}^1} S_2$.

$$\begin{tikzcd} & S_1 \times_{\mathbb{P}^1} S_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S_1 \arrow[dr, "\pi_1"'] & & S_2 \arrow[dl, "\pi_2"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

We assume that $\pi_1^{-1}(p)$ or $\pi_2^{-1}(p)$ is smooth for every point $p \in \mathbb{P}^1$. Then it is easy to see that X is a smooth projective threefold with $K_X \sim 0$ by using the canonical bundle formula for rational elliptic surfaces (see [Scho] and [BHPV, Chapter V. (12.3) Corollary]). We can directly check $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Therefore, X is a Calabi–Yau threefold. Let l be a (-1)-curve on S_1 and let $\{m_\lambda\}_{\lambda \in \Lambda}$ be the set of all (-1)-curves on S_2. Then $C_\lambda = l \times_{\mathbb{P}^1} m_\lambda$ is a $(-1,-1)$-curve, that is, a rational curve whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, on X for every $\lambda \in \Lambda$. We take a semi-ample Cartier divisor H on S_1 which is a supporting Cartier divisor of $\mathbb{R}_{\geq 0}[l] \subset \overline{NE}(S_1)$. Let H_λ be a semi-ample Cartier divisor on S_2 which is a supporting Cartier divisor of $\mathbb{R}_{\geq 0}[m_\lambda] \subset \overline{NE}(S_2)$ for every $\lambda \in \Lambda$. Then $p_1^*H + p_2^*H_\lambda$ induces a contraction morphism $\varphi_\lambda : X \to W_\lambda$ such that $\mathrm{Exc}(\varphi_\lambda) = C_\lambda$ for every $\lambda \in \Lambda$. Therefore, $\mathbb{R}_{\geq 0}[C_\lambda]$ is an extremal ray of $\overline{NE}(X)$. We put $D = l \times_{\mathbb{P}^1} S_2$. Then it is easy to see that $(K_X + \varepsilon D) \cdot C_\lambda = -\varepsilon$ for every $\lambda \in \Lambda$. Therefore, $(X, \varepsilon D)$ is a klt threefold which has infinitely many $(K_X + \varepsilon D)$-negative extremal rays for $0 < \varepsilon \ll 1$. Note that we have the following flopping diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi_\lambda"] \arrow[dr, "\varphi_\lambda"'] & & X_\lambda^+ \arrow[dl, "\varphi_\lambda^+"] \\ & W_\lambda & \end{tikzcd}$$

where X_λ^+ is a smooth projective threefold with $K_{X_\lambda^+} \sim 0$. Although we have infinitely many flops $\phi_\lambda : X \dashrightarrow X_\lambda^+$, Namikawa (see [**Nam**]) proved that there are only finitely many X_λ^+ up to isomorphisms. For the details, see [**Nam**].

4.6. Lengths of extremal rays

In this section, which is essentially a reproduction of [**F28**, Section 18], we discuss estimates of lengths of extremal rays. It is indispensable for the log minimal model program with scaling (see, for example, [**BCHM**]) and the geography of log models (see, for example, [**Sh3**] and [**ShCh**]). The results in this section were obtained in [**Ko4**], [**Ko5**], and [**Ka4**], [**Sh3**], [**Sh5**], and [**Bir2**] with some extra assumptions.

Let us recall the following easy lemma.

LEMMA 4.6.1 (cf. [**Sh5**, Lemma 1]). *Let (X, Δ) be a log canonical pair, where Δ is an $\mathbb{R}$-divisor. Then there are positive real numbers r_i, effective $\mathbb{Q}$-divisors Δ_i for $1 \leq i \leq l$, and a positive integer m such that $\sum_{i=1}^{l} r_i = 1$,*

$$K_X + \Delta = \sum_{i=1}^{l} r_i (K_X + \Delta_i),$$

(X, Δ_i) is log canonical for every i, and $m(K_X + \Delta_i)$ is Cartier for every i.

PROOF. Let $\sum_k D_k$ be the irreducible decomposition of $\operatorname{Supp}\Delta$. We consider the finite dimensional real vector space $V = \bigoplus_k \mathbb{R}D_k$. We put

$$\mathcal{Q} = \{D \in V \mid K_X + D \text{ is } \mathbb{R}\text{-Cartier}\}.$$

Then, it is easy to see that $\mathcal{Q}$ is an affine subspace of V defined over $\mathbb{Q}$. We put

$$\mathcal{L} = \{D \in \mathcal{Q} \mid K_X + D \text{ is log canonical}\}.$$

Thus, by the definition of log canonicity, it is also easy to check that $\mathcal{L}$ is a closed convex rational polytope in V. We note that $\mathcal{L}$ is compact in the classical topology of V. By assumption, $\Delta \in \mathcal{L}$. Therefore, we can find the desired $\mathbb{Q}$-divisors $\Delta_i \in \mathcal{L}$ and positive real numbers r_i. $\square$

The next result is essentially due to [Ka4] and [Sh5, Proposition 1]. We will prove a more general result in Theorem 4.6.7 whose proof depends on Theorem 4.6.2.

THEOREM 4.6.2. *Let X be a normal variety such that (X, Δ) is log canonical and let $\pi : X \to S$ be a projective morphism onto a variety S. Let R be a $(K_X + \Delta)$-negative extremal ray. Then we can find a (possibly singular) rational curve C on X such that $[C] \in R$ and*

$$0 < -(K_X + \Delta) \cdot C \leq 2 \dim X.$$

PROOF. By shrinking S, we may assume that S is quasi–projective. By replacing $\pi : X \to S$ with the extremal contraction $\varphi_R : X \to Y$ over S (see Theorem 4.5.2 (4)), we may assume that the relative Picard number $\rho(X/S) = 1$. In particular, $-(K_X + \Delta)$ is π-ample. Let

$$K_X + \Delta = \sum_{i=1}^{l} r_i(K_X + \Delta_i)$$

be as in Lemma 4.6.1. We assume that $-(K_X + \Delta_1)$ is π-ample and $-(K_X + \Delta_i) = -s_i(K_X + \Delta_1)$ in $N^1(X/S)$ with $s_i \leq 1$ for every $i \geq 2$. Thus, it is sufficient to find a rational curve C such that $\pi(C)$ is a point and that

$$-(K_X + \Delta_1) \cdot C \leq 2 \dim X.$$

So, we may assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and log canonical. By taking a dlt blow-up (see Theorem 4.4.21), there is a birational morphism $f : (Y, \Delta_Y) \to (X, \Delta)$ such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$, Y is $\mathbb{Q}$-factorial, and (Y, Δ_Y) is dlt. By [Ka4, Theorem 1] and [Ma, Theorem 10-2-1] (see also [Deb, Section 7.11]), we can find a rational curve

C' on Y such that

$$-(K_Y + \Delta_Y) \cdot C' \leq 2 \dim Y = 2 \dim X$$

and that C' spans a $(K_Y + \Delta_Y)$-negative extremal ray. By the projection formula, the f-image of C' is a desired rational curve. So, we finish the proof. □

REMARK 4.6.3. It is conjectured that the estimate $\leq 2 \dim X$ in Theorem 4.6.2 should be replaced by $\leq \dim X + 1$. When X is smooth projective, it is true by Mori's famous result (see [Mo2], Theorem 1.1.1, and [KoMo, Theorem 1.13]). When X is a toric variety, it is also true by [F4] and [F10].

REMARK 4.6.4. In the proof of Theorem 4.6.2, we need Kawamata's estimate on the length of an extremal rational curve (see, for example, [Ka4, Theorem 1], [Ma, Theorem 10-2-1], and [Deb, Section 7.11]). It depends on Mori's bend and break technique to create rational curves. So, we need the mod p reduction technique there.

REMARK 4.6.5. Let (X, D) be a log canonical pair such that D is an $\mathbb{R}$-divisor. Let $\phi : X \to Y$ be a projective morphism and let H be a Cartier divisor on X. Assume that $H - (K_X + D)$ is f-ample. By the Kawamata–Viehweg type vanishing theorem for log canonical pairs (see Theorem 5.6.4), $R^q \phi_* \mathcal{O}_X(H) = 0$ for every $q > 0$ if X and Y are *algebraic* varieties. If this vanishing theorem holds for analytic spaces X and Y, then Kawamata's original argument in [Ka4] works directly for log canonical pairs. In that case, we do not need dlt blow-ups (see Theorem 4.4.21), which follows from [BCHM], in the proof of Theorem 4.6.2.

We consider the proof of [Ma, Theorem 10-2-1] when (X, D) is $\mathbb{Q}$-factorial dlt. We need $R^1 \phi_* \mathcal{O}_X(H) = 0$ after shrinking X and Y analytically. In our situation, $(X, D - \varepsilon \lfloor D \rfloor)$ is klt for $0 < \varepsilon \ll 1$. Therefore, $H - (K_X + D - \varepsilon \lfloor D \rfloor)$ is ϕ-ample and $(X, D - \varepsilon \lfloor D \rfloor)$ is klt for $0 < \varepsilon \ll 1$. Thus, we can apply the analytic version of the relative Kawamata–Viehweg vanishing theorem (see, for example, [F31]). So, we do not need the analytic version of the Kawamata–Viehweg type vanishing theorem for *log canonical* pairs.

REMARK 4.6.6. We give a remark on [BCHM]. We use the same notation as in [BCHM, 3.8]. In the proof of [BCHM, Corollary 3.8.2], we may assume that $K_X + \Delta$ is klt by [BCHM, Lemma 3.7.4]. By perturbing the coefficients of B slightly, we can further assume that B is a $\mathbb{Q}$-divisor. By applying the usual cone theorem to the klt pair (X, B), we obtain that there are only finitely many $(K_X + \Delta)$-negative extremal rays of $\overline{NE}(X/U)$. We note that [BCHM, Theorem 3.8.1] is only used in the proof of [BCHM, Corollary 3.8.2]. Therefore, we do not need the estimate of lengths of extremal rays in [BCHM]. In particular, we do not need mod p reduction arguments for the proof of the main results in [BCHM].

The final result in this section is an estimate of lengths of extremal rays which are relatively ample at non-lc loci (see also [Ko4] and [Ko5]).

THEOREM 4.6.7 (Theorem 4.5.2 (5)). *Let X be a normal variety, let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier, and let $\pi : X \to S$ be a projective morphism onto a variety S. Let R be a $(K_X+\Delta)$-negative extremal ray of $\overline{NE}(X/S)$ which is relatively ample at* $\mathrm{Nlc}(X,\Delta)$*, that is, $R \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \{0\}$. Then we can find a (possibly singular) rational curve C on X such that $[C] \in R$ and*

$$0 < -(K_X+\Delta)\cdot C \leq 2\dim X.$$

PROOF. By shrinking S, we may assume that S is quasi-projective. By replacing $\pi : X \to S$ with the extremal contraction $\varphi_R : X \to Y$ over S (see Theorem 4.5.2 (4)), we may assume that the relative Picard number $\rho(X/S) = 1$ and that π is an isomorphism in a neighborhood of $\mathrm{Nlc}(X,\Delta)$. In particular, $-(K_X+\Delta)$ is π-ample. By taking a dlt blow-up (see Theorem 4.4.21), there is a projective birational morphism $f : Y \to X$ such that

(i) $K_Y+\Delta_Y = f^*(K_X+\Delta) + \sum_{a(E,X,\Delta)<-1}(a(E,X,\Delta)+1)E$, where
$$\Delta_Y = f_*^{-1}\Delta + \sum_{E: f\text{-exceptional}} E,$$
(ii) (Y,Δ_Y) is a $\mathbb{Q}$-factorial dlt pair, and
(iii) $K_Y+D = f^*(K_X+\Delta)$ with $D = \Delta_Y + F$, where
$$F = -\sum_{a(E,X,\Delta)<-1}(a(E,X,\Delta)+1)E \geq 0.$$

Therefore, we have

$$f_*(\overline{NE}(Y/S)_{K_Y+D\geq 0}) \subseteq \overline{NE}(X/S)_{K_X+\Delta\geq 0} = \{0\}.$$

We also note that

$$f_*(\overline{NE}(Y/S)_{\mathrm{Nlc}(Y,D)}) = \{0\}.$$

Thus, there is a (K_Y+D)-negative extremal ray R' of $\overline{NE}(Y/S)$ which is relatively ample at $\mathrm{Nlc}(Y,D)$. By Theorem 4.5.2, R' is spanned by a curve $C^\dagger$. Since $-(K_Y+D)\cdot C^\dagger > 0$, we see that $f(C^\dagger)$ is a curve. If

$C^\dagger \subset \operatorname{Supp} F$, then $f(C^\dagger) \subset \operatorname{Nlc}(X, \Delta)$. This is a contradiction because $\pi \circ f(C^\dagger)$ is a point. Thus, $C^\dagger \not\subset \operatorname{Supp} F$. Since

$$-(K_Y + \Delta_Y) = -(K_Y + D) + F,$$

we can see that R' is a $(K_Y + \Delta_Y)$-negative extremal ray of $\overline{NE}(Y/S)$. Therefore, we can find a rational curve C' on Y such that C' spans R' and that

$$0 < -(K_Y + \Delta_Y) \cdot C' \leq 2 \dim X$$

by Theorem 4.6.2. By the above argument, we can easily see that $C' \not\subset \operatorname{Supp} F$. Therefore, we obtain

$$\begin{aligned} 0 < -(K_Y + D) \cdot C' &= -(K_Y + \Delta_Y) \cdot C' - F \cdot C' \\ &\leq -(K_Y + \Delta_Y) \cdot C' \leq 2 \dim X. \end{aligned}$$

Since $K_Y + D = f^*(K_X + \Delta)$, $C = f(C')$ is a rational curve on X such that $\pi(C)$ is a point and $0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$. □

REMARK 4.6.8. In Theorem 4.6.7, we can prove $0 < -(K_X + \Delta) \cdot C \leq \dim X + 1$ when $\dim X \leq 2$. For the details, see [**F29**, Proposition 3.7].

4.7. Shokurov polytope

In this section, we discuss a very important result obtained by Shokurov (cf. [**Sh3**, 6.2. First Main Theorem]), which is an application of Theorem 4.6.2. We closely follow Birkar's treatment in [**Bir3**, Section 3].

4.7.1. Let $\pi : X \to S$ be a projective morphism from a normal variety X to a variety S. A curve Γ on X is called *extremal* over S if the following properties hold.

(1) Γ generates an extremal ray R of $\overline{NE}(X/S)$.
(2) There is a π-ample Cartier divisor H on X such that

$$H \cdot \Gamma = \min\{H \cdot C\},$$

where C ranges over curves generating R.

We note that every $(K_X + \Delta)$-negative extremal ray R of $\overline{NE}(X/S)$ is spanned by a curve if Δ is an effective $\mathbb{R}$-divisor on X such that (X, Δ) is log canonical. It is a consequence of the cone and contraction theorem (see Theorem 4.5.2).

Let Δ be an effective $\mathbb{R}$-divisor on X such that (X, Δ) is log canonical and let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/S)$. Then we can take a rational curve C such that C spans R and that

$$0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$$

by Theorem 4.6.2. Let Γ be an extremal curve generating R. Then we have
$$\frac{-(K_X+\Delta)\cdot\Gamma}{H\cdot\Gamma}=\frac{-(K_X+\Delta)\cdot C}{H\cdot C}.$$
Therefore,
$$-(K_X+\Delta)\cdot\Gamma=(-(K_X+\Delta)\cdot C)\cdot\frac{H\cdot\Gamma}{H\cdot C}\leq 2\dim X.$$

Let F be a reduced divisor on X. We consider the finite dimensional real vector space $V=\bigoplus_k \mathbb{R}F_k$ where $F=\sum_k F_k$ is the irreducible decomposition. We have already seen that
$$\mathcal{L}=\{D\in V\,|\,(X,D)\text{ is log canonical}\}$$
is a rational polytope in V, that is, it is the convex hull of finitely many rational points in V (see Lemma 4.6.1).

Let $D_1,\cdots,D_r$ be the vertices of $\mathcal{L}$ and let m be a positive integer such that $m(K_X+D_j)$ is Cartier for every j. We take an $\mathbb{R}$-divisor $\Delta\in\mathcal{L}$. Then we can find non-negative real numbers $a_1,\cdots,a_r$ such that $\Delta=\sum_j a_jD_j$, $\sum_j a_j=1$, and (X,D_j) is log canonical for every j (see Lemma 4.6.1). For every curve C on X, the intersection number $-(K_X+\Delta)\cdot C$ can be written as
$$\sum_j a_j\frac{n_j}{m}$$
such that $n_j\in\mathbb{Z}$ for every j. If C is an extremal curve, then we can see that $n_j\leq 2m\dim X$ for every j by the above arguments.

On the real vector space V, we consider the following norm
$$\|\Delta\|=\max_j\{|b_j|\},$$
where $\Delta=\sum_j b_jF_j$.

We explain Shokurov's important results (cf. [**Sh3**]) following [**Bir3**, Proposition 3.2].

Theorem 4.7.2. *We use the same notation as in 4.7.1. We fix an $\mathbb{R}$-divisor $\Delta\in\mathcal{L}$. Then we can find positive real numbers α and δ, which depend on (X,Δ) and F, with the following properties.*

(1) *If Γ is any extremal curve over S and $(K_X+\Delta)\cdot\Gamma>0$, then $(K_X+\Delta)\cdot\Gamma>\alpha$.*
(2) *If $D\in\mathcal{L}$, $\|D-\Delta\|<\delta$, and $(K_X+D)\cdot R\leq 0$ for an extremal curve Γ, then $(K_X+\Delta)\cdot\Gamma\leq 0$.*

(3) *Let* $\{R_t\}_{t\in T}$ *be any set of extremal rays of* $\overline{NE}(X/S)$. *Then*

$$\mathcal{N}_T = \{D \in \mathcal{L} \,|\, (K_X + D) \cdot R_t \geq 0 \text{ for every } t \in T\}$$

is a rational polytope in V.

PROOF. (1) If Δ is a $\mathbb{Q}$-divisor, then the claim is obvious even if Γ is not extremal. We assume that Δ is not a $\mathbb{Q}$-divisor. Then we can write $K_X + \Delta = \sum_j a_j(K_X + D_j)$ as in 4.7.1. Then $(K_X + \Delta) \cdot \Gamma = \sum_j a_j(K_X + D_j) \cdot \Gamma$. If $(K_X + \Delta) \cdot \Gamma < 1$, then

$$\begin{aligned} -2\dim X \leq (K_X + D_{j_0}) \cdot \Gamma &< \frac{1}{a_{j_0}}\{-\sum_{j\neq j_0} a_j(K_X + D_j) \cdot \Gamma + 1\} \\ &\leq \frac{2\dim X + 1}{a_{j_0}} \end{aligned}$$

for $a_{j_0} \neq 0$. This is because $(K_X + D_j) \cdot \Gamma \geq -2\dim X$ for every j. Thus there are only finitely many possibilities of the intersection numbers $(K_X + D_j) \cdot \Gamma$ for $a_j \neq 0$ when $(K_X + \Delta) \cdot \Gamma < 1$. Therefore, the existence of α is obvious.

(2) If we take δ sufficiently small, then, for every $D \in \mathcal{L}$ with $\|D - \Delta\| < \delta$, we can always find $D' \in \mathcal{L}$ such that

$$K_X + D = (1 - s)(K_X + \Delta) + s(K_X + D')$$

with

$$0 \leq s \leq \frac{\alpha}{\alpha + 2\dim X}.$$

Since Γ is extremal, we have $(K_X+D')\cdot\Gamma \geq -2\dim X$ for every $D' \in \mathcal{L}$. We assume that $(K_X + \Delta) \cdot \Gamma > 0$. Then $(K_X + \Delta) \cdot \Gamma > \alpha$ by (1). Therefore,

$$\begin{aligned} (K_X + D) \cdot \Gamma &= (1 - s)(K_X + \Delta) \cdot \Gamma + s(K_X + D') \cdot \Gamma \\ &> (1 - s)\alpha + s(-2\dim X) \geq 0. \end{aligned}$$

This is a contradiction. Therefore, we obtain $(K_X + \Delta) \cdot \Gamma \leq 0$. We complete the proof of (2).

(3) For every $t \in T$, we may assume that there is some $D_t \in \mathcal{L}$ such that $(K_X + D_t) \cdot R_t < 0$. We note that $(K_X + D) \cdot R_t < 0$ for some $D \in \mathcal{L}$ implies $(K_X + D_j) \cdot R_t < 0$ for some j. Therefore, we may assume that T is contained in $\mathbb{N}$. This is because there are only countably many $(K_X + D_j)$-negative extremal rays for every j by the cone theorem (see Theorem 4.5.2). We note that $\mathcal{N}_T$ is a closed convex subset of $\mathcal{L}$ by definition. If T is a finite set, then the claim is obvious. Thus, we may assume that $T = \mathbb{N}$. By (2) and by the compactness of

$\mathcal{N}_T$, we can take $\Delta_1, \cdots, \Delta_n \in \mathcal{N}_T$ and $\delta_1, \cdots, \delta_n > 0$ such that $\mathcal{N}_T$ is covered by

$$\mathcal{B}_i = \{D \in \mathcal{L} \,|\, \|D - \Delta_i\| < \delta_i\}$$

and that if $D \in \mathcal{B}_i$ with $(K_X + D) \cdot R_t < 0$ for some t, then $(K_X + \Delta_i) \cdot R_t = 0$. If we put

$$T_i = \{t \in T \,|\, (K_X + D) \cdot R_t < 0 \text{ for some } D \in \mathcal{B}_i\},$$

then $(K_X + \Delta_i) \cdot R_t = 0$ for every $t \in T_i$ by the above construction. Since $\{\mathcal{B}_i\}_{i=1}^n$ gives an open covering of $\mathcal{N}_T$, we have $\mathcal{N}_T = \bigcap_{1 \le i \le n} \mathcal{N}_{T_i}$ by the following claim.

Claim. $\mathcal{N}_T = \bigcap_{1 \le i \le n} \mathcal{N}_{T_i}$.

Proof of Claim. We note that $\mathcal{N}_T \subset \bigcap_{1 \le i \le n} \mathcal{N}_{T_i}$ is obvious. We assume that $\mathcal{N}_T \subsetneq \bigcap_{1 \le i \le n} \mathcal{N}_{T_i}$. We take $D \in \bigcap_{1 \le i \le n} \mathcal{N}_{T_i} \setminus \mathcal{N}_T$ which is very close to $\mathcal{N}_T$. Since $\mathcal{N}_T$ is covered by $\{\mathcal{B}_i\}_{i=1}^n$, there is some i_0 such that $D \in \mathcal{B}_{i_0}$. Since $D \notin \mathcal{N}_T$, there is some $t_0 \in T$ such that $(K_X + D) \cdot R_{t_0} < 0$. Thus, $t_0 \in T_{i_0}$. This is a contradiction because $D \in \mathcal{N}_{T_{i_0}}$. Therefore, $\mathcal{N}_T = \bigcap_{1 \le i \le n} \mathcal{N}_{T_i}$. □

So, it is sufficient to see that each $\mathcal{N}_{T_i}$ is a rational polytope in V. By replacing T with T_i, we may assume that there is some $D \in \mathcal{N}_T$ such that $(K_X + D) \cdot R_t = 0$ for every $t \in T$.

If $\dim_{\mathbb{R}} \mathcal{L} = 1$, then this already implies the claim. We assume $\dim_{\mathbb{R}} \mathcal{L} > 1$. Let $\mathcal{L}^1, \cdots, \mathcal{L}^p$ be the proper faces of $\mathcal{L}$. Then $\mathcal{N}_T^i = \mathcal{N}_T \cap \mathcal{L}^i$ is a rational polytope by induction on dimension. Moreover, for each $D'' \in \mathcal{N}_T$ which is not D, there is D' on some proper face of $\mathcal{L}$ such that D'' is on the line segment determined by D and D'. Note that $(K_X + D) \cdot R_t = 0$ for every $t \in T$. Therefore, if $D' \in \mathcal{L}^i$, then $D' \in \mathcal{N}_T^i$. Thus, $\mathcal{N}_T$ is the convex hull of D and all the $\mathcal{N}_T^i$. So there is a finite subset $T' \subset T$ such that

$$\bigcup_i \mathcal{N}_T^i = \mathcal{N}_{T'} \cap (\bigcup_i \mathcal{L}^i).$$

Therefore, the convex hull of D and $\bigcup_i \mathcal{N}_T^i$ is just $\mathcal{N}_{T'}$. We complete the proof of (3). □

By Theorem 4.7.2 (3), Lemma 2.6 in [**Bir2**] holds for log canonical pairs. It may be useful for the minimal model program with scaling.

Theorem 4.7.3 (cf. [**Bir2**, Lemma 2.6]). *Let (X, Δ) be a log canonical pair, let Δ be an $\mathbb{R}$-divisor, and let $\pi : X \to S$ be a projective morphism between algebraic varieties. Let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $K_X + \Delta + H$ is π-nef and $(X, \Delta + H)$ is log* *canonical. Then, either $K_X + \Delta$ is also π-nef or there is a $(K_X + \Delta)$-negative extremal ray R such that $(K_X + \Delta + \lambda H) \cdot R = 0$, where*

$$\lambda := \inf\{t \geq 0 \,|\, K_X + \Delta + tH \ \textit{is}\ \pi\textit{-nef}\}.$$

Of course, $K_X + \Delta + \lambda H$ is π-nef.

Note that Theorem 4.7.3 is nothing but Theorem 4.5.2 (6).

Proof. Assume that $K_X + \Delta$ is not π-nef. Let $\{R_j\}$ be the set of $(K_X + \Delta)$-negative extremal rays over S. Let C_j be an extremal curve spanning R_j for every j. We put $\mu = \sup_j \{\mu_j\}$, where

$$\mu_j = \frac{-(K_X + \Delta) \cdot C_j}{H \cdot C_j}.$$

Obviously, $\lambda = \mu$ and $0 < \mu \leq 1$. So, it is sufficient to prove that $\mu = \mu_{j_0}$ for some j_0. There are positive real numbers $r_1, \cdots, r_l$ such that $\sum_i r_i = 1$ and a positive integer m, which are independent of j, such that

$$-(K_X + \Delta) \cdot C_j = \sum_{i=1}^{l} \frac{r_i n_{ij}}{m} > 0$$

(see Lemma 4.6.1, Theorem 4.6.2, and 4.7.1). Since C_j is extremal, n_{ij} is an integer with $n_{ij} \leq 2m \dim X$ for every i and j. If $(K_X + \Delta + H) \cdot R_{j_0} = 0$ for some j_0, then there are nothing to prove since $\lambda = 1$ and $(K_X + \Delta + H) \cdot R = 0$ with $R = R_{j_0}$. Thus, we assume that $(K_X + \Delta + H) \cdot R_j > 0$ for every j. We put $F = \mathrm{Supp}(\Delta + H)$. Let $F = \sum_k F_k$ be the irreducible decomposition. We put $V = \bigoplus_k \mathbb{R} F_k$,

$$\mathcal{L} = \{D \in V \,|\, (X, D) \text{ is log canonical}\},$$

and

$$\mathcal{N} = \{D \in \mathcal{L} \,|\, (K_X + D) \cdot R_j \geq 0 \text{ for every } j\}.$$

Then $\mathcal{N}$ is a rational polytope in V by Theorem 4.7.2 (3) and $\Delta + H$ is in the relative interior of $\mathcal{N}$ by the above assumption. Therefore, we can write

$$K_X + \Delta + H = \sum_{p=1}^{q} r'_p (K_X + D_p),$$

where $r'_1, \cdots, r'_q$ are positive real numbers such that $\sum_p r'_p = 1$, (X, D_p) is log canonical for every p, $m'(K_X + D_p)$ is Cartier for some positive integer m' and every p, and $(K_X + D_p) \cdot C_j > 0$ for every p and j. So, we obtain

$$(K_X + \Delta + H) \cdot C_j = \sum_{p=1}^{q} \frac{r'_p n'_{pj}}{m'}$$

with $0 < n'_{pj} = m'(K_X + D_p) \cdot C_j \in \mathbb{Z}$. Note that m' and r'_p are independent of j for every p. We also note that

$$\begin{aligned}\frac{1}{\mu_j} = \frac{H \cdot C_j}{-(K_X+\Delta)\cdot C_j} &= \frac{(K_X+\Delta+H)\cdot C_j}{-(K_X+\Delta)\cdot C_j} + 1 \\ &= \frac{m\sum_{p=1}^{q} r'_p n'_{pj}}{m'\sum_{i=1}^{l} r_j n_{ij}} + 1.\end{aligned}$$

Since

$$\sum_{i=1}^{l} \frac{r_i n_{ij}}{m} > 0$$

for every j and $n_{ij} \leq 2m \dim X$ with $n_{ij} \in \mathbb{Z}$ for every i and j, the number of the set $\{n_{ij}\}_{i,j}$ is finite. Thus,

$$\inf_j \left\{ \frac{1}{\mu_j} \right\} = \frac{1}{\mu_{j_0}}$$

for some j_0. Therefore, we obtain $\mu = \mu_{j_0}$. We finish the proof. □

Let us recall the abundance conjecture, which is one of the most important conjectures in the minimal model theory for higher-dimensional algebraic varieties.

4.7.4 (Abundance conjecture). We treat some applications of Theorem 4.7.2 (3) to the abundance conjecture for $\mathbb{R}$-divisors (see [**Sh3**, 2.7. Theorem on log semi-ampleness for 3-folds]).

Conjecture 4.7.5 (Abundance conjecture). *Let (X, Δ) be a log canonical pair and let $f : X \to Y$ be a projective morphism between varieties. If $K_X + \Delta$ is f-nef, then $K_X + \Delta$ is f-semi-ample.*

For the recent developments of the abundance conjecture, see, for example, [**FG1**].

The following proposition is a useful application of Theorem 4.7.2 (see [**Sh3**, 2.7]).

Proposition 4.7.6. *Let $f : X \to Y$ be a projective morphism between algebraic varieties. Let Δ be an effective $\mathbb{R}$-divisor on X such that (X, Δ) is log canonical and that $K_X + \Delta$ is f-nef. Assume that the abundance conjecture holds for $\mathbb{Q}$-divisors. More precisely, we assume that $K_X + D$ is f-semi-ample if $D \in \mathcal{L}$, D is a $\mathbb{Q}$-divisor, and $K_X + D$ is f-nef, where*

$$\mathcal{L} = \{D \in V \mid (X, D) \text{ is log canonical}\},$$

$V = \bigoplus_k \mathbb{R}F_k$, and $\sum_k F_k$ is the irreducible decomposition of $\operatorname{Supp}\Delta$. *Then $K_X + \Delta$ is f-semi-ample.*

Proof. Let $\{R_t\}_{t\in T}$ be the set of all extremal rays of $\overline{NE}(X/Y)$. We consider $\mathcal{N}_T$ as in Theorem 4.7.2 (3). Then $\mathcal{N}_T$ is a rational polytope in $\mathcal{L}$ by Theorem 4.7.2 (3). We can easily see that

$$\mathcal{N}_T = \{D \in \mathcal{L} \,|\, K_X + D \text{ is } f\text{-nef}\}.$$

By assumption, $\Delta \in \mathcal{N}_T$. Let $\mathcal{F}$ be the minimal face of $\mathcal{N}_T$ containing Δ. Then we can find $\mathbb{Q}$-divisors $D_1, \cdots, D_l$ on X such that D_i is in the relative interior of $\mathcal{F}$,

$$K_X + \Delta = \sum_i d_i(K_X + D_i),$$

where d_i is a positive real number for every i and $\sum_i d_i = 1$. By assumption, $K_X + D_i$ is f-semi-ample for every i. Therefore, $K_X + \Delta$ is f-semi-ample. □

Remark 4.7.7 (Stability of Iitaka fibrations). In the proof of Proposition 4.7.6, we note the following property. If C is a curve on X such that $f(C)$ is a point and $(K_X + D_{i_0}) \cdot C = 0$ for some i_0, then $(K_X + D_i) \cdot C = 0$ for every i. This is because we can find $\Delta' \in \mathcal{F}$ such that $(K_X+\Delta')\cdot C < 0$ if $(K_X+D_i)\cdot C > 0$ for some $i \neq i_0$. This is a contradiction. Therefore, there exists a contraction morphism $g : X \to Z$ over Y and h-ample $\mathbb{Q}$-divisors $A_1, \cdots, A_l$ on Z, where $h : Z \to Y$, such that $K_X + D_i \sim_{\mathbb{Q}} g^*A_i$ for every i. In particular,

$$K_X + \Delta \sim_{\mathbb{R}} g^*(\sum_i d_i A_i).$$

Note that $\sum_i d_i A_i$ is h-ample. Roughly speaking, the Iitaka fibration of $K_X + \Delta$ is the same as that of $K_X + D_i$ for every i.

Corollary 4.7.8. *Let $f : X \to Y$ be a projective morphism between algebraic varieties. Assume that (X, Δ) is log canonical and that $K_X + \Delta$ is f-nef. We further assume one of the following conditions.*

(i) $\dim X \leq 3$.
(ii) $\dim X = 4$ *and* $\dim Y \geq 1$.

Then $K_X + \Delta$ is f-semi-ample.

Proof. It is obvious by Proposition 4.7.6 and the log abundance theorems for threefolds and fourfolds (see, for example, [**KeMM**, 1.1. Theorem] and [**F22**, Theorem 3.10]). □

Corollary 4.7.9. *Let $f : X \to Y$ be a projective morphism between algebraic varieties. Assume that (X, Δ) is klt and $K_X + \Delta$ is f-nef. We further assume that* $\dim X - \dim Y \leq 3$. *Then $K_X + \Delta$ is f-semi-ample.*

Proof. If Δ is a $\mathbb{Q}$-divisor, then it is well known that $K_{X_\eta}+\Delta_\eta$ is semi-ample, where X_η is the generic fiber of f and $\Delta_\eta=\Delta|_{X_\eta}$ (see, for example, [**KeMM**, 1.1. Theorem]). Therefore, $K_X+\Delta$ is f-semi-ample by [**F24**, Theorem 1.1]. When Δ is an $\mathbb{R}$-divisor, we can take $\mathbb{Q}$-divisors $D_1,\cdots,D_l\in\mathcal{F}$ as in the proof of Proposition 4.7.6 such that (X,D_i) is klt for every i. Since K_X+D_i is f-semi-ample by the above argument, we obtain that $K_X+\Delta$ is f-semi-ample. $\square$

4.8. MMP for lc pairs

In this section, we discuss the minimal model program for log canonical pairs and some related topics.

Let us start with the definition of log canonical models.

Definition 4.8.1 (Log canonical model). Let (X,Δ) be a log canonical pair and let $\pi: X\to S$ be a proper morphism. A pair (X',Δ') sitting in a diagram

$$\begin{tikzcd} (X,\Delta) \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & (X',\Delta') \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

is called a *log canonical model* of (X,Δ) over S if

(i) π' is proper,
(ii) ϕ^{-1} has no exceptional divisors,
(iii) $\Delta'=\phi_*\Delta$,
(iv) $K_{X'}+\Delta'$ is π'-ample, and
(v) $a(E,X,\Delta)\le a(E,X',\Delta')$ for every ϕ-exceptional divisor $E\subset X$.

Lemma 4.8.2. *Let (X,Δ) be a log canonical pair and let $\pi: X\to S$ be a proper morphism. Let (X',Δ') be a log canonical model of (X,Δ) over S. Then $a(E,X,\Delta)\le a(E,X',\Delta')$ for every prime divisor E over X. We assume that Δ is a $\mathbb{Q}$-divisor. Then*

$$X'=\mathrm{Proj}_X\bigoplus_{m\ge 0}\pi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor).$$

PROOF. By the same argument as Lemma 4.3.2, we have $a(E, X, \Delta) \leq a(E, X', \Delta')$ for every prime divisor E over X. We take a common resolution

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

of X and X'. We put $\Delta_W = p_*^{-1}\Delta + E$, where E is the sum of all p-exceptional divisors. Then we can write

$$K_W + \Delta_W = p^*(K_X + \Delta) + F$$

and

$$K_W + \Delta_W = q^*(K_{X'} + \Delta') + G$$

where F is effective and p-exceptional, and G is effective and q-exceptional. Therefore,

$$\begin{aligned} X' &= \operatorname{Proj}_S \bigoplus_{m \geq 0} \pi'_* \mathcal{O}_{X'}(\lfloor m(K_{X'} + \Delta') \rfloor) \\ &= \operatorname{Proj}_S \bigoplus_{m \geq 0} (\pi' \circ q)_* \mathcal{O}_W(\lfloor m(K_W + \Delta_W) \rfloor) \\ &= \operatorname{Proj}_S \bigoplus_{m \geq 0} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor). \end{aligned}$$

This is the desired description of X'. $\square$

LEMMA 4.8.3. *Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a proper morphism onto a variety. Let (X^m, Δ^m) be a minimal model of (X, Δ) over S and let (X^{lc}, Δ^{lc}) be a log canonical model of (X, Δ) over S. Then there is a natural morphism $\alpha : X^m \to X^{lc}$ such that*

$$K_{X^m} + \Delta^m = \alpha^*(K_{X^{lc}} + \Delta^{lc}).$$

In particular, $K_{X^m} + \Delta^m$ is semi-ample over S, that is, (X^m, Δ^m) is a good minimal model of (X, Δ) over S.

PROOF. We take a common resolution

$$\begin{tikzcd} X & W \arrow[l, "p"'] \arrow[r, "r"] \arrow[d, "q"] & X^{lc} \\ & X^m & \end{tikzcd}$$

of X, X^m, and X^{lc}. Let E be the sum of all p-exceptional divisors. We put $\Delta_W = p_*^{-1}\Delta + E$. Then we have

$$K_W + \Delta_W = q^*(K_{X^m} + \Delta^m) + F$$

and

$$K_W + \Delta_W = r^*(K_{X^{lc}} + \Delta^{lc}) + G$$

where F is effective and q-exceptional and G is effective and r-exceptional by the negativity lemma (see Lemma 2.3.26). Therefore, we obtain

$$q^*(K_{X^m} + \Delta^m) + F = r^*(K_{X^{lc}} + \Delta^{lc}) + G.$$

Note that $q_*(G - F)$ is effective and $-(G - F)$ is q-nef. This implies $G - F \geq 0$ by the negativity lemma (see Lemma 2.3.26). Similarly, $r_*(F - G)$ is effective and $-(F - G)$ is r-nef. This implies $F - G \geq 0$ by the negativity lemma (see Lemma 2.3.26) again. Therefore, $F = G$. So, we have

$$q^*(K_{X^m} + \Delta^m) = r^*(K_{X^{lc}} + \Delta^{lc}).$$

We assume that $r \circ q^{-1} : X^m \dashrightarrow X^{lc}$ is not a morphism. Then we can find a curve C on W such that $q(C)$ is a point and that $r(C)$ is a curve. In this case,

$$0 = C \cdot q^*(K_{X^m} + \Delta^m) = C \cdot r^*(K_{X^{lc}} + \Delta^{lc}) > 0.$$

This is a contradiction. Therefore, $\alpha : r \circ q^{-1} : X^m \dashrightarrow X^{lc}$ is a morphism and $K_{X^m} + \Delta^m = \alpha^*(K_{X^{lc}} + \Delta^{lc})$. □

By the proof of Lemma 4.8.3, we have:

Corollary 4.8.4. *Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a proper morphism onto a variety. Let (X_1, Δ_1) and (X_2, Δ_2) be log canonical models of (X, Δ) over S. Then (X_1, Δ_1) is isomorphic to (X_2, Δ_2) over S. Therefore, the log canonical model of (X, Δ) over S is unique.*

In order to discuss the minimal model program for log canonical pairs, it is convenient to use the following definitions of minimal models and Mori fiber spaces due to Birkar–Shokurov.

Definition 4.8.5 (Minimal models, see [Bir4, Definition 2.1]). Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a projective morphism onto a variety S. let $\phi : X \dashrightarrow Y$ be a birational map over S. We put $\Delta_Y = \widetilde{\Delta} + E$ where $\widetilde{\Delta}$ is the birational transform of Δ on Y and E is the reduced exceptional divisor of ϕ^{-1}, that is, $E = \sum_j E_j$ where E_j is a prime divisor on Y which is exceptional over X for every j. We assume that

(i) (Y, Δ_Y) is a $\mathbb{Q}$-factorial dlt pair and Y is projective over S,

(ii) $K_Y + \Delta_Y$ is nef over S, and
(iii) for any prime divisor E on X which is exceptional over Y, we have
$$a(E, X, \Delta) < a(E, Y, \Delta_Y).$$

Then (Y, Δ_Y) is called a *minimal model of* (X, Δ) *over* S. Furthermore, if $K_Y + \Delta_Y$ is semi-ample over S, then (Y, Δ_Y) is called a *good minimal model of* (X, Δ) *over* S.

Remark 4.8.6. By the same argument as Lemma 4.3.2, we can prove that $a(E, X, \Delta) \leq a(E, Y, \Delta_Y)$ for every prime divisor E over X in Definition 4.8.5. Therefore, if (X, Δ) is plt in Definition 4.8.5, then (Y, Δ_Y) is plt and ϕ^{-1} has no exceptional divisors.

Definition 4.8.7 (Mori fiber spaces, see [**Bir4**, Definition 2.2]). Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a projective morphism onto a variety S. let $\phi : X \dashrightarrow Y$ be a birational map over S. We put $\Delta_Y = \widetilde{\Delta} + E$ where $\widetilde{\Delta}$ is the birational transform of Δ on Y and E is the reduced exceptional divisor of ϕ^{-1}, that is, $E = \sum_j E_j$ where E_j is a prime divisor on Y which is exceptional over X for every j. We assume that

(i) (Y, Δ_Y) is a $\mathbb{Q}$-factorial dlt pair and Y is projective over S,
(ii) there is a $(K_Y+\Delta_Y)$-negative extremal contraction $\varphi : Y \to Z$, that is, $-(K_Y+\Delta_Y)$ is φ-ample, $\rho(Y/Z) = 1$, and $\varphi_*\mathcal{O}_Y \simeq \mathcal{O}_Z$, over S with $\dim Y > \dim Z$, and
(iii) we have
$$a(E, X, \Delta) \leq a(E, Y, \Delta_Y).$$
for any prime divisor E over X and strct inequality holds if E is on X and ϕ contracts E.

Then (Y, Δ_Y) is called a *Mori fiber space of* (X, Δ) *over* S.

Let us quickly recall some results in [**Bir4**] and [**HaX1**]. For the details, see the original papers [**Bir4**] and [**HaX1**].

Theorem 4.8.8 (cf. [**Bir4**, Theorem 1.1] and [**HaX1**, Theorem 1.6]). *Let (X, Δ) be a $\mathbb{Q}$-factorial log canonical pair such that Δ is a $\mathbb{Q}$-divisor and let $\pi : X \to S$ be a projective morphism between quasi-projective varieties. Assume that there is an effective $\mathbb{Q}$-divisor Δ' on X such that $(X, \Delta + \Delta')$ is log canonical and $K_X + \Delta + \Delta' \sim_{\mathbb{Q},\pi} 0$. Then (X, Δ) has a Mori fiber space or a good minimal model over S.*

As a direct consequence of Theorem 4.8.8, we have:

Corollary 4.8.9. *In Theorem 4.8.8, we further assume that $K_X + \Delta$ is π-big. Then (X, Δ) has a log canonical model over S.*

PROOF. Let (Y, Δ_Y) be a good minimal model of (X, Δ) over S. Then

$$\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor) \simeq \bigoplus_{m\geq 0} \pi_{Y*}\mathcal{O}_Y(\lfloor m(K_Y+\Delta_Y)\rfloor)$$

as $\mathcal{O}_S$-algebras (see Lemma 4.8.2), where $\pi_Y : Y \to S$, and

$$\bigoplus_{m\geq 0} \pi_{Y*}\mathcal{O}_Y(\lfloor m(K_Y+\Delta_Y)\rfloor)$$

is a finitely generated $\mathcal{O}_S$-algebra since $K_Y + \Delta_Y$ is a π_Y-semi-ample $\mathbb{Q}$-divisor. We put

$$X' = \mathrm{Proj}_S \bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor).$$

Then (X', Δ'), where Δ' is the strict transform of Δ on X', is a log canonical model of (X, Δ) over S by Lemma 4.8.2. $\square$

COROLLARY 4.8.10 (cf. [**Bir4**, Corollary 1.2] and [**HaX1**, Corollary 1.8]). *Let $\varphi : (X, \Delta) \to W$ be a log canonical flipping contraction associated to a $(K_X+\Delta)$-negative extremal ray. Then the $(K_X+\Delta)$-flip of $\varphi : (X, \Delta) \to W$ exists.*

PROOF. Since $\rho(X/W) = 1$, we may assume that Δ is a $\mathbb{Q}$-divisor by perturbing Δ slightly. By taking an affine cover of W, we may assume that W is affine. Then we can find an effective $\mathbb{Q}$-divisor Δ' on X such that $K_X + \Delta + \Delta' \sim_{\mathbb{Q},\varphi} 0$. Then (X, Δ) has a log canonical model over W. It is nothing but a flip of $\varphi : (X, \Delta) \to W$. $\square$

REMARK 4.8.11. By Corollary 4.8.10, log canonical flips always exist. On the other hand, log canonical flops do not always exist. For the details, see [**F38**, Section 7], where Kollár's examples are described in details.

4.8.12 (MMP for $\mathbb{Q}$-factorial log canonical pairs). Let (X, Δ) be a $\mathbb{Q}$-factorial log canonical pair and let $f : X \to S$ be a projective morphism onto a variety S. By Theorem 4.5.2 and Corollary 4.8.10, we can run the minimal model program for (X, Δ) over S. This means that the minimal model program discussed in 4.3.5 works by replacing *dlt* with *log canonical*. Moreover, by Theorem 4.5.2 (6), we can run the minimal model program with scaling discussed in 4.4.11 for $\mathbb{Q}$-factorial log canonical pairs. Note that the termination of the above minimal model programs is an important open problem of the minimal model theory (see Conjecture 4.3.6 and Lemma 4.9.3 below).

We note the following well-known lemma.

LEMMA 4.8.13 (see [KoMo, Corollary 3.44]). *Let* (X,Δ) *be a dlt (resp. klt or lc) pair. Let* $g : X \dashrightarrow X'$ *be either a divisorial contraction of a* $(K_X+\Delta)$*-negative extremal ray or a* $(K_X+\Delta)$*-flip. We put* $\Delta' = g_*\Delta$*. Then* (X',Δ') *is also dlt (resp. klt or lc).*

PROOF. This lemma easily follows from Lemma 2.3.27 when (X,Δ) is klt or lc. From now on, we treat the case when (X,Δ) is dlt. Let $Z \subset X$ be as in Proposition 2.3.20. We put $Z' = g(Z) \cup \mathrm{Exc}(g^{-1})$ such that $\mathrm{Exc}(g^{-1})$ is the closed subset of X' where g^{-1} is not an isomorphism. Then $X' \setminus Z'$ is isomorphic to an open subset of $X \setminus Z$. Therefore, $X' \setminus Z'$ is smooth and $\Delta'|_{X'\setminus Z'}$ has a simple normal crossing support. Let E be an exceptional divisor over X' such that $c_{X'}(E)$, the center of E on X', is contained in Z'. Then $c_X(E)$, the center of E on X, is contained in $Z \cup \mathrm{Exc}(g)$, where $\mathrm{Exc}(g)$ is the closed subset of X where g is not an isomorphism. We have

$$a(E,X',\Delta') \geq a(E,X,\Delta) \geq -1$$

by Lemma 2.3.27. If $c_X(E)$ is contained in Z, then the second inequality is strict by the definition of dlt pairs. If $c_X(E)$ is contained in $\mathrm{Exc}(g)$, then the first inequality is strict by Lemma 2.3.27. Anyway, (X',Δ') is dlt by Proposition 2.3.20. □

We also note the following easy two propositions.

PROPOSITION 4.8.14 (cf. [KoMo, Proposition 3.36]). *Let* (X,Δ) *be a* $\mathbb{Q}$*-factorial log canonical pair and let* $\pi : X \to S$ *be a projective morphism. Let* $\varphi_R : X \to Y$ *be the contraction of a* $(K_X+\Delta)$*-negative extremal ray* R *of* $\overline{NE}(X/S)$*. Assume that* φ_R *is either a divisorial or a Fano contraction. Then we have*

(i) Y *is* $\mathbb{Q}$*-factorial, and*
(ii) $\rho(Y/S) = \rho(X/S) - 1$.

PROOF. This proposition directly follows from Corollary 4.5.3 and Corollary 4.5.4. □

REMARK 4.8.15. If $\varphi_R : X \to Y$ is a Fano contraction in Proposition 4.8.14, then we know that Y has only log canonical singularities by [F38]. We further assume that X has only log terminal singularities. Then Y has only log terminal singularities. For the details and some related topics, see [F38].

PROPOSITION 4.8.16 (cf. [KoMo, Proposition 3.37]). *Let* (X,Δ) *be a* $\mathbb{Q}$*-factorial log canonical pair and let* $\pi : X \to S$ *be a projective* *morphism. Let $\varphi_R : X \to W$ be the flipping contraction of a $(K_X + \Delta)$-negative extremal ray R of $\overline{NE}(X/S)$ and let $\varphi_R^+ : X^+ \to W$ be the flip.*

$$\begin{tikzcd}
X \arrow[rr, dashed, "\phi"] \arrow[dr, "\varphi_R"] \arrow[ddr, "\pi"'] & & X^+ \arrow[dl, "\varphi_R^+"'] \arrow[ddl] \\
& W \arrow[d] & \\
& S &
\end{tikzcd}$$

Then we have

(i) X^+ *is $\mathbb{Q}$-factorial, and*
(ii) $\rho(X^+/S) = \rho(X/S)$.

Proof. By perturbing Δ slightly, we may assume that Δ is a $\mathbb{Q}$-divisor. Sine $\phi : X \dashrightarrow X^+$ is an isomorphism in codimension one, it induces a natural isomorphism between the group of Weil divisors on X and the group of Weil divisors on X^+. Let D^+ be a Weil divisor on X^+ and let D be the strict transform of D^+ on X. Then there is a rational number r such that

$$(((D + r(K_X + \Delta)) \cdot R) = 0.$$

We take a positive integer m such that $m(D + r(K_X + \Delta))$ is Cartier. By Theorem 4.5.2 (4), there is a Cartier divisor D_W on W such that $m(D + r(K_X + \Delta)) \sim \varphi_R^* D_W$. Thus we obtain that

$$mD^+ = m\phi_* D \sim (\varphi_R^+)^* D_W - (mr)(K_{X^+} + \Delta^+)$$

is $\mathbb{Q}$-Cartier. This means that X^+ is $\mathbb{Q}$-factorial. It is easy to see that $\rho(X/S) = \rho(X^+/S)$ by the above argument. $\square$

4.8.17 (Conjectures concerning MMP for lc pairs). The following conjecture is one of the most important open problems of the minimal model program for log canonical pairs.

Conjecture 4.8.18. *Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor on X. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

It is known that Conjecture 4.8.18 holds when $\dim X \leq 4$. When $\dim X \geq 5$, Conjecture 4.8.18 is still an open problem.

THEOREM 4.8.19 (cf. [F22, Theorem 1.2]). *Let* (X, Δ) *be a projective log canonical pair such that* Δ *is a* $\mathbb{Q}$*-divisor with* $\dim X \le 4$*. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \ge 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor))$$

is a finitely generated $\mathbb{C}$*-algebra.*

Let us recall the good minimal model conjecture.

CONJECTURE 4.8.20 (Good minimal model conjecture). *Let* (X, Δ) *be a* $\mathbb{Q}$*-factorial projective dlt pair and let* Δ *be an* $\mathbb{R}$*-divisor. If* $K_X + \Delta$ *is pseudo-effective, then* (X, Δ) *has a good minimal model.*

In [FG2], we obtained:

THEOREM 4.8.21. *Conjecture 4.8.18 with* $\dim X = n$ *and Conjecture 4.8.20 with* $\dim X \le n - 1$ *are equivalent.*

Moreover, in [FG2], we proved:

THEOREM 4.8.22. *Conjecture 4.8.20 with* $\dim X \le n - 1$ *is equivalent to Conjecture 4.8.23 with* $\dim X = n$*.*

CONJECTURE 4.8.23. *Let* (X, Δ) *be a* $\mathbb{Q}$*-factorial projective plt pair such that* Δ *is a* $\mathbb{Q}$*-divisor on* X *and that* $\lfloor \Delta \rfloor$ *is irreducible. We further assume that* $K_X + \Delta$ *is big. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \ge 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor))$$

is a finitely generated $\mathbb{C}$*-algebra.*

Therefore, Conjecture 4.8.18 is equivalent to Conjecture 4.8.23 by Theorems 4.8.21 and 4.8.22.

Let us recall some related conjectures. For the details, see [FG1] and [FG2].

CONJECTURE 4.8.24 (Non-vanishing conjecture). *Let* X *be a smooth projective variety. If* K_X *is pseudo-effective, then there exists some effective* $\mathbb{Q}$*-divisor* D *such that* $K_X \sim_{\mathbb{Q}} D$*.*

REMARK 4.8.25. Let X be a smooth projective variety. Then K_X is pseudo-effective if and only if X is not uniruled by [BDPP]. For the proof, see, for example, [La2, Corollary 11.4.20].

CONJECTURE 4.8.26 (DLT extension conjecture, see [DHP, Conjecture 1.3] and [FG2, Conjecture G]). *Let* (X, Δ) *be a projective* *dlt pair such that Δ is a $\mathbb{Q}$-divisor, $\lfloor \Delta \rfloor = S$, $K_X + \Delta$ is nef, and $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$ where $S \subset \operatorname{Supp} D$. Then the restriction map*

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$$

is surjective for all sufficiently divisible integers $m \geq 2$.

Note that the restriction map

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$$

in Conjecture 4.8.26 is surjective for every positive integer m such that $m(K_X + \Delta)$ is Cartier when $K_X + \Delta$ is semi-ample (see [**FG1**, Proposition 5.12]). Therefore, Conjecture 4.8.26 follows from the abundance conjecture (see Conjecture 4.7.5).

Conjecture 4.8.27. *Let (X, Δ) be a projective klt pair such that Δ is a $\mathbb{Q}$-divisor with $\kappa(X, K_X + \Delta) = 0$. Then $\kappa_\sigma(X, K_X + \Delta) = 0$, where κ_σ denotes Nakayama's numerical dimension in Definition 2.4.8.*

Conjecture 4.8.27 is known as a special case of the generalized abundance conjecture for klt pairs.

In [**FG2**], we obtained the following two theorems.

Theorem 4.8.28. *Conjecture 4.8.20 with $\dim X \leq n$ follows from Conjecture 4.8.24 with $\dim X \leq n$ and Conjecture 4.8.26 with $\dim X \leq n$.*

Theorem 4.8.29. *Conjecture 4.8.20 with $\dim X \leq n$ follows from Conjecture 4.8.24 with $\dim X \leq n$ and Conjecture 4.8.27 with $\dim X \leq n$.*

Anyway, Conjecture 4.8.24 seems to be the hardest open problem in the minimal model program.

4.9. Non-$\mathbb{Q}$-factorial MMP

In this section, we explain the minimal model program for non-$\mathbb{Q}$-factorial log canonical pairs, that is, the minimal model program for (not necessarily $\mathbb{Q}$-factorial) log canonical pairs. It is the most general minimal model program in the usual sense. Although it is essentially the same as 4.3.5, we describe it for the reader's convenience.

Let us explain the minimal model program for non-$\mathbb{Q}$-factorial log canonical pairs (cf. [**F13**, 4.4]).

4.9.1 (MMP for non-$\mathbb{Q}$-factorial log canonical pairs). We start with a pair $(X, \Delta) = (X_0, \Delta_0)$. Let $f_0 : X_0 \to S$ be a projective morphism. The aim is to set up a recursive procedure which creates intermediate pairs (X_i, Δ_i) and projective morphisms $f_i : X_i \to S$. After some steps, it should stop with a final pair (X', Δ') and $f' : X' \to S$.

Step 0 (Initial datum). Assume that we have already constructed (X_i, Δ_i) and $f_i : X_i \to S$ with the following properties:

(1) (X_i, Δ_i) is log canonical,
(2) f_i is projective, and
(3) X_i is not necessarily $\mathbb{Q}$-factorial.

If X_i is $\mathbb{Q}$-factorial, then it is easy to see that X_k is also $\mathbb{Q}$-factorial for every $k \geq i$. Even when X_i is not $\mathbb{Q}$-factorial, X_{i+1} sometimes becomes $\mathbb{Q}$-factorial (see, for example, Example 7.5.1 below.)

Step 1 (Preparation). If $K_{X_i} + \Delta_i$ is f_i-nef, then we go directly to Step 3 (2). If $K_{X_i} + \Delta_i$ is not f_i-nef, then we have already established the following two results (see Theorem 4.5.2):

(1) (Cone theorem). We have the following equality.
$$\overline{NE}(X_i/S) = \overline{NE}(X_i/S)_{(K_{X_i}+\Delta_i)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$
(2) (Contraction theorem). Any $(K_{X_i} + \Delta_i)$-negative extremal ray $R_i \subset \overline{NE}(X_i/S)$ can be contracted. Let $\varphi_{R_i} : X_i \to Y_i$ denote the corresponding contraction. It sits in a commutative diagram.
$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{R_i}"] \arrow[dr, "f_i"'] & & Y_i \arrow[dl, "g_i"] \\ & S & \end{tikzcd}$$

Step 2 (Birational transformations). If $\varphi_{R_i} : X_i \to Y_i$ is birational, then we take an effective $\mathbb{Q}$-divisor Δ_i' on X_i such that (X_i, Δ_i') is log canonical and $-(K_{X_i} + \Delta_i')$ is φ_{R_i}-ample. Note that $\rho(X_i/S) = 1$. By Corollary 4.8.9, the relative log canonical ring
$$\bigoplus_{m\geq 0} (\varphi_{R_i})_* \mathcal{O}_{X_i}(\lfloor m(K_{X_i} + \Delta_i')\rfloor)$$
is a finitely generated $\mathcal{O}_{Y_i}$-algebra. We put
$$X_{i+1} = \mathrm{Proj}_{Y_i} \bigoplus_{m\geq 0} (\varphi_{R_i})_* \mathcal{O}_{X_i}(\lfloor m(K_{X_i} + \Delta_i')\rfloor).$$
Then we have the following diagram.
$$\begin{tikzcd} X_i \arrow[rr, dashed] \arrow[dr, "\varphi_{R_i}"'] & & X_{i+1} \arrow[dl, "\varphi_{R_i}^+"] \\ & Y_i & \end{tikzcd}$$

Let Δ_{i+1} be the pushforward of Δ_i on X_{i+1}. We note that (X_{i+1}, Δ_{i+1}) is the log canonical model of (X_i, Δ_i) over Y_i (see Definition 4.8.1). Therefore, $\varphi^+_{R_i} : X_{i+1} \to Y_i$ is a projective birational morphism, $K_{X_{i+1}} + \Delta_{i+1}$ is $\varphi^+_{R_i}$-ample, and (X_{i+1}, Δ_{i+1}) is log canonical. Then we go back to Step 0 with (X_{i+1}, Δ_{i+1}), $f_{i+1} = g_i \circ \varphi^+_{R_i}$ and start anew.

If X_i is $\mathbb{Q}$-factorial, then so is X_{i+1}. If X_i is $\mathbb{Q}$-factorial and φ_{R_i} is not small, then $\varphi^+_{R_i} : X_{i+1} \to Y_i$ is an isomorphism. It may happen that $\rho(X_i/S) < \rho(X_{i+1}/S)$ when X_i is not $\mathbb{Q}$-factorial (see, for example, Example 7.5.1 below).

Step 3 (Final outcome). We expect that eventually the procedure stops, and we get one of the following two possibilities:

(1) (Mori fiber space). If φ_{R_i} is a Fano contraction, that is, $\dim Y_i < \dim X_i$, then we set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$. We usually call $f' : (X', \Delta') \to Y_i$ a *Mori fiber space of* (X, Δ) *over* S.

(2) (Minimal model). If $K_{X_i} + \Delta_i$ is f_i-nef, then we again set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$. We can easily check that (X', Δ') is a minimal model of (X, Δ) over S in the sense of Definition 4.3.1.

We can always run the minimal model program discussed in 4.9.1 for non-$\mathbb{Q}$-factorial log canonical pairs. Unfortunately, the termination of this minimal model program is widely open.

Remark 4.9.2. By Theorem 4.5.2 (6), we can run the minimal model program with scaling discussed in 4.4.11 for non-$\mathbb{Q}$-factorial log canonical pairs. Of course, the termination of this minimal model program is an important open problem.

The following lemma is well known. It is essentially contained in [**F13**, Proof of Theorem 4.2.1].

Lemma 4.9.3. *We assume that Conjecture 4.3.6 holds for* $\mathbb{Q}$*-factorial dlt pairs in dimension* n. *Then the minimal model program discussed in 4.9.1 terminates after finitely many steps in dimension* n.

Proof. Let

$$(X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_k, \Delta_k) \dashrightarrow \cdots$$

be a minimal model program discussed in 4.9.1 with $\dim X_0 = n$. Let $\alpha_0 : X^0_0 \to X_0$ be a dlt blow-up, that is, (X^0_0, Δ^0_0) is $\mathbb{Q}$-factorial and dlt such that $K_{X^0_0} + \Delta^0_0 = \alpha^*_0(K_{X_0} + \Delta_0)$ (see Theorem 4.4.21). We run the minimal model program with respet to $K_{X^0_0} + \Delta^0_0$ over Y_0

$$X^0_0 \dashrightarrow X^1_0 \dashrightarrow X^2_0 \dashrightarrow \cdots,$$

and finally get a minimal model $(X_0^{k_0}, \Delta_0^{k_0})$ of (X_0^0, Δ_0^0) over Y_0. Since $(X_1, \Delta_1) \to Y_0$ is the log canonical model of $(X_0^0, \Delta_0^0) \to Y_0$, we have a natural morphism $\alpha_1 : X_0^{k_0} \to X_1$ (see Lemma 4.8.3). We note that $K_{X_0^{k_0}} + \Delta_0^{k_0} = \alpha_1^*(K_{X_1} + \Delta_1)$ by Lemma 4.8.3. We put $(X_1^0, \Delta_1^0) = (X_0^{k_0}, \Delta_0^{k_0})$. We run the minimal model program with respect to $K_{X_1^0} + \Delta_1^0$ over Y_1. Then we obtain a sequence

$$X_1^0 \dashrightarrow X_1^1 \dashrightarrow X_1^2 \dashrightarrow \cdots,$$

and finally get a minimal model $(X_1^{k_1}, \Delta_1^{k_1})$ of (X_1^0, Δ_1^0) over Y_1. By the same reason as above, we have a natural morphism $\alpha_2 : X_1^{k_1} \to X_2$ such that $K_{X_1^{k_1}} + \Delta_1^{k_1} = \alpha_2^*(K_{X_2} + \Delta_2)$ by Lemma 4.8.3. By repeating this procedure, we obtain a $(K_{X_0^0} + \Delta_0^0)$-minimal model program over S:

$$X_0^0 \dashrightarrow \cdots \dashrightarrow X_0^{k_0} = X_1^0 \dashrightarrow \cdots \dashrightarrow X_1^{k_1} = X_2^0 \dashrightarrow \cdots.$$

It terminates by the assumption of this lemma. Therefore, the original minimal model program must terminate after finitely many steps. □

By combining Lemma 4.9.3 with Lemma 4.3.8, it is sufficient to prove Conjecture 4.3.6 for klt pairs.

4.10. MMP for log surfaces

In this section, we discuss the minimal model theory for log surfaces, which is an application of Theorem 4.5.2.

Let us recall the definition of log surfaces.

DEFINITION 4.10.1 (Log surfaces). Let X be a normal algebraic surface and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair (X, Δ) is called a *log surface*. We recall that a *boundary* $\mathbb{R}$-divisor is an effective $\mathbb{R}$-divisor whose coefficients are less than or equal to one.

The following theorem is a special case of Theorem 4.5.2. Note that the non-lc locus of a log surface (X, Δ) is zero-dimensional. Therefore, no curve is contained in the non-lc locus $\mathrm{Nlc}(X, \Delta)$ of (X, Δ).

THEOREM 4.10.2. *Let (X, Δ) be a log surface and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Then we have*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+\Delta\geq 0} + \sum R_j$$

with the following properties.

(1) *R_j is a $(K_X+\Delta)$-negative extremal ray of $\overline{NE}(X/S)$ for every j.*

(2) *Let H be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(K_X+\Delta+H)_{<0}$. In particular, the R_j's are discrete in the half-space $(K_X+\Delta)_{<0}$.*

(3) *Let R be a $(K_X+\Delta)$-negative extremal ray of $\overline{NE}(X/S)$. Then there exists a contraction morphism $\varphi_R : X \to Y$ over S with the following properties.*
   (i) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_R(C)$ is a point if and only if $[C] \in R$.*
   (ii) $\mathcal{O}_Y \simeq (\varphi_R)_*\mathcal{O}_X$.
   (iii) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in R$. Then there exists a line bundle L_Y on Y such that $L \simeq \varphi_R^* L_Y$.*

Theorem 4.10.3 and Theorem 4.10.4 are the main results of [**F29**].

THEOREM 4.10.3 (Minimal model program for log surfaces ([**F29**, Theorem 3.3])). *Let (X,Δ) be a log surface and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. We assume one of the following conditions:*

(A) *X is $\mathbb{Q}$-factorial.*
(B) *(X,Δ) is log canonical.*

Then, by Theorem 4.10.2, we can run the minimal model program over S with respect to $K_X+\Delta$. So, there is a sequence of at most $\rho(X/S)-1$ contractions

$$(X,\Delta)=(X_0,\Delta_0) \xrightarrow{\varphi_0} (X_1,\Delta_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} (X_k,\Delta_k)=(X^*,\Delta^*)$$

over S such that one of the following holds:

(1) (Minimal model). *if $K_X+\Delta$ is pseudo-effective over S, then $K_{X^*}+\Delta^*$ is nef over S. In this case, (X^*,Δ^*) is called a* minimal model of (X,Δ) over S.

(2) (Mori fiber space). *if $K_X+\Delta$ is not pseudo-effective over S, then there is a morphism $g : X^* \to C$ over S such that $-(K_{X^*}+\Delta^*)$ is g-ample, $\dim C < 2$, and $\rho(X^*/C)=1$. We usually call $g : (X^*,\Delta^*) \to C$ a* Mori fiber space of (X,Δ) over S.

We note that X_i is $\mathbb{Q}$-factorial (resp. (X_i,Δ_i) is log canonical) for every i in Case (A) *(resp.* (B)*).*

THEOREM 4.10.4 (Abundance theorem ([**F29**, Theorem 8.1])). *Let (X,Δ) be a log surface and let $\pi : X \to S$ be a proper surjective morphism onto a variety S. Assume that X is $\mathbb{Q}$-factorial or that (X,Δ) is log canonical. We further assume that $K_X+\Delta$ is π-nef. Then $K_X+\Delta$ is π-semi-ample.*

As an easy consequence of Theorem 4.10.3 and Theorem 4.10.4, we have:

Theorem 4.10.5. *Let X be a normal $\mathbb{Q}$-factorial projective surface. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

As a corollary of Theorem 4.10.5, we obtain:

Theorem 4.10.6 ([**F29**, Corollary 4.6]). *Let X be a normal projective surface with only rational singularities. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

Remark 4.10.7. If X is a surface with only rational singularities, then it is well known that X is $\mathbb{Q}$-factorial. If X has only rational singularities in Theorem 4.10.3, then we can check that X_i has only rational singularities for every i (see [**F29**, Proposition 3.7]).

The following theorem, which is not covered by Theorem 4.10.5, is well known to the experts (see, for example, [**Bă**, Theorem 14.42]).

Theorem 4.10.8. *Let X be a normal projective Gorenstein surface. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

Here, we give a proof of Theorem 4.10.8 by using Theorem 4.5.2.

Proof. If $\kappa(X, K_X) \leq 0$, then the statement is obvious. Therefore, we may assume $\kappa(X, K_X) \geq 1$. By taking some crepant resolutions of rational Gorenstein singularities, we may assume that every singularity of X is not log terminal. Let $f : Y \to X$ be a minimal resolution of singularities of X. Then we can write $K_Y + E = f^*K_X$ where E is an effective Cartier divisor with $\mathrm{Exc}(f) = \mathrm{Supp}\, E$ by the negativity lemma (see Lemma 2.3.26). We assume that K_X is not nef. Then $K_Y + E$ is obviously not nef. By Theorem 4.5.2, there is an irreducible rational curve C' on Y such that $C' \cdot (K_Y + E) < 0$ and $(C')^2 < 0$. Note that $f(C') = C$ is not a point by $C' \cdot (K_Y + E) < 0$. Therefore, $C' \cdot E \geq 0$. This implies $C' \cdot K_Y < 0$. Thus, we have $(C')^2 = C' \cdot K_Y = -1$.

Note that E is an effective Cartier divisor. So, we have $C' \cdot E = 0$ by $C' \cdot (K_Y + E) < 0$ and $C' \cdot K_Y = -1$. This implies that $C' \cap E = \emptyset$. Thus C is contained in the smooth locus of X. Note that $C \subset \mathbf{B}(K_X) \subsetneq X$. Let $\varphi : X \to X'$ be the contraction morphism which contracts C to a smooth point. We can replace X with X'. By repeating this process finitely many times, we may assume that K_X is nef. Note that $R(X)$ is preserved by this process. If $\kappa(X, K_X) = 2$, then K_X is semi-ample by the basepoint-free theorem (see Corollary 4.5.6). Note that the non-klt locus of X is zero-dimensional. If $\kappa(X, K_X) = 1$, then it is easy to see that K_X is semi-ample. Anyway, we obtain that the canonical ring $R(X)$ is a finitely generated $\mathbb{C}$-algebra. $\square$

We do not know if Theorem 4.10.8 holds true or not under the weaker assumption that X is only $\mathbb{Q}$-Gorenstein. The following theorem is a partial result for $\mathbb{Q}$-Gorenstein surfaces.

THEOREM 4.10.9. *Let X be a normal projective surface such that K_X is $\mathbb{Q}$-Cartier. Assume that there exists an effective $\mathbb{Q}$-divisor $D = \sum_i d_i D_i$ such that $K_X \sim_{\mathbb{Q}} D$ and that D_i is a $\mathbb{Q}$-Cartier prime divisor on X for every i. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

PROOF. Let $R = \mathbb{R}_{\geq 0}[C]$ be a K_X-negative extremal ray of $\overline{NE}(X)$. Then $C \cdot K_X < 0$ implies that $C \subset \operatorname{Supp} D$. So, $\overline{NE}(X)$ has only finitely many K_X-negative extremal rays. Moreover, the contraction morphism $\varphi_R : X \to Y$ is birational. We note that the exceptional locus of φ_R is an irreducible curve contained in $\operatorname{Supp} D$. This is because each irreducible component of D is $\mathbb{Q}$-Cartier. Therefore, we can check that $K_Y = \varphi_{R*} K_X$ is $\mathbb{Q}$-Cartier, $K_Y \sim_{\mathbb{Q}} \sum_i d_i \varphi_{R*} D_i$, and $\varphi_{R*} D_i$ is $\mathbb{Q}$-Cartier if $\varphi_{R*} D_i \neq 0$. After finitely many contraction morphisms, this program terminates. Since $R(X)$ is preserved by the above process, we may assume that K_X is nef by replacing X with its final model. When $\kappa(X, K_X) = 0$ or 1, $R(X)$ is obviously a finitely generated $\mathbb{C}$-algebra. So, we may assume that K_X is big. Since the non-klt locus of X is zero-dimensional, K_X is semi-ample by the basepoint-free theorem (see Corollary 4.5.6). In particular, $R(X)$ is a finitely generated $\mathbb{C}$-algebra. $\square$

For the details of the minimal model theory for log surfaces, see [**F29**]. For the minimal model theory for log surfaces in positive characteristic, see [**FT**], [**Tana1**], and [**Tana2**]. In positive characteristic, we note the following contraction theorem of Artin–Keel (see [**Ar**] and [**Ke**]).

THEOREM 4.10.10 (Artin–Keel). *Let X be a complete normal algebraic surface defined over an algebraically closed field k of positive characteristic and let H be a nef and big Cartier divisor on X. We put*

$$\mathcal{E}(H) = \{C \,|\, C \text{ is a curve on } X \text{ and } C \cdot H = 0\}.$$

Then $\mathcal{E}(H)$ consists of finitely many irreducible curves on X. Assume that $H|_{\mathbf{E}(H)}$ is semi-ample where

$$\mathbf{E}(H) = \bigcup_{C \in \mathcal{E}(H)} C$$

with the reduced scheme structure. Then H is semi-ample. Therefore,

$$\Phi_{|mH|} : X \to Y$$

is a proper birational morphism onto a normal projective surface Y which contracts $\mathbf{E}(H)$ and is an isomorphism outside $\mathbf{E}(H)$ for a sufficiently large and divisible positive integer m.

For the proof of Theorem 4.10.10, we recommend the reader to see [**FT**, Theorem 2.1], where we gave two different proofs using the Fujita vanishing theorem (see Theorem 3.8.1). Note that Theorem 4.10.10 does not hold in characteristic zero. By Theorem 4.10.10, the minimal model theory for log surfaces is easier in characteristic $p > 0$ than in characteristic zero.

We close this section with an example of a non-$\mathbb{Q}$-factorial log canonical surface.

EXAMPLE 4.10.11 (Non-$\mathbb{Q}$-factorial log canonical surface). Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and let $Y \subset \mathbb{P}^3$ be a cone over C. Then Y is log canonical. In this case, Y is not $\mathbb{Q}$-factorial. We can check it as follows. Let $f : X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \to Y$ be a resolution such that $K_X + E = f^*K_Y$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)|_C$ and E is the exceptional curve. We take $P, Q \in C$ such that $\mathcal{O}_C(P - Q)$ is not a torsion in $\mathrm{Pic}^0(C)$. We consider $D = \pi^*P - \pi^*Q$, where $\pi : X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \to C$. We put $D' = f_*D$. If D' is $\mathbb{Q}$-Cartier, then $mD = f^*mD' + aE$ for some $a \in \mathbb{Z}$ and $m \in \mathbb{Z}_{>0}$. Restrict it to E. Then

$$\mathcal{O}_C(m(P - Q)) \simeq \mathcal{O}_E(aE) \simeq (\mathcal{L}^{-1})^{\otimes a}.$$

Therefore, we obtain that $a = 0$ and $m(P - Q) \sim 0$. This is a contradiction. Thus, D' is not $\mathbb{Q}$-Cartier. In particular, Y is not $\mathbb{Q}$-factorial.

4.11. On semi log canonical pairs

In this final section, we quickly review the main theorem of [**F33**], which says that every quasi-projective semi log canonical pair has a natural quasi-log structure compatible with the original semi log canonical structure, without proof. For the details, see [**F33**].

The notion of semi log canonical singularities was introduced in [**KSB**] in order to investigate deformations of surface singularities and compactifications of moduli spaces for surfaces of general type. By the recent developments of the minimal model program, we know that the appropriate singularities to permit on the varieties at the boundaries of moduli spaces are semi log canonical (see, for example, [**Ale1**], [**Ale2**], [**Ko13**], [**HaKo**, Part III], [**Kv4**], [**Kv7**], [**KSB**], and so on).

First, let us recall the definition of conductors.

DEFINITION 4.11.1 (Conductor). Let X be an equidimensional variety which satisfies Serre's S_2 condition and is normal crossing in codimension one and let $\nu : X^\nu \to X$ be the normalization. Then the *conductor ideal* of X is defined by

$$\mathfrak{cond}_X := \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \subset \mathcal{O}_X.$$

The *conductor* $\mathcal{C}_X$ of X is the subscheme defined by $\mathfrak{cond}_X$. Since X satisfies Serre's S_2 condition and X is normal crossing in codimension one, $\mathcal{C}_X$ is a reduced closed subscheme of pure codimension one in X.

Although we do not use the notion of double normal crossing points and pinch points explicitly in this book, it plays crucial roles for the study of semi log canonical pairs.

DEFINITION 4.11.2 (Double normal crossing points and pinch points). An n-dimensional singularity $(x \in X)$ is called a *double normal crossing point* if it is analytically (or formally) isomorphic to

$$(0 \in (x_0x_1 = 0)) \subset (0 \in \mathbb{C}^{n+1}).$$

It is called a *pinch point* if it is analytically (or formally) isomorphic to

$$(0 \in (x_0^2 = x_1x_2^2)) \subset (0 \in \mathbb{C}^{n+1}).$$

We recall the definition of semi log canonical pairs.

DEFINITION 4.11.3 (Semi log canonical pairs). Let X be an equidimensional algebraic variety that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor whose support does not contain any irreducible components of the conductor of X. The pair (X, Δ) is called a *semi log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) (X^ν, Θ) is log canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$.

Note that if X has only smooth points, double normal crossing points and pinch points then it is easy to see that X is semi log canonical.

The following examples are obvious by the definition of semi log canonical pairs.

EXAMPLE 4.11.4. Let (X, Δ) be a log canonical pair. Then (X, Δ) is a semi log canonical pair.

EXAMPLE 4.11.5. Let (X, Δ) be a semi log canonical pair. Assume that X is normal. Then (X, Δ) is log canonical.

EXAMPLE 4.11.6. Let X be a nodal curve. More generally, X is a normal crossing variety. Then X is semi log canonical.

We introduce the notion of semi log canonical centers. It is a direct generalization of the notion of log canonical centers for log canonical pairs.

DEFINITION 4.11.7 (Slc center). Let (X, Δ) be a semi log canonical pair and let $\nu : X^\nu \to X$ be the normalization. We set

$$K_{X^\nu} + \Theta = \nu^*(K_X + \Delta).$$

A closed subvariety W of X is called a *semi log canonical center* (an *slc center*, for short) *with respect to* (X, Δ) if there exist a resolution of singularities $f : Y \to X^\nu$ and a prime divisor E on Y such that the discrepancy coefficient $a(E, X^\nu, \Theta) = -1$ and $\nu \circ f(E) = W$.

For our purposes, it is very convenient to introduce the notion of slc strata for semi log canonical pairs.

DEFINITION 4.11.8 (Slc stratum). Let (X, Δ) be a semi log canonical pair. A subvariety W of X is called a *semi log canonical stratum* (an *slc stratum*, for short) of the pair (X, Δ) if W is a semi log canonical center with respect to (X, Δ) or W is an irreducible component of X.

The following theorem is the main theorem of [**F33**].

THEOREM 4.11.9 ([**F33**, Theorem 1.2]). *Let (X, Δ) be a quasi-projective semi log canonical pair. Then we can construct a smooth quasi-projective variety M with $\dim M = \dim X + 1$, a simple normal crossing divisor Z on M, a subboundary $\mathbb{R}$-divisor B on M, and a projective surjective morphism $h : Z \to X$ with the following properties.*

(i) *B and Z have no common irreducible components.*
(ii) $\mathrm{Supp}(Z+B)$ *is a simple normal crossing divisor on M.*
(iii) $K_Z+\Delta_Z \sim_{\mathbb{R}} h^*(K_X+\Delta)$ *with* $\Delta_Z = B|_Z$.
(iv) $h_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1}\rceil) \simeq \mathcal{O}_X$.

By properties (i), (ii), (iii), *and* (iv), $[X, K_X+\Delta]$ *has a quasi-log structure with only qlc singularities.*

(v) *The set of slc strata of (X,Δ) gives the set of qlc strata of $[X, K_X+\Delta]$. This means that W is an slc stratum of (X,Δ) if and only if W is the h-image of some stratum of the simple normal crossing pair (Z,Δ_Z).*

By property (v), *the above quasi-log structure of $[X, K_X+\Delta]$ is compatible with the original semi log canonical structure of (X,Δ).*

We note that $h_*\mathcal{O}_Z \simeq \mathcal{O}_X$ by condition (iv).

For the details of quasi-log structures, see Chapter 6.

REMARK 4.11.10. In Theorem 4.11.9, $h: Z \to X$ is not necessarily birational. Note that Z is not always irreducible even when X is irreducible.

EXAMPLE 4.11.11. Let us consider the Whitney umbrella

$$X = (x^2 - y^2 z = 0) \subset \mathbb{C}^3.$$

It is easy to see that X has only semi log canonical singularities. Let $M \to \mathbb{C}^3$ be the blow-up along $C = (x = y = 0)$. We put $Z = X' + E$, where X' is the strict transform of X on M and E is the exceptional divisor of $M \to \mathbb{C}^3$. Then Z is a simple normal crossing variety on a smooth quasi-projective variety M. Let $h: Z \to X$ be the natural morphism. Then we can easily check that $h_*\mathcal{O}_Z \simeq \mathcal{O}_X$ and $K_Z = h^*K_X$. In this case, $h: Z \to X$ gives a natural quasi-log structure which is compatible with the original semi log canonical structure. Note that $h: Z \to X$ is not birational.

By Theorem 4.11.9, we can apply the fundamental theorems for quasi-log schemes (see Chapter 6) to quasi-projective semi log canonical pairs. For the details, see [**F33**]. Moreover, Theorem 4.11.9 drastically increased the importance of the theory of quasi-log schemes.

We note that the proof of Theorem 4.11.9 heavily depends on the recent developments of the theory of partial resolution of singularities for reducible varieties (see [**BM**] and [**BVP**]).

We close this section with the definition of stable varieties.

DEFINITION 4.11.12 (Stable varieties). Let X be a projective variety with only semi log canonical singularities. If K_X is ample, then X is called a *stable* variety.

Definition 4.11.12 is a generalization of the notion of *stable curves*. Roughly speaking, the notion of semi log canonical singularities was introduced in [KSB] in order to define stable surfaces for the compactification problem of moduli spaces of canonically polarized surfaces. Although the approach to the moduli problems in [KSB] is not directly related to Mumford's geometric invariant theory, the notion of semi log canonical singularities appears to be natural from the geometric invariant theoretic viewpoint by [Od].

CHAPTER 5

Injectivity and vanishing theorems

The main purpose of this chapter is to establish the vanishing and torsion-free theorem for simple normal crossing pair (see Theorem 5.1.3), which is indispensable for the theory of quasi-log schemes discussed in Chapter 6.

In Section 5.1, we explain the main results of this chapter. In Section 5.2, we review the notion of $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors again for the reader's convenience. This is because we have to treat reducible varieties from this chapter. In Section 5.3, we quickly review Du Bois complexes and Du Bois singularities. We use them in Section 5.4. In Section 5.4, we prove the Hodge theoretic injectivity theorem. It is a correct and powerful generalization of Kollár's injectivity theorem from the Hodge theoretic viewpoint. In Section 5.5, we generalize the Hodge theoretic injectivity theorem for the relative setting. The relative version of the Hodge theoretic injectivity theorem drastically simplifies the proof of the injectivity, vanishing, and torsion-free theorems for simple normal crossing pairs in Section 5.6. Section 5.6 is devoted to the proof of the injectivity, vanishing, and torsion-free theorems for simple normal crossing pairs. In Section 5.7, we treat the vanishing theorem of Reid–Fukuda type for embedded simple normal crossing pairs. In Section 5.8, we treat embedded normal crossing pairs. Note that the results in Section 5.8 are not necessary for the theory of quasi-log schemes discussed in Chapter 6. So the reader can skip Section 5.8. Section 5.9 contains many nontrivial examples, which help us understand the results discussed in this chapter.

5.1. Main results

In this chapter, we prove the following theorems. Theorem 5.1.1 is a complete generalization of Lemma 3.1.1.

THEOREM 5.1.1 (Hodge theoretic injectivity theorem, see Theorem 5.4.1). *Let (X, Δ) be a simple normal crossing pair such that X is proper and that Δ is a boundary $\mathbb{R}$-divisor on X. Let L be a Cartier divisor on X and let D be an effective Weil divisor on X whose support is contained in* $\operatorname{Supp}\Delta$. *Assume that $L \sim_{\mathbb{R}} K_X + \Delta$. Then the natural* *homomorphism*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ *is injective for every* q.

After we generalize Theorem 5.1.1 for the relative setting, we prove Theorem 5.1.2 as an application. It is a generalization of Kollár's injectivity theorem: Theorem 3.6.2.

THEOREM 5.1.2 (Injectivity theorem for simple normal crossing pairs, see Theorem 5.6.2). *Let* (X, Δ) *be a simple normal crossing pair such that* Δ *is a boundary* $\mathbb{R}$*-divisor on* X*, and let* $\pi : X \to V$ *be a proper morphism between schemes. Let* L *be a Cartier divisor on* X *and let* D *be an effective Cartier divisor that is permissible with respect to* (X, Δ)*. Assume the following conditions.*

(i) $L \sim_{\mathbb{R},\pi} K_X + \Delta + H$,
(ii) H *is a* π*-semi-ample* $\mathbb{R}$*-divisor, and*
(iii) $tH \sim_{\mathbb{R},\pi} D + D'$ *for some positive real number* t*, where* D' *is an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor that is permissible with respect to* (X, Δ).

Then the homomorphisms

$$R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L + D),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$*, are injective for all* q.

By using Theorem 5.1.2, we obtain Theorem 5.1.3.

THEOREM 5.1.3 (see Theorem 5.6.3). *Let* (Y, Δ) *be a simple normal crossing pair such that* Δ *is a boundary* $\mathbb{R}$*-divisor on* Y*. Let* $f : Y \to X$ *be a proper morphism to a scheme* X *and let* L *be a Cartier divisor on* Y *such that* $L - (K_Y + \Delta)$ *is* f*-semi-ample. Let* q *be an arbitrary non-negative integer. Then we have the following properties.*

(i) *Every associated prime of* $R^qf_*\mathcal{O}_Y(L)$ *is the generic point of the* f*-image of some stratum of* (Y, Δ).
(ii) *Let* $\pi : X \to V$ *be a projective morphism to a scheme* V *such that*
$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^*H$$
for some π*-ample* $\mathbb{R}$*-divisor* H *on* X*. Then* $R^qf_*\mathcal{O}_Y(L)$ *is* π_**-acyclic, that is,*
$$R^p\pi_*R^qf_*\mathcal{O}_Y(L) = 0$$
for every $p > 0$.

Theorem 5.1.3 will play crucial roles in the theory of quasi-log schemes discussed in Chapter 6.

We give an easy example, which shows a trouble in [**Am1**].

Example 5.1.4. Let X be a smooth projective variety and let H be a Cartier divisor on X. Let A be a smooth irreducible member of $|2H|$ and let S be a smooth divisor on X such that S and A are disjoint. We put $B = \frac{1}{2}A + S$ and $L = H + K_X + S$. Then $L \sim_{\mathbb{Q}} K_X + B$ and $2L \sim 2(K_X + B)$. We define

$$\mathcal{E} = \mathcal{O}_X(-L + K_X)$$

as in the proof of [**Am1**, Theorem 3.1]. Apply the argument in the proof of [**Am1**, Theorem 3.1]. Then we have a double cover $\pi : Y \to X$ corresponding to $2B \in |\mathcal{E}^{-2}|$. Then

$$\pi_*\Omega^p_Y(\log \pi^*B) \simeq \Omega^p_X(\log B) \oplus \Omega^p_X(\log B) \otimes \mathcal{E}(S).$$

Note that $\Omega^p_X(\log B) \otimes \mathcal{E}$ is not a direct summand of $\pi_*\Omega^p_Y(\log \pi^*B)$. Theorem 3.1 in [**Am1**] claims that the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

are injective for all q. Here, we used the notation in [**Am1**, Theorem 3.1]. In our case, $D = mA$ for some positive integer m. However, Ambro's argument just implies that

$$H^q(X, \mathcal{O}_X(L - \lfloor B \rfloor)) \to H^q(X, \mathcal{O}_X(L - \lfloor B \rfloor + D))$$

is injective for every q. Therefore, his proof works only for the case when $\lfloor B \rfloor = 0$ even if X is smooth.

The proof of [**Am1**, Theorem 3.1] seems to contain a conceptual mistake. The trouble discussed in Example 5.1.4 is serious for applications to the theory of quasi-log schemes. Ambro's proof of the injectivity theorem in [**Am1**] is based on the mixed Hodge structure of

$$H^i(Y - \pi^*B, \mathbb{Z}).$$

It is a standard technique for injectivity and vanishing theorems in the minimal model program. In this chapter, we use the mixed Hodge structure of

$$H^i_c(Y - \pi^*S, \mathbb{Z}),$$

where $H^i_c(Y - \pi^*S, \mathbb{Z})$ is the cohomology group with compact support.

5.1.5 (Observation). Let us explain the main idea of this chapter. Let X be a smooth projective variety with $\dim X = n$ and let Δ be a simple normal crossing divisor on X. The decomposition

$$H^i_c(X-\Delta, \mathbb{C}) = \bigoplus_{p+q=i} H^q(X, \Omega^p_X(\log \Delta) \otimes \mathcal{O}_X(-\Delta)).$$

is suitable for our purposes. The dual statement

$$H^{2n-i}(X-\Delta, \mathbb{C}) = \bigoplus_{p+q=i} H^{n-q}(X, \Omega^{n-p}_X(\log \Delta)),$$

which is well known and is commonly used is not useful for our purposes. Note that the paper [**FF**] supports our approach in this chapter.

Anyway, [**Am1**, 3. Vanishing theorems] seems to be quite short. In this chapter, we establish the injectivity, vanishing, and torsion-free theorems sufficient for the theory of quasi-log schemes discussed in Chapter 6. This chapter covers all the results in [**Am1**, Section 3] and contains several nontrivial generalizations. In [**Am1**, Section 3], Ambro closely followed Esnault–Viehweg's arguments in [**EsVi2**] (see also [**F17**, Chapter 2]). On the other hand, our approach in this chapter is more similar to Kollár's (see, for example, [**Ko6**, Chapter 9] and [**KoMo**, Section 2.4]).

5.2. Simple normal crossing pairs

We quickly recall basic definitions of divisors again. We note that we have to deal with reducible schemes in this paper. For details, see, for example, [**Har5**, Section 2] and [**Li**, Section 7.1].

5.2.1. Let X be a scheme with structure sheaf $\mathcal{O}_X$ and let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$. Let $\mathcal{K}^*_X$ denote the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$, and $\mathcal{O}^*_X$ the sheaf of invertible elements in $\mathcal{O}_X$. We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}^*_X \subset \mathcal{K}^*_X$.

5.2.2 (Cartier, $\mathbb{Q}$-Cartier, and $\mathbb{R}$-Cartier divisors). A *Cartier divisor* D on X is a global section of $\mathcal{K}^*_X/\mathcal{O}^*_X$, that is, D is an element of $H^0(X, \mathcal{K}^*_X/\mathcal{O}^*_X)$. A $\mathbb{Q}$-*Cartier divisor* (resp. $\mathbb{R}$-*Cartier divisor*) is an element of $H^0(X, \mathcal{K}^*_X/\mathcal{O}^*_X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $H^0(X, \mathcal{K}^*_X/\mathcal{O}^*_X) \otimes_{\mathbb{Z}} \mathbb{R}$).

5.2.3 (Linear, $\mathbb{Q}$-linear, and $\mathbb{R}$-linear equivalence). Let D_1 and D_2 be two $\mathbb{R}$-Cartier divisors on X. Then D_1 is *linearly* (resp. $\mathbb{Q}$-*linearly*, or $\mathbb{R}$-*linearly*) *equivalent* to D_2, denoted by $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$, or $D_1 \sim_{\mathbb{R}} D_2$) if

$$D_1 = D_2 + \sum_{i=1}^{k} r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by

$$\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Let $f : X \to Y$ be a morphism. If there is an $\mathbb{R}$-Cartier divisor B on Y such that $D_1 \sim_{\mathbb{R}} D_2 + f^*B$, then D_1 is said to be *relatively* $\mathbb{R}$*-linearly equivalent* to D_2. It is denoted by $D_1 \sim_{\mathbb{R},f} D_2$ or $D_1 \sim_{\mathbb{R},Y} D_2$.

5.2.4 (Supports). Let D be a Cartier divisor on X. The *support* of D, denoted by $\operatorname{Supp} D$, is the subset of X consisting of points x such that a local equation for D is not in $\mathcal{O}_{X,x}^*$. The support of D is a closed subset of X.

5.2.5 (Weil divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors). Let X be an equidimensional variety. We note that X is not necessarily regular in codimension one. A (*Weil*) *divisor* D on X is a finite formal sum

$$\sum_{i=1}^{n} d_i D_i$$

where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is an integer for every i such that $D_i \neq D_j$ for $i \neq j$.

If $d_i \in \mathbb{Q}$ (resp. $d_i \in \mathbb{R}$) for every i, then D is called a $\mathbb{Q}$*-divisor* (resp. $\mathbb{R}$*-divisor*). We define the *round-up* $\lceil D \rceil = \sum_{i=1}^{r} \lceil d_i \rceil D_i$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{i=1}^{r} \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x+1$ (resp. $x-1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We define $D^{<1} = \sum_{d_i<1} d_i D_i$, and so on. We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i.

Let us define simple normal crossing pairs.

Definition 5.2.6 (Simple normal crossing pairs). We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular system of parameters $(x_1, \cdots, x_p, y_1, \cdots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If X is a simple normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. Therefore, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$ (cf. [**Li**, Section 7.1 Corollary 1.19]). It is a Cartier divisor on X and is well-defined up to linear equivalence.

We say that a simple normal crossing pair is *embedded* if there exists a closed embedding $\iota : X \hookrightarrow M$, where M is a smooth variety of dimension $\dim X + 1$. We call M the *ambient space* of (X, Δ).

The author learned the following interesting example from Kento Fujita (cf. [**Ko13**, Remark 1.9]).

EXAMPLE 5.2.7. Let $X_1 = \mathbb{P}^2$ and let C_1 be a line on X_1. Let $X_2 = \mathbb{P}^2$ and let C_2 be a smooth conic on X_2. We fix an isomorphism $\tau : C_1 \to C_2$. By gluing X_1 and X_2 along $\tau : C_1 \to C_2$, we obtain a simple normal crossing surface X such that $-K_X$ is ample (cf. [**Fk1**]). We can check that X can not be embedded into any smooth varieties as a simple normal crossing divisor.

We note that a simple normal crossing pair is called a *semi-snc pair* in [**Ko13**, Definition 1.9].

DEFINITION 5.2.8 (Strata and permissibility). Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \cdots, i_k\} \subset I$. A Cartier divisor D on X is *permissible* if D contains no strata of X in its support. A finite $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisors is called a *permissible* $\mathbb{Q}$*-divisor* (resp. $\mathbb{R}$*-divisor*) on X.

5.2.9. Let X be a simple normal crossing variety. Let $\mathrm{PerDiv}(X)$ be the abelian group generated by permissible Cartier divisors on X and let $\mathrm{Weil}(X)$ be the abelian group generated by Weil divisors on X. Then we can define natural injective homomorphisms of abelian groups

$$\psi : \mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} \to \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}$$

for $\mathbb{K} = \mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Let $\nu : \widetilde{X} \to X$ be the normalization. Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Div}(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} \arrow[r, "\sim", "\widetilde{\psi}"'] & \mathrm{Weil}(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} \arrow[d, "\nu_*"] \\
\mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} \arrow[u, "\nu^*"] \arrow[r, "\psi"'] & \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}
\end{tikzcd}$$

Note that $\mathrm{Div}(\widetilde{X})$ is the abelian group generated by Cartier divisors on $\widetilde{X}$ and that $\widetilde{\psi}$ is an isomorphism since $\widetilde{X}$ is smooth.

By ψ, every permissible Cartier divisor (resp. $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor) can be considered as a Weil divisor (resp. $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor). For $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors, see 5.2.5. Therefore, various operations, for example, $\lfloor D \rfloor$, $D^{<1}$, and so on, make sense for a permissible $\mathbb{R}$-divisor D on X.

We note the following easy example.

EXAMPLE 5.2.10. Let X be a simple normal crossing variety in $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ defined by $xy = 0$. We set $D_1 = (x + z = 0) \cap X$ and $D_2 = (x - z = 0) \cap X$. Then $D = \frac{1}{2}D_1 + \frac{1}{2}D_2$ is a permissible $\mathbb{Q}$-divisor on X. In this case, $\lfloor D \rfloor = (x = z = 0)$ on X. Therefore, $\lfloor D \rfloor$ is not a Cartier divisor on X.

DEFINITION 5.2.11 (Simple normal crossing divisors). Let X be a simple normal crossing variety and let D be a Cartier divisor on X. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X.

REMARK 5.2.12. Let X be a simple normal crossing variety and let D be a $\mathbb{K}$-divisor on X where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. If $\operatorname{Supp} D$ is a simple normal crossing divisor on X and D is $\mathbb{K}$-Cartier, then $\lfloor D \rfloor$ and $\lceil D \rceil$ (resp. $\{D\}$, $D^{<1}$, and so on) are Cartier (resp. $\mathbb{K}$-Cartier) divisors on X (cf. [**BVP**, Section 8]).

The following lemma is easy but important.

LEMMA 5.2.13. *Let X be a simple normal crossing variety and let B be a permissible $\mathbb{R}$-divisor on X such that $\lfloor B \rfloor = 0$. Let A be a Cartier divisor on X. Assume that $A \sim_{\mathbb{R}} B$. Then there exists a permissible $\mathbb{Q}$-divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\lfloor C \rfloor = 0$, and* $\operatorname{Supp} C = \operatorname{Supp} B$.

PROOF. We can write $B = A + \sum_{i=1}^{k} r_i(f_i)$, where $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{R}$ for every i. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of

$\mathcal{O}_X$ (see 5.2.1). Let $P \in X$ be a scheme theoretic point corresponding to some stratum of X. We consider the following affine map

$$\mathbb{K}^k \to H^0(X_P, \mathcal{K}^*_{X_P}/\mathcal{O}^*_{X_P}) \otimes_{\mathbb{Z}} \mathbb{K}$$

given by $(a_1, \cdots, a_k) \mapsto A + \sum_{i=1}^k a_i(f_i)$, where $X_P = \operatorname{Spec} \mathcal{O}_{X,P}$ and $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Then we can check that

$$\mathcal{P} = \{(a_1, \cdots, a_k) \in \mathbb{R}^k \,|\, A + \sum_i a_i(f_i) \text{ is permissible}\} \subset \mathbb{R}^k$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Therefore, we see that

$$\mathcal{S} = \{(a_1, \cdots, a_k) \in \mathcal{P} \,|\, \operatorname{Supp}(A + \sum_i a_i(f_i)) \subset \operatorname{Supp} B\} \subset \mathcal{P}$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Since $(r_1, \cdots, r_k) \in \mathcal{S}$, we know that $\mathcal{S} \neq \emptyset$. We take a point $(s_1, \cdots, s_k) \in \mathcal{S} \cap \mathbb{Q}^k$ which is general in $\mathcal{S}$ and sufficiently close to $(r_1, \cdots, r_k)$ and set

$$C = A + \sum_{i=1}^k s_i(f_i).$$

By construction, C is a permissible $\mathbb{Q}$-divisor such that $C \sim_{\mathbb{Q}} A$, $\lfloor C \rfloor = 0$, and $\operatorname{Supp} C = \operatorname{Supp} B$. $\square$

We need the following important definition in Section 5.6.

Definition 5.2.14 (Strata and permissibility for pairs). Let (X, D) be a simple normal crossing pair. Let $\nu : X^\nu \to X$ be the normalization. We define Θ by the formula

$$K_{X^\nu} + \Theta = \nu^*(K_X + D).$$

Then a *stratum* of (X, D) is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Θ). When $D = 0$, this definition is compatible with Definition 5.2.8. A Cartier divisor B on X is *permissible with respect to* (X, D) if B contains no strata of (X, D) in its support. A finite $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisors with respect to (X, D) is called a *permissible* $\mathbb{Q}$-*divisor* (*resp.* $\mathbb{R}$-*divisor*) *with respect to* (X, D).

5.2.15 (Partial resolution of singularities for reducible varieties). In this chapter, we will repeatedly use the following results on the partial resolution of singularities for reducible varieties.

Theorem 5.2.16 is a special case of [BM, Theorem 1.5].

THEOREM 5.2.16 (Bierstone–Milman). *Let X be an equidimensional variety and let X^{snc} denote the simple normal crossings locus of X. Then there is a morphism $\sigma : X' \to X$ which is a composite of finitely many blow-ups such that*

(1) *X' is a simple normal crossing variety,*
(2) *σ is an isomorphism over X^{snc}, and*
(3) *σ maps $\mathrm{Sing} X'$ birationally onto the closure of $\mathrm{Sing} X^{\mathrm{snc}}$.*

Theorem 5.2.17 is a special case of [BVP, Theorem 1.4].

THEOREM 5.2.17 (Bierstone–Vera Pacheco). *Let X be an equidimensional variety and let Δ be an $\mathbb{R}$-divisor on X. Assume that no component of Δ lies in the singular locus of X. Let $U \subset X$ be the largest open subset such that $(U, \Delta|_U)$ is a simple normal crossing pair. Then there is a morphism $f : \widetilde{X} \to X$ given by a composite of blow-ups such that*

(1) *f is an isomorphism over U,*
(2) *$(\widetilde{X}, \widetilde{\Delta})$ is a simple normal crossing pair, where $\widetilde{\Delta} = f_*^{-1}\Delta + \mathrm{Exc}(f)$.*

For the precise statements and the proof of Theorem 5.2.16 and Theorem 5.2.17, see [BM] and [BVP]. We also recommend the reader to see [BVP, Section 4, Algorithm for the main theorem].

Finally, we recall Grothendieck's Quot scheme for the reader's convenience. For the details, see, for example, [Ni, Theorem 5.14] and [AltKle, Section 2]. We will use it in the proof of Theorem 5.5.1.

THEOREM 5.2.18 (Grothendieck). *Let S be a noetherian scheme, let $\pi : X \to S$ be a projective morphism, and let L be a relatively very ample line bundle on X. Then for any coherent $\mathcal{O}_X$-module E and any polynomial $\Phi \in \mathbb{Q}[\lambda]$, the functor $\mathfrak{Quot}^{\Phi,L}_{E/X/S}$ is representable by a projective S-scheme $\mathrm{Quot}^{\Phi,L}_{E/X/S}$.*

5.3. Du Bois complexes and Du Bois pairs

In this section, we quickly review Du Bois complexes and Du Bois singularities. For the details, see, for example, [Du], [St], [GNPP, Exposé V], [Sa], [PS], [Kv5], and [Ko13, Chapter 6].

5.3.1 (Du Bois complexes). Let X be an algebraic variety. Then we can associate a filtered complex $(\underline{\Omega}^{\bullet}_X, F)$ called the *Du Bois complex* of X in a suitable derived category $D^b_{\mathrm{diff,coh}}(X)$ (see [Du, 1. Complexes filtrés d'opérateurs différentiels d'ordre ≤ 1] and Remark 5.3.2 below). We put

$$\underline{\Omega}^0_X = \mathrm{Gr}^0_F \underline{\Omega}^\bullet_X.$$

There is a natural map $(\Omega^\bullet_X, \sigma) \to (\underline{\Omega}^\bullet_X, F)$. It induces $\mathcal{O}_X \to \underline{\Omega}^0_X$. If $\mathcal{O}_X \to \underline{\Omega}^0_X$ is a quasi-isomorphism, then X is said to have *Du Bois singularities*. We sometimes simply say that X is *Du Bois*. Let Σ be a reduced closed subvariety of X. Then there is a natural map $\rho : (\underline{\Omega}^\bullet_X, F) \to (\underline{\Omega}^\bullet_\Sigma, F)$ in $D^b_{\mathrm{diff,coh}}(X)$. By taking the cone of ρ with a shift by one, we obtain a filtered complex $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$ in $D^b_{\mathrm{diff,coh}}(X)$. Note that $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$ was essentially introduced by Steenbrink in [St, Section 3]. We put

$$\underline{\Omega}^0_{X,\Sigma} = \mathrm{Gr}^0_F \underline{\Omega}^\bullet_{X,\Sigma}.$$

Then there are a map $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$, where $\mathcal{J}_\Sigma$ is the defining ideal sheaf of Σ on X, and the following commutative diagram

$$\begin{tikzcd}
\mathcal{J}_\Sigma \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_\Sigma \arrow[r, "+1"] \arrow[d] & {} \\
\underline{\Omega}^0_{X,\Sigma} \arrow[r] & \underline{\Omega}^0_X \arrow[r] & \underline{\Omega}^0_\Sigma \arrow[r, "+1"] & {}
\end{tikzcd}$$

in the derived category $D^b_{\mathrm{coh}}(X)$ (see also Remark 5.3.4 below).

For completeness, we include the definitions of the derived categories $D^b_{\mathrm{coh}}(X)$, $D^b_{\mathrm{diff,coh}}(X)$, and so on.

REMARK 5.3.2 (Derived categories). Let X be a variety. Then $D(X)$ denotes the derived category of $\mathcal{O}_X$-modules and $D^b_{\mathrm{coh}}(X)$ is the full subcategory of $D(X)$ consisting of complexes whose cohomologies are all coherent and vanish in sufficiently negative and positive degrees. For the details, see [**Har1**].

Let us consider the category $C_{\mathrm{diff}}(X)$. Each object of $C_{\mathrm{diff}}(X)$ is a triple $(K^\bullet, d, F)$ consisting of a complex $(K^\bullet, d)$ of $\mathcal{O}_X$-modules and a decreasing filtration F on $K^\bullet$ such that

(i) $K^\bullet$ is bounded below,
(ii) the filtration F is biregular, that is, for each component K^i of $K^\bullet$, there exist integers m and n such that $F^mK^i = K^i$ and $F^nK^i = 0$,
(iii) d is a differential operator of order at most one and preserves the filtration F, and
(iv) $\mathrm{Gr}^p_F(d) : \mathrm{Gr}^p_F(K^i) \to \mathrm{Gr}^p_F(K^{i+1})$ is $\mathcal{O}_X$-linear for any integers p and i.

Let $D_{\mathrm{diff}}(X)$ be the derived category of the category $C_{\mathrm{diff}}(X)$. For the details, see [**Du**]. In this situation, $D^b_{\mathrm{diff,coh}}(X)$ is the full subcategory of $D_{\mathrm{diff}}(X)$ consisting of $(K^\bullet, d, F)$ such that $\mathrm{Gr}^p_F(K^\bullet)$ is an object of $D^b_{\mathrm{coh}}(X)$ for every p.

By using the theory of mixed Hodge structures on cohomology with compact support, we have the following theorem.

Theorem 5.3.3. *Let X be a variety and let Σ be a reduced closed subvariety of X. We put $j : X - \Sigma \hookrightarrow X$. Then we have the following properties.*

(1) *The complex $(\underline{\Omega}^\bullet_{X,\Sigma})^{\mathrm{an}}$ is a resolution of $j_!\mathbb{C}_{X^{\mathrm{an}}-\Sigma^{\mathrm{an}}}$.*

(2) *If in addition X is proper, then the spectral sequence*

$$E_1^{p,q} = \mathbb{H}^q(X, \underline{\Omega}^p_{X,\Sigma}) \Rightarrow H^{p+q}(X^{\mathrm{an}}, j_!\mathbb{C}_{X^{\mathrm{an}}-\Sigma^{\mathrm{an}}})$$

degenerates at E_1, where $\underline{\Omega}^p_{X,\Sigma} = \mathrm{Gr}^p_F \underline{\Omega}^\bullet_{X,\Sigma}[p]$.

From now on, we will simply write X (resp. $\mathcal{O}_X$ and so on) to express X^{an} (resp. $\mathcal{O}_{X^{\mathrm{an}}}$ and so on) if there is no risk of confusion.

Proof. Here, we use the formulation of [**PS**, §3.3 and §3.4]. We assume that X is proper. We take cubical hyperresolutions $\pi_X : X_\bullet \to X$ and $\pi_\Sigma : \Sigma_\bullet \to \Sigma$ fitting in a commutative diagram.

$$\begin{tikzcd} \Sigma_\bullet \arrow[r] \arrow[d, "\pi_\Sigma"'] & X_\bullet \arrow[d, "\pi_X"] \\ \Sigma \arrow[r, "\iota"'] & X \end{tikzcd}$$

Let $\mathcal{H}dg(X) := R\pi_{X*}\mathcal{H}dg^\bullet(X_\bullet)$ be a mixed Hodge complex of sheaves on X giving the natural mixed Hodge structure on $H^\bullet(X, \mathbb{Z})$ (see [**PS**, Definition 5.32 and Theorem 5.33]). We can obtain a mixed Hodge complex of sheaves $\mathcal{H}dg(\Sigma) := R\pi_{\Sigma*}\mathcal{H}dg^\bullet(\Sigma_\bullet)$ on Σ analogously. Roughly speaking, by forgetting the weight filtration and the $\mathbb{Q}$-structure of $\mathcal{H}dg(X)$ and considering it in $D^b_{\mathrm{diff,coh}}(X)$, we obtain the Du Bois complex $(\underline{\Omega}^\bullet_X, F)$ of X (see [**GNPP**, Exposé V (3.3) Théoréme]). We can also obtain the Du Bois complex $(\underline{\Omega}^\bullet_\Sigma, F)$ of Σ analogously. By taking the mixed cone of $\mathcal{H}dg(X) \to \iota_*\mathcal{H}dg(\Sigma)$ with a shift by one, we obtain a mixed Hodge complex of sheaves on X giving the natural mixed Hodge structure on $H^\bullet_c(X - \Sigma, \mathbb{Z})$ (see [**PS**, 5.5 Relative Cohomology]). Roughly speaking, by forgetting the weight filtration and the $\mathbb{Q}$-structure, we obtain the desired filtered complex $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$ in $D^b_{\mathrm{diff,coh}}(X)$. When X is not proper, we take completions of $\overline{X}$ and $\overline{\Sigma}$ of X and Σ and apply the above arguments to $\overline{X}$ and $\overline{\Sigma}$. Then we restrict everything to X. The properties (1) and (2) obviously hold by the above description of $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$. By the above construction and description of $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$, we know that the map $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ in $D^b_{\mathrm{coh}}(X)$ is induced by natural maps of complexes. $\square$

REMARK 5.3.4. Note that the Du Bois complex $\underline{\Omega}^\bullet_X$ is nothing but the filtered complex $R\pi_{X*}(\Omega^\bullet_{X_\bullet}, F)$. For the details, see [**GNPP**, Exposé V (3.3) Théoréme and (3.5) Définition]. Therefore, the Du Bois complex of the pair (X, Σ) is given by

$$\mathrm{Cone}^\bullet(R\pi_{X*}(\Omega^\bullet_{X_\bullet}, F) \to \iota_* R\pi_{\Sigma*}(\Omega^\bullet_{\Sigma_\bullet}, F))[-1].$$

By the construction of $\underline{\Omega}^\bullet_X$, there is a natural map $a_X : \mathcal{O}_X \to \underline{\Omega}^\bullet_X$ which induces $\mathcal{O}_X \to \underline{\Omega}^0_X$ in $D^b_{\mathrm{coh}}(X)$. Moreover, the composition of $a^{\mathrm{an}}_X : \mathcal{O}_{X^{\mathrm{an}}} \to (\underline{\Omega}^\bullet_X)^{\mathrm{an}}$ with the natural inclusion $\mathbb{C}_{X^{\mathrm{an}}} \subset \mathcal{O}_{X^{\mathrm{an}}}$ induces a quasi-isomorphism $\mathbb{C}_{X^{\mathrm{an}}} \xrightarrow{\simeq} (\underline{\Omega}^\bullet_X)^{\mathrm{an}}$. We have a natural map $a_\Sigma : \mathcal{O}_\Sigma \to \underline{\Omega}^\bullet_\Sigma$ with the same properties as a_X and the following commutative diagram.

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma \\ {\scriptstyle a_X}\downarrow & & \downarrow{\scriptstyle a_\Sigma} \\ \underline{\Omega}^\bullet_X & \longrightarrow & \underline{\Omega}^\bullet_\Sigma \end{array}$$

Therefore, we have a natural map $b : \mathcal{J}_\Sigma \to \underline{\Omega}^\bullet_{X,\Sigma}$ such that b induces $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ in $D^b_{\mathrm{coh}}(X)$ and that the composition of $b^{\mathrm{an}} : (\mathcal{J}_\Sigma)^{\mathrm{an}} \to (\underline{\Omega}^\bullet_{X,\Sigma})^{\mathrm{an}}$ with the natural inclusion $j_!\mathbb{C}_{X^{\mathrm{an}}-\Sigma^{\mathrm{an}}} \subset (\mathcal{J}_\Sigma)^{\mathrm{an}}$ induces a quasi-isomorphism $j_!\mathbb{C}_{X^{\mathrm{an}}-\Sigma^{\mathrm{an}}} \xrightarrow{\simeq} (\underline{\Omega}^\bullet_{X,\Sigma})^{\mathrm{an}}$. We need the weight filtration and the $\mathbb{Q}$-structure in order to prove the E_1-degeneration of Hodge to de Rham type spectral sequence. We used the framework of [**PS**, §3.3 and §3.4] because we had to check that various diagrams related to comparison morphisms are commutative (see [**PS**, Remark 3.23]) for the proof of Theorem 5.3.3 (2) and so on.

Let us recall the definition of Du Bois pairs by [**Kv5**, Definition 3.13].

DEFINITION 5.3.5 (Du Bois pairs). With the notation of 5.3.1 and Theorem 5.3.3, if the map $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ is a quasi-isomorphism, then the pair (X, Σ) is called a *Du Bois pair*.

By the definitions, we can easily check the following useful proposition.

PROPOSITION 5.3.6. *With the notation of 5.3.1 and Theorem 5.3.3, we assume that both X and Σ are Du Bois. Then the pair (X, Σ) is a Du Bois pair, that is, $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ is a quasi-isomorphism.*

Let us recall the following well-known results on Du Bois singularities.

Theorem 5.3.7. *Let X be a normal algebraic variety with only quotient singularities. Then X is Du Bois. Note that X has only rational singularities.*

Theorem 5.3.7 follows from, for example, [**Du**, 5.2. Théorème], [**Kv1**], and so on.

Lemma 5.3.8. *Let X be a variety with closed subvarieties X_1 and X_2 such that $X = X_1 \cup X_2$. Assume that X_1, X_2, and $X_1 \cap X_2$ are Du Bois. Note that, in particular, we assume that $X_1 \cap X_2$ is reduced. Then X is Du Bois.*

For the proof of Lemma 5.3.8, see, for example, [**Schw**, Lemma 3.4].

Although it is dispensable, we will use the notion of Du Bois complexes for the proof of the Hodge theoretic injectivity theorem: Theorem 5.4.1.

5.4. Hodge theoretic injectivity theorems

The main theorem of this section is:

Theorem 5.4.1 (Hodge theoretic injectivity theorem, see [**F36**, Theorem 1.1]). *Let (X, Δ) be a simple normal crossing pair such that X is proper and that Δ is a boundary $\mathbb{R}$-divisor on X. Let L be a Cartier divisor on X and let D be an effective Weil divisor on X whose support is contained in* $\operatorname{Supp}\Delta$. *Assume that $L \sim_{\mathbb{R}} K_X + \Delta$. Then the natural homomorphism*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective for every q.

Theorem 5.4.1 is nothing but [**F36**, Theorem 1.1]. As a very useful special case of Theorem 5.4.1, we have:

Theorem 5.4.2 (see [**Am2**, Theorem 2.3] and [**F36**, Theorem 1.4]). *Let X be a proper smooth algebraic variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that* $\operatorname{Supp}\Delta$ *is a simple normal crossing divisor on X. Let L be a Cartier divisor on X and let D be an effective Cartier divisor on X whose support is contained in* $\operatorname{Supp}\Delta$. *Assume that $L \sim_{\mathbb{R}} K_X + \Delta$. Then the natural homomorphism*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective for every q.

Theorem 5.4.2 is a very useful generalization of Lemma 3.1.1. It is sufficient for many applications of the minimal model program. However, we need Theorem 5.4.1 for the theory of quasi-log schemes discussed in Chapter 6. For various applications of Theorem 5.4.2, see [**F36**, Section 5].

First, let us prove Theorem 5.4.2.

Proof of Theorem 5.4.2. Without loss of generality, we may assume that X is connected. We set $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. By perturbing B, we may assume that B is a $\mathbb{Q}$-divisor (cf. Lemma 5.2.13). We set $\mathcal{M} = \mathcal{O}_X(L - K_X - S)$. Let N be the smallest positive integer such that $NL \sim N(K_X + S + B)$. In particular, NB is an integral Weil divisor. We take the N-fold cyclic cover

$$\pi' : Y' = \mathrm{Spec}_X \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i} \to X$$

associated to the section $NB \in |\mathcal{M}^N|$. More precisely, let $s \in H^0(X, \mathcal{M}^N)$ be a section whose zero divisor is NB. Then the dual of $s : \mathcal{O}_X \to \mathcal{M}^N$ defines an $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}$. Let $Y \to Y'$ be the normalization and let $\pi : Y \to X$ be the composition morphism. It is well known that

$$Y = \mathrm{Spec}_X \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}(\lfloor iB \rfloor).$$

For the details, see [**EsVi3**, 3.5. Cyclic covers]. Note that Y has only quotient singularities by construction. We set $T = \pi^* S$. Let $T = \sum_{i \in I} T_i$ be the irreducible decomposition. Then every irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k}$ has only quotient singularities for every $\{i_1, \cdots, i_k\} \subset I$. Hence it is easy to see that both Y and T have only Du Bois singularities by Theorem 5.3.7 and Lemma 5.3.8 (see also [**I**]). Therefore, the pair (Y, T) is a Du Bois pair by Proposition 5.3.6. This means that $\mathcal{O}_Y(-T) \to \underline{\Omega}^0_{Y,T}$ is a quasi-isomorphism (see also [**FFS**, 3.4]). We note that T is Cartier. Hence $\mathcal{O}_Y(-T)$ is the defining ideal sheaf of T on Y. The E_1-degeneration of

$$E_1^{p,q} = \mathbb{H}^q(Y, \underline{\Omega}^p_{Y,T}) \Rightarrow H^{p+q}(Y, j_!\mathbb{C}_{Y-T})$$

implies that the homomorphism

$$H^q(Y, j_!\mathbb{C}_{Y-T}) \to H^q(Y, \mathcal{O}_Y(-T))$$

induced by the natural inclusion

$$j_!\mathbb{C}_{Y-T} \subset \mathcal{O}_Y(-T)$$

is surjective for every q (see Remark 5.3.4). By taking a suitable direct summand

$$\mathcal{C} \subset \mathcal{M}^{-1}(-S)$$

of

$$\pi_*(j_!\mathbb{C}_{Y-T}) \subset \pi_*\mathcal{O}_Y(-T),$$

we obtain a surjection

$$H^q(X, \mathcal{C}) \to H^q(X, \mathcal{M}^{-1}(-S))$$

induced by the natural inclusion $\mathcal{C} \subset \mathcal{M}^{-1}(-S)$ for every q. We can check the following simple property by examining the monodromy action of the Galois group $\mathbb{Z}/N\mathbb{Z}$ of $\pi : Y \to X$ on $\mathcal{C}$ around $\operatorname{Supp} B$.

LEMMA 5.4.3 (cf. [**KoMo**, Corollary 2.54]). *Let $U \subset X$ be a connected open set such that $U \cap \operatorname{Supp} \Delta \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

PROOF OF LEMMA 5.4.3. If $U \cap \operatorname{Supp} B \neq \emptyset$, then $H^0(U, \mathcal{C}|_U) = 0$ since the monodromy action on $\mathcal{C}$ around $\operatorname{Supp} B$ is nontrivial. If $U \cap \operatorname{Supp} S \neq \emptyset$, then $H^0(U, \mathcal{C}|_U) = 0$ since $\mathcal{C}$ is a direct summand of $\pi_*(j_!\mathbb{C}_{Y-T})$ and $T = \pi^*S$. $\square$

This property is utilized by the following fact. The proof of Lemma 5.4.4 is obvious.

LEMMA 5.4.4 (cf. [**KoMo**, Lemma 2.55]). *Let F be a sheaf of Abelian groups on a topological space V and let F_1 and F_2 be subsheaves of F. Let Z be a closed subset of V. Assume that*

(1) *$F_2|_{V-Z} = F|_{V-Z}$, and*
(2) *if U is connected, open and $U \cap Z \neq \emptyset$, then $H^0(U, F_1|U) = 0$.*

Then F_1 is a subsheaf of F_2.

As a corollary, we obtain:

COROLLARY 5.4.5 (cf. [**KoMo**, Corollary 2.56]). *Let $M \subset \mathcal{M}^{-1}(-S)$ be a subsheaf such that $M|_{X-\operatorname{Supp}\Delta} = \mathcal{M}^{-1}(-S)|_{X-\operatorname{Supp}\Delta}$. Then the injection*

$$\mathcal{C} \to \mathcal{M}^{-1}(-S)$$

factors as

$$\mathcal{C} \to M \to \mathcal{M}^{-1}(-S).$$

Therefore,

$$H^q(X, M) \to H^q(X, \mathcal{M}^{-1}(-S))$$

is surjective for every q.

Proof of Corollary 5.4.5. The first part is clear from Lemma 5.4.3 and Lemma 5.4.4. This implies that we have maps

$$H^q(X, \mathcal{C}) \to H^q(X, M) \to H^q(X, \mathcal{M}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right. $\square$

Therefore, $H^q(X, \mathcal{M}^{-1}(-S-D)) \to H^q(X, \mathcal{M}^{-1}(-S))$ is surjective for every q. By Serre duality, we obtain that

$$H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{M}(S)) \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{M}(S+D))$$

is injective for every q. This means that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

is injective for every q. $\square$

Next, let us prove Theorem 5.4.1, the main theorem of this section. The proof of Theorem 5.4.2 given above works for Theorem 5.4.1 with some minor modifications.

Proof of Theorem 5.4.1. Without loss of generality, we may assume that X is connected. We can take an effective Cartier divisor D' on X such that $D'-D$ is effective and $\operatorname{Supp} D' \subset \operatorname{Supp} \Delta$. Therefore, by replacing D with D', we may assume that D is a Cartier divisor. We set $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. By Lemma 5.2.13, we may assume that B is a $\mathbb{Q}$-divisor. We set $\mathcal{M} = \mathcal{O}_X(L - K_X - S)$. Let N be the smallest positive integer such that $NL \sim N(K_X + S + B)$. We define an $\mathcal{O}_X$-algebra structure of $\bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}(\lfloor iB \rfloor)$ by $s \in H^0(X, \mathcal{M}^N)$ with $(s=0) = NB$. We set

$$\pi : Y = \operatorname{Spec}_X \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}(\lfloor iB \rfloor) \to X$$

and $T = \pi^* S$. Let $Y = \sum_{j \in J} Y_j$ be the irreducible decomposition. Then every irreducible component of $Y_{j_1} \cap \cdots \cap Y_{j_l}$ has only quotient singularities for every $\{j_1, \cdots, j_l\} \subset J$ by construction. Let $T = \sum_{i \in I} T_i$ be the irreducible decomposition. Then every irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k}$ has only quotient singularities for every $\{i_1, \cdots, i_k\} \subset I$ by construction. Hence it is easy to see that both Y and T are Du Bois by Theorem 5.3.7 and Lemma 5.3.8 (see also [I]). Therefore, the pair (Y, T) is a Du Bois pair by Proposition 5.3.6. This means that $\mathcal{O}_Y(-T) \to \underline{\Omega}^0_{Y,T}$ is a quasi-isomorphism (see also [FFS, 3.4]). We note that T is Cartier. Hence $\mathcal{O}_Y(-T)$ is the defining ideal sheaf of T on Y. The E_1-degeneration of

$$E_1^{p,q} = \mathbb{H}^q(Y, \underline{\Omega}^p_{Y,T}) \Rightarrow H^{p+q}(Y, j_!\mathbb{C}_{Y-T})$$

implies that the homomorphism

$$H^q(Y, j_!\mathbb{C}_{Y-T}) \to H^q(Y, \mathcal{O}_Y(-T))$$

induced by the natural inclusion

$$j_!\mathbb{C}_{Y-T} \subset \mathcal{O}_Y(-T)$$

is surjective for every q (see Remark 5.3.4). By taking a suitable direct summand

$$\mathcal{C} \subset \mathcal{M}^{-1}(-S)$$

of

$$\pi_*(j_!\mathbb{C}_{Y-T}) \subset \pi_*\mathcal{O}_Y(-T),$$

we obtain a surjection

$$H^q(X, \mathcal{C}) \to H^q(X, \mathcal{M}^{-1}(-S))$$

induced by the natural inclusion $\mathcal{C} \subset \mathcal{M}^{-1}(-S)$ for every q. It is easy to see that Lemma 5.4.3 holds for this new setting. Hence Corollary 5.4.5 also holds without any modifications. Therefore,

$$H^q(X, \mathcal{M}^{-1}(-S-D)) \to H^q(X, \mathcal{M}^{-1}(-S))$$

is surjective for every q. By Serre duality, we obtain that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

is injective for every q. $\square$

We close this section with an easy application of Theorem 5.4.1.

Corollary 5.4.6 (Kodaira vanishing theorem for simple normal crossing varieties). *Let X be a projective simple normal crossing variety and let $\mathcal{L}$ be an ample line bundle on X. Then $H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}) = 0$ for every $q > 0$.*

Proof. We take a general member $\Delta \in |\mathcal{L}^l|$ for some positive large number l. Then we can find a Cartier divisor M on X such that $M \sim_{\mathbb{Q}} K_X + \frac{1}{l}\Delta$ and that $\mathcal{O}_X(K_X) \otimes \mathcal{L} \simeq \mathcal{O}_X(M)$. Then, by Theorem 5.4.1,

$$H^q(X, \mathcal{O}_X(M)) \to H^q(X, \mathcal{O}_X(M + m\Delta))$$

is injective for every q and any positive integer m. Since Δ is ample, Serre's vanishing theorem implies that $H^q(X, \mathcal{O}_X(M)) = 0$ for every $q > 0$. $\square$

5.5. Relative Hodge theoretic injectivity theorem

In this section, we generalize Theorem 5.4.1 for the relative setting. It is much more useful than Theorem 5.4.1.

THEOREM 5.5.1 (Relative Hodge theoretic injectivity theorem, see [**F36**, Theorem 6.1]). *Let (X, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $\pi : X \to V$ be a proper morphism between schemes and let L be a Cartier divisor on X and let D be an effective Weil divisor on X whose support is contained in* $\operatorname{Supp}\Delta$. *Assume that $L \sim_{\mathbb{R},\pi} K_X + \Delta$, that is, there is an $\mathbb{R}$-Cartier divisor B on V such that $L \sim_{\mathbb{R}} K_X + \Delta + \pi^*B$. Then the natural homomorphism*

$$R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L+D)$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective for every q.

By using [**BVP**] (see Theorem 5.2.17 and Lemma 5.5.2), we can reduce Theorem 5.5.1 to Theorem 5.4.1.

LEMMA 5.5.2. *Let $f : Z \to W$ be a proper morphism from a simple normal crossing pair (Z, Δ) to a scheme W. Let $\overline{W}$ be a projective scheme such that $\overline{W}$ contains W as a Zariski open dense subset. Then there exist a proper simple normal crossing pair $(\overline{Z}, \overline{\Delta})$ that is a compactification of (Z, Δ) and a proper morphism $\overline{f} : \overline{Z} \to \overline{W}$ that compactifies $f : Z \to W$. Moreover, $\overline{Z} \setminus Z$ is a divisor on $\overline{Z}$,* $\operatorname{Supp}\overline{\Delta} \cup \operatorname{Supp}(\overline{Z} \setminus Z)$ *is a simple normal crossing divisor on $\overline{Z}$, and $\overline{Z} \setminus Z$ has no common irreducible components with $\overline{\Delta}$. We note that we can make $\overline{\Delta}$ a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on $\overline{Z}$ when so is Δ on Z, where $\mathbb{K}$ is $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. When f is projective, we can make $\overline{Z}$ projective.*

PROOF OF LEMMA 5.5.2. Let $\overline{\Delta} \subset \overline{Z}$ be any compactification of $\Delta \subset Z$. By blowing up $\overline{Z}$ inside $\overline{Z} \setminus Z$, we may assume that $f : Z \to W$ extends to $\overline{f} : \overline{Z} \to \overline{W}$, $\overline{Z}$ is a simple normal crossing variety, and $\overline{Z} \setminus Z$ is of pure codimension one (see Theorem 5.2.16 and [**BM**, Theorem 1.5]). By Theorem 5.2.17 (see also [**BVP**, Theorem 1.4]), we can construct a desired compactification. Note that we can make $\overline{\Delta}$ a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor by the argument in [**BVP**, Section 8]. $\square$

REMARK 5.5.3. We put $X = (x^2 - zy^2 = 0) \subset \mathbb{C}^3$. Then $X \setminus \{0\}$ is a normal crossing variety (see Definition 5.8.3 below). In this case, there is no normal crossing complete variety which contains $X \setminus \{0\}$ as a Zariski open subset. For the details, see [**F12**, 3.6 Whitney umbrella]. Therefore, we can not directly apply the arguments in this section to normal crossing varieties. For the details of the injectivity, torsion-free, and vanishing theorems for normal crossing pairs, see Section 5.8 below.

Let us start the proof of Theorem 5.5.1.

Proof of Theorem 5.5.1. By shrinking V, we may assume that V is affine and $L \sim_{\mathbb{R}} K_X + \Delta$. Without loss of generality, we may assume that X is connected. Let $\overline{V}$ be a projective compactification of V. By Lemma 5.5.2, we can compactify $\pi : X \to V$ to $\overline{\pi} : \overline{X} \to \overline{V}$. We put $\lfloor \Delta \rfloor = S$ and $B = \{\Delta\}$. Let $\overline{B}$ (resp. $\overline{S}$) be the closure of B (resp. S) on $\overline{X}$. We may assume that $\overline{S}$ is Cartier and $\overline{B}$ is $\mathbb{R}$-Cartier (see Lemma 5.5.2). We construct a coherent sheaf $\mathcal{F}$ on $\overline{X}$ which is an extension of $\mathcal{O}_X(L)$. We consider Grothendieck's Quot scheme $\mathrm{Quot}^{1,\mathcal{O}_{\overline{X}}}_{\mathcal{F}/\overline{X}/\overline{X}}$ (see Theorem 5.2.18). Note that the restriction of $\mathrm{Quot}^{1,\mathcal{O}_{\overline{X}}}_{\mathcal{F}/\overline{X}/\overline{X}}$ to X is nothing but X because $\mathcal{F}|_X = \mathcal{O}_X(L)$ is a line bundle on X. Therefore, the structure morphism from $\mathrm{Quot}^{1,\mathcal{O}_{\overline{X}}}_{\mathcal{F}/\overline{X}/\overline{X}}$ to $\overline{X}$ has a section s over X. By taking the closure of $s(X)$ in $\mathrm{Quot}^{1,\mathcal{O}_{\overline{X}}}_{\mathcal{F}/\overline{X}/\overline{X}}$, we have a compactification $X^\dagger$ of X and a line bundle $\mathcal{L}$ on $X^\dagger$ with $\mathcal{L}|_X = \mathcal{O}_X(L)$. If necessary, we take more blow-ups of $X^\dagger$ outside X by Theorem 5.2.17 (see also [BVP, Theorem 1.4]). Then we obtain a new compactification $\overline{X}$ and a Cartier divisor $\overline{L}$ on $\overline{X}$ with $\overline{L}|_X = L$ (cf. Lemma 5.5.2). In this situation, $\overline{L} \sim_{\mathbb{R}} (K_{\overline{X}} + \overline{\Delta})$, where $\overline{\Delta} = \overline{S} + \overline{B}$, does not necessarily hold. We can write

$$\sum_i b_i(f_i) = L - (K_Y + \Delta),$$

where b_i is a real number and $f_i \in \Gamma(X, \mathcal{K}^*_X)$ for every i. We set

$$E = \sum_i b_i(f_i) - (\overline{L} - (K_{\overline{X}} + \overline{\Delta})).$$

We note that we can see $f_i \in \Gamma(\overline{X}, \mathcal{K}^*_{\overline{X}})$ for every i (cf. [Li, Section 7.1 Proposition 1.15]). Then we have

$$\overline{L} + \lceil E \rceil \sim_{\mathbb{R}} K_{\overline{X}} + \overline{\Delta} + \{-E\}.$$

By the above construction, it is obvious that $\mathrm{Supp}\, E \subset \overline{X} \setminus X$. Let $\overline{D}$ be the closure of D in $\overline{X}$. It is sufficient to prove that the map

$$\varphi^q : R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil) \to R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{D})$$

induced by the natural inclusion $\mathcal{O}_{\overline{X}} \to \mathcal{O}_{\overline{X}}(\overline{D})$ is injective for every q. Suppose that φ^q is not injective for some q. Let A be a sufficiently ample general Cartier divisor on $\overline{V}$ such that $H^0(\overline{V}, \mathrm{Ker}\varphi^q \otimes \mathcal{O}_{\overline{V}}(A)) \neq 0$. In this case, the map

$$\begin{aligned}&H^0(\overline{V}, R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil) \otimes \mathcal{O}_{\overline{V}}(A))\\ &\to H^0(\overline{V}, R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{D}) \otimes \mathcal{O}_{\overline{V}}(A))\end{aligned}$$

induced by φ^q is not injective. Since A is sufficiently ample, this implies that

$$\begin{aligned}&H^q(\overline{X}, \mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{\pi}^*A))\\ &\to H^q(\overline{X}, \mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{\pi}^*A + \overline{D}))\end{aligned}$$

is not injective. Since

$$\overline{L} + \lceil E \rceil + \overline{\pi}^*A \sim_{\mathbb{R}} K_{\overline{X}} + \overline{\Delta} + \{-E\} + \overline{\pi}^*A,$$

this contradicts Theorem 5.4.1. Hence φ^q is injective for every q. $\square$

5.6. Injectivity, vanishing, and torsion-free theorems

The next lemma is an easy generalization of the vanishing theorem of Reid–Fukuda type for simple normal crossing pairs, which is a very special case of Theorem 5.6.3 (i). However, we need Lemma 5.6.1 for our proof of Theorem 5.6.3.

LEMMA 5.6.1 (Relative vanishing lemma). *Let $f : Y \to X$ be a proper morphism from a simple normal crossing pair (Y, Δ) to a scheme X such that Δ is a boundary $\mathbb{R}$-divisor on Y. We assume that f is an isomorphism at the generic point of any stratum of the pair (Y, Δ). Let L be a Cartier divisor on Y such that $L \sim_{\mathbb{R},f} K_Y + \Delta$. Then $R^qf_*\mathcal{O}_Y(L) = 0$ for every $q > 0$.*

PROOF. By shrinking X, we may assume that $L \sim_{\mathbb{R}} K_Y + \Delta$. By applying Lemma 5.2.13 to $\{\Delta\}$, we may further assume that Δ is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_Y + \Delta$.

STEP 1. We assume that Y is irreducible. In this case, $L-(K_Y+\Delta)$ is nef and log big over X with respect to the pair (Y, Δ), that is, $L - (K_Y + \Delta)$ is nef and big over X and $(L - (K_Y + \Delta))|_W$ is big over $f(W)$ for every log canonical center W of the pair (Y, Δ) (see Definition 3.2.10 and Definition 5.7.2 below). Therefore, $R^qf_*\mathcal{O}_Y(L) = 0$ for every $q > 0$ by the vanishing theorem of Reid–Fukuda type (see, for example, Theorem 3.2.11).

STEP 2. Let Y_1 be an irreducible component of Y and let Y_2 be the union of the other irreducible components of Y. Then we have a short exact sequence

$$0 \to \mathcal{O}_{Y_1}(-Y_2|_{Y_1}) \to \mathcal{O}_Y \to \mathcal{O}_{Y_2} \to 0.$$

We set $L' = L|_{Y_1} - Y_2|_{Y_1}$. Then we have a short exact sequence

$$0 \to \mathcal{O}_{Y_1}(L') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y_2}(L|_{Y_2}) \to 0$$

and $L' \sim_{\mathbb{Q}} K_{Y_1} + \Delta|_{Y_1}$. On the other hand, we can check that

$$L|_{Y_2} \sim_{\mathbb{Q}} K_{Y_2} + Y_1|_{Y_2} + \Delta|_{Y_2}.$$

We have already known that $R^q f_* \mathcal{O}_{Y_1}(L') = 0$ for every $q > 0$ by Step 1. By induction on the number of the irreducible components of Y, we have $R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) = 0$ for every $q > 0$. Therefore, $R^q f_* \mathcal{O}_Y(L) = 0$ for every $q > 0$ by the exact sequence:

$$\cdots \to R^q f_* \mathcal{O}_{Y_1}(L') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) \to \cdots.$$

So, we finish the proof of Lemma 5.6.1. □

It is the time to state the main injectivity theorem for simple normal crossing pairs. Our formulation of Theorem 5.6.2 is indispensable for the proof of our main theorem: Theorem 5.6.3.

THEOREM 5.6.2 (Injectivity theorem for simple normal crossing pairs). *Let (X, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X and let $\pi : X \to V$ be a proper morphism between schemes. Let L be a Cartier divisor on X and let D be an effective Cartier divisor that is permissible with respect to (X, Δ). Assume the following conditions.*

(i) $L \sim_{\mathbb{R},\pi} K_X + \Delta + H$,
(ii) *H is a π-semi-ample $\mathbb{R}$-divisor, and*
(iii) *$tH \sim_{\mathbb{R},\pi} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to (X, Δ).*

Then the homomorphisms

$$R^q \pi_* \mathcal{O}_X(L) \to R^q \pi_* \mathcal{O}_X(L + D),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Theorem 5.6.2 is new and is a relative version of [**F32**, Theorem 3.4].

PROOF OF THEOREM 5.6.2. We set $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$ throughout this proof. We obtain a projective birational morphism $f : Y \to X$ from a simple normal crossing variety Y such that f is an isomorphism over $X \setminus \mathrm{Supp}(D + D' + B)$, and that the union of the support of $f^*(S + B + D + D')$ and the exceptional locus of f has a simple normal crossing support on Y by Theorem 5.2.17 (see also [**BVP**, Theorem

1.4]). Let B' be the strict transform of B on Y. We may assume that $\operatorname{Supp} B'$ is disjoint from any strata of Y that are not irreducible components of Y by taking blow-ups. We write

$$K_Y + S' + B' = f^*(K_X + S + B) + E,$$

where S' is the strict transform of S and E is f-exceptional. By the construction of $f : Y \to X$, S' is Cartier and B' is $\mathbb{R}$-Cartier. Therefore, E is also $\mathbb{R}$-Cartier. It is easy to see that $E_+ = \lceil E \rceil \geq 0$. We set $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is $\mathbb{R}$-Cartier because $\operatorname{Supp} E$ is simple normal crossing on Y (cf. Remark 5.2.12). Without loss of generality, we may assume that V is affine. Since f^*H is an $\mathbb{R}_{>0}$-linear combination of semi-ample Cartier divisors, we can write $f^*H \sim_{\mathbb{R}} \sum_i a_i H_i$, where $0 < a_i < 1$ and H_i is a general Cartier divisor on Y for every i. We set

$$B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D + D') + (1 - \varepsilon) \sum_i a_i H_i$$

for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{R}} K_Y + S' + B''$. By construction, $\lfloor B'' \rfloor = 0$, the support of $S' + B''$ is simple normal crossing on Y, and $\operatorname{Supp} B'' \supset \operatorname{Supp} f^*D$. So, Theorem 5.5.1 implies that the homomorphisms

$$R^q(\pi \circ f)_* \mathcal{O}_Y(L') \to R^q(\pi \circ f)_* \mathcal{O}_Y(L' + f^*D)$$

are injective for all q. By Lemma 5.6.1, $R^q f_* \mathcal{O}_Y(L') = 0$ for every $q > 0$ and it is easy to see that $f_* \mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms

$$R^q \pi_* \mathcal{O}_X(L) \to R^q \pi_* \mathcal{O}_X(L + D)$$

are injective for all q. $\square$

Since we formulated Theorem 5.6.2 in the relative setting, the proof of Theorem 5.6.3, which is nothing but [**F32**, Theorem 1.1], is much simpler than the proof given in [**F32**].

Theorem 5.6.3 (Vanishing and torsion-free theorem for simple normal crossing pairs, see [**F32**, Theorem 1.1]). *Let (Y, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on Y. Let $f : Y \to X$ be a proper morphism to a scheme X and let L be a Cartier divisor on Y such that $L - (K_Y + \Delta)$ is f-semi-ample. Let q be an arbitrary non-negative integer. Then we have the following properties.*

(i) *Every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ).*

(ii) *Let $\pi : X \to V$ be a projective morphism to a scheme V such that*

$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^*H$$

for some π-ample $\mathbb{R}$-divisor H on X. Then $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic, that is,

$$R^p\pi_*R^qf_*\mathcal{O}_Y(L) = 0$$

for every $p > 0$.

Proof of Theorem 5.6.3 (i). Without loss of generality, we may assume that X is affine. Suppose that $R^qf_*\mathcal{O}_Y(L)$ has a local section whose support does not contain the f-images of any strata of (Y,Δ). More precisely, let U be a non-empty Zariski open set and let $s \in \Gamma(U, R^qf_*\mathcal{O}_Y(L))$ be a non-zero section of $R^qf_*\mathcal{O}_Y(L)$ on U whose support $V \subset U$ does not contain the f-images of any strata of (Y,Δ). Without loss of generality, we may further assume that U is affine and $X = U$ by shrinking X. Then we can find a Cartier divisor A on X with the following properties:

(a) f^*A is permissible with respect to (Y,Δ), and
(b) $R^qf_*\mathcal{O}_Y(L) \to R^qf_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

This contradicts Theorem 5.6.2. Therefore, the support of every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ contains the f-image of some stratum of (Y,Δ), equivalently, the support of every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ is equal to the union of the f-images of some strata of (Y,Δ). This means that every associated prime of $R^qf_*\mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y,Δ). $\square$

From now on, we prove Theorem 5.6.3 (ii).

Proof of Theorem 5.6.3 (ii). Without loss of generality, we may assume that V is affine. In this case, we can write $H \sim_{\mathbb{R}} H_1 + H_2$, where H_1 (resp. H_2) is a π-ample $\mathbb{Q}$-divisor (resp. a π-ample $\mathbb{R}$-divisor) on X. So, we can write $H_2 \sim_{\mathbb{R}} \sum_i a_iA_i$, where $0 < a_i < 1$ and A_i is a general very ample Cartier divisor over V on X for every i. Replacing B (resp. H) with $B + \sum_i a_if^*A_i$ (resp. H_1), we may assume that H is a π-ample $\mathbb{Q}$-divisor. We take a general member $A \in |mH|$, where m is a sufficiently large and divisible positive integer, such that $A' = f^*A$ and $R^qf_*\mathcal{O}_Y(L + A')$ is π_*-acyclic for all q. By Theorem 5.6.3 (i), we have the following short exact sequences

$$0 \to R^qf_*\mathcal{O}_Y(L) \to R^qf_*\mathcal{O}_Y(L + A') \to R^qf_*\mathcal{O}_{A'}(L + A') \to 0.$$

for all q. Note that $R^qf_*\mathcal{O}_{A'}(L+A')$ is π_*-acyclic by induction on $\dim X$ and that $R^qf_*\mathcal{O}_Y(L + A')$ is also π_*-acyclic by the above assumption.

Thus, $E_2^{p,q}=0$ for $p\geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{tikzcd} E_2^{p,q}=R^p\pi_*R^qf_*\mathcal{O}_Y(L) \arrow[r, Rightarrow] \arrow[d, "\varphi^{pq}"'] & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L) \arrow[d, "\varphi^{p+q}"'] \\ \overline{E}_2^{p,q}=R^p\pi_*R^qf_*\mathcal{O}_Y(L+A') \arrow[r, Rightarrow] & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L+A') \end{tikzcd}$$

We note that φ^{1+q} is injective by Theorem 5.6.2. We have that

$$E_2^{1,q}\xrightarrow{\alpha}R^{1+q}(\pi\circ f)_*\mathcal{O}_Y(L)$$

is injective by the fact that $E_2^{p,q}=0$ for $p\geq 2$. We also have that $\overline{E}_2^{1,q}=0$ by the above assumption. Therefore, we obtain $E_2^{1,q}=0$ since the injection

$$E_2^{1,q}\xrightarrow{\alpha}R^{1+q}(\pi\circ f)_*\mathcal{O}_Y(L)\xrightarrow{\varphi^{1+q}}R^{1+q}(\pi\circ f)_*\mathcal{O}_Y(L+A')$$

factors through $\overline{E}_2^{1,q}=0$. This implies that $R^p\pi_*R^qf_*\mathcal{O}_Y(L)=0$ for every $p>0$. □

As an application of Theorem 5.6.3, we have:

THEOREM 5.6.4 (Kodaira vanishing theorem for log canonical pairs, see [**F18**, Theorem 4.4]). *Let (X,Δ) be a log canonical pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let L be a $\mathbb{Q}$-Cartier Weil divisor on X such that $L-(K_X+\Delta)$ is π-ample, where $\pi: X\to V$ is a projective morphism. Then $R^q\pi_*\mathcal{O}_X(L)=0$ for every $q>0$.*

PROOF. Let $f: Y\to X$ be a resolution of singularities of X such that

$$K_Y=f^*(K_X+\Delta)+\sum_i a_iE_i$$

with $a_i\geq -1$ for every i. We may assume that $\sum_i E_i\cup \operatorname{Supp} f^*L$ is a simple normal crossing divisor on Y. We put

$$E=\sum_i a_iE_i$$

and

$$F=\sum_{a_j=-1}(1-b_j)E_j,$$

where $b_j=\operatorname{mult}_{E_j}\{f^*L\}$. We note that $A=L-(K_X+\Delta)$ is π-ample by assumption. We have

$$\begin{aligned} f^*A&=f^*L-f^*(K_X+\Delta)\\ &=\lceil f^*L+E+F\rceil-(K_Y+F+\{-(f^*L+E+F)\}). \end{aligned}$$

We can easily check that

$$f_*\mathcal{O}_Y(\lceil f^*L+E+F\rceil)\simeq \mathcal{O}_X(L)$$

and that $F+\{-(f^*L+E+F)\}$ has a simple normal crossing support and is a boundary $\mathbb{R}$-divisor on Y. By Theorem 5.6.3 (ii), we obtain that $\mathcal{O}_X(L)$ is π_*-acyclic. Thus, we have $R^q\pi_*\mathcal{O}_X(L)=0$ for every $q>0$. $\square$

We note that Theorem 5.6.4 contains a complete form of [**Kv2**, Theorem 0.3] as a corollary. For the related topics, see [**KSS**, Corollary 1.3].

Corollary 5.6.5 (Kodaira vanishing theorem for log canonical varieties). *Let X be a projective log canonical variety and let L be an ample Cartier divisor on X. Then*

$$H^q(X,\mathcal{O}_X(K_X+L))=0$$

for every $q>0$. Furthermore, if we assume that X is Cohen–Macaulay, then $H^q(X,\mathcal{O}_X(-L))=0$ for every $q<\dim X$.

Remark 5.6.6. We can see that Corollary 5.6.5 is contained in [**F6**, Theorem 2.6], which is a very special case of Theorem 5.6.3 (ii). The author forgot to state Corollary 5.6.5 explicitly in [**F6**]. There, we do not need embedded simple normal crossing pairs.

Note that there are typos in the proof of [**F6**, Theorem 2.6]. In the commutative diagram, $R^if_*\omega_X(D)$'s should be replaced by $R^jf_*\omega_X(D)$'s.

We close this section with an easy example.

Example 5.6.7. Let X be a projective log canonical threefold which has the following properties: (i) there exists a projective birational morphism $f:Y\to X$ from a smooth projective threefold, and (ii) the exceptional locus E of f is an Abelian surface with $K_Y=f^*K_X-E$. For example, X is a cone over a normally projective Abelian surface in $\mathbb{P}^N$ and $f:Y\to X$ is the blow-up at the vertex of X. Let L be an ample Cartier divisor on X. By the Leray spectral sequence, we have

$$\begin{aligned}0&\to H^1(X,f_*f^*\mathcal{O}_X(-L))\to H^1(Y,f^*\mathcal{O}_X(-L))\\&\to H^0(X,R^1f_*f^*\mathcal{O}_X(-L))\to H^2(X,f_*f^*\mathcal{O}_X(-L))\\&\to H^2(Y,f^*\mathcal{O}_X(-L))\to\cdots.\end{aligned}$$

Therefore, we obtain

$$H^2(X,\mathcal{O}_X(-L))\simeq H^0(X,\mathcal{O}_X(-L)\otimes R^1f_*\mathcal{O}_Y),$$

because $H^1(Y, f^*\mathcal{O}_X(-L)) = H^2(Y, f^*\mathcal{O}_X(-L)) = 0$ by the Kawamata–Viehweg vanishing theorem (see Theorem 3.2.1). On the other hand, we have

$$R^q f_*\mathcal{O}_Y \simeq H^q(E, \mathcal{O}_E)$$

for every $q > 0$ since $R^q f_*\mathcal{O}_Y(-E) = 0$ for every $q > 0$ by the Grauert–Riemenschneider vanishing theorem (see Theorem 3.2.7). Thus, we obtain $H^2(X, \mathcal{O}_X(-L)) \simeq \mathbb{C}^2$. In particular, $H^2(X, \mathcal{O}_X(-L)) \neq 0$. We note that X is not Cohen–Macaulay. In the above example, if we assume that E is a $K3$-surface, then $H^q(X, \mathcal{O}_X(-L)) = 0$ for $q < 3$ and X is Cohen–Macaulay. For the details, see Section 7.2, especially, Lemma 7.2.7.

5.7. Vanishing theorems of Reid–Fukuda type

Here, we treat some generalizations of Theorem 5.6.3. First, we introduce the notion of nef and log big divisors.

Definition 5.7.1 (Nef and log big divisors). Let $f : (Y, \Delta) \to X$ be a proper morphism from a simple normal crossing pair (Y, Δ) to a scheme X. Let $\pi : X \to V$ be a proper morphism between schemes and let H be an $\mathbb{R}$-Cartier divisor on X. We say that H is *nef and log big over V with respect to $f : (Y, \Delta) \to X$* if and only if $H|_C$ is nef and big over V for any C, where C is the image of a stratum of (Y, Δ).

We also need the notion of nef and log big divisors for normal pairs.

Definition 5.7.2 (Nef and log big divisors for normal pairs). Let (X, Δ) be a normal pair and let $\pi : X \to V$ be a proper morphism. Let D be an $\mathbb{R}$-Cartier divisor on X. We say that D is *nef and log big over V with respect to* (X, Δ) if and only if D is nef and big over V and $D|_C$ is big over V for every log canonical center C of (X, Δ).

We can generalize Theorem 5.6.3 as follows. It is [**Am1**, Theorem 7.4] for embedded *simple* normal crossing pairs.

Theorem 5.7.3 (cf. [**Am1**, Theorem 7.4]). *Let $f : (Y, \Delta) \to X$ be a proper morphism from an embedded simple normal crossing pair (Y, Δ) to a scheme X such that Δ is a boundary $\mathbb{R}$-divisor. Let L be a Cartier divisor on Y and let $\pi : X \to V$ be a proper morphism between schemes. Assume that*

$$f^*H \sim_{\mathbb{R}} L - (K_Y + \Delta),$$

where H is nef and log big over V with respect to $f : (Y, \Delta) \to X$. Let q be an arbitrary non-negative integer. Then we have the following properties.

(i) *Every associated prime of $R^q f_*\mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ).*

(ii) *We have*

$$R^p\pi_* R^q f_*\mathcal{O}_Y(L) = 0$$

for every $p > 0$.

Proof. Note that $L - (K_Y + \Delta)$ is f-semi-ample. Therefore, (i) is a special case of Theorem 5.6.3 (i).

From now on, we will prove (ii). We note that we may assume that V is affine without loss of generality.

Step 1. We assume that every stratum of (Y, Δ) dominates some irreducible component of X. By taking the Stein factorization, we may assume that f has connected fibers. Then we may further assume that X is irreducible and every stratum of (Y, Δ) dominates X. By Chow's lemma, there exists a projective birational morphism $\mu : X' \to X$ such that $\pi' : X' \to V$ is projective. By taking a proper birational morphism $\varphi : Y' \to Y$ that is an isomorphism over the generic point of any stratum of (Y, Δ), we have the following commutative diagram.

$$\begin{tikzcd}
Y' \arrow[r, "\varphi"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\
X' \arrow[r, "\mu"'] \arrow[dr, "\pi'"'] & X \arrow[d, "\pi"] \\
 & V
\end{tikzcd}$$

Then, by Theorem 5.2.17 (see also [BVP, Theorem 1.4]), we can write

$$K_{Y'} + \Delta' = \varphi^*(K_Y + \Delta) + E,$$

where

(1) (Y', Δ') is an embedded simple normal crossing pair such that Δ' is a boundary $\mathbb{R}$-divisor.

(2) E is an effective φ-exceptional Cartier divisor.

(3) Every stratum of (Y', Δ') dominates X'.

We note that every stratum of (Y, Δ) dominates X. Therefore,

$$\varphi^*L + E \sim_{\mathbb{R}} K_{Y'} + \Delta' + \varphi^* f^* H.$$

We note that

$$\varphi_*\mathcal{O}_{Y'}(\varphi^*L + E) \simeq \mathcal{O}_Y(L)$$

and

$$R^i\varphi_*\mathcal{O}_{Y'}(\varphi^*L + E) = 0$$

for every $i>0$ by Theorem 5.6.3 (i). Thus, by replacing Y and L with Y' and φ^*L+E, we may assume that $\varphi: Y'\to Y$ is the identity, that is, we have

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "\mu"] \arrow[dr, "\pi'"'] & X \arrow[d, "\pi"] \\ & V. \end{tikzcd}$$

We put $\mathcal{F}=R^qg_*\mathcal{O}_Y(L)$. Since μ^*H is nef and big over V and $\pi': X'\to V$ is projective, we can write $\mu^*H=E+A$, where A is a π'-ample $\mathbb{R}$-divisor on X' and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor by Kodaira (see Lemma 2.1.18). By the same arguments as above, we take some blow-ups and may further assume that $(Y, \Delta+g^*E)$ is an embedded simple normal crossing pair. If k is a sufficiently large positive integer, then

$$\lfloor \{\Delta\}+\frac{1}{k}g^*E\rfloor=0,$$

$$\mu^*H=\frac{1}{k}E+\frac{1}{k}A+\frac{k-1}{k}\mu^*H,$$

and

$$\frac{1}{k}A+\frac{k-1}{k}\mu^*H$$

is π'-ample. Thus, $\mathcal{F}$ is μ_*-acyclic and $(\pi\circ\mu)_*=\pi'_*$-acyclic by Theorem 5.6.3 (ii). We note that

$$L-\Big(K_Y+\Delta+\frac{1}{k}g^*E\Big)\sim_{\mathbb{R}} g^*\Big(\frac{1}{k}A+\frac{k-1}{k}\mu^*H\Big).$$

So, we have $R^qf_*\mathcal{O}_Y(L)\simeq \mu_*\mathcal{F}$ and $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic. Thus, we finish the proof when every stratum of (Y,Δ) dominates some irreducible component of X.

Step 2. We treat the general case by induction on $\dim f(Y)$. By taking some embedded log transformations (see Lemma 5.7.4 below), we can decompose $Y=Y'\cup Y''$ as follows: Y' is the union of all strata of (Y,Δ) that are not mapped to irreducible components of X and $Y''=Y-Y'$. We put

$$K_{Y''}+\Delta_{Y''}=(K_Y+\Delta)|_{Y''}-Y'|_{Y''}.$$

Then $f:(Y'',\Delta_{Y''})\to X$ and $L''=L|_{Y''}-Y'|_{Y''}$ satisfy the assumption in Step 1. We consider the following short exact sequence

$$0\to\mathcal{O}_{Y''}(L'')\to\mathcal{O}_Y(L)\to\mathcal{O}_{Y'}(L)\to 0.$$

By taking $R^q f_*$, we have short exact sequence

$$0 \to R^q f_* \mathcal{O}_{Y''}(L'') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y'}(L) \to 0$$

for every q. This is because the connecting homomorphisms

$$R^q f_* \mathcal{O}_{Y'}(L) \to R^{q+1} f_* \mathcal{O}_{Y''}(L'')$$

are zero maps for every q by (i). Since (ii) holds for the first and third members by Step 1 and by induction on the dimension, respectively, it also holds for $R^q f_* \mathcal{O}_Y(L)$.

So, we finish the proof. □

We have already used Lemma 5.7.4 in the proof of Theorem 5.7.3. Lemma 5.7.4 is easy to check. So we omit the proof.

Lemma 5.7.4 (cf. [**Am1**, p.218 embedded log transformation]). *Let (X, Δ) be an embedded simple normal crossing pair and let M be an ambient space of (X, Δ). Let C be a smooth stratum of (X, Δ). Let $\sigma : N \to M$ be the blow-up along C. Let Y denote the reduced structure of the total transform of X in N. We put*

$$K_Y + \Delta_Y = f^*(K_X + \Delta),$$

where $f = \sigma|_Y$. Then we have the following properties.

- (i) *(Y, Δ_Y) is an embedded simple normal crossing pair with an ambient space N.*
- (ii) *$f_* \mathcal{O}_Y \simeq \mathcal{O}_X$ and $R^i f_* \mathcal{O}_Y = 0$ for every $i > 0$.*
- (iii) *The strata of (X, Δ) are exactly the images of the strata of (Y, Δ_Y).*
- (iv) *$\sigma^{-1}(C)$ is a maximal (with respect to the inclusion) stratum of (Y, Δ_Y).*
- (v) *If Δ is a boundary $\mathbb{R}$-divisor on X, then Δ_Y is a boundary $\mathbb{R}$-divisor on Y.*

Remark 5.7.5. We need the notion of *embedded* simple normal crossing pairs to prove Theorem 5.7.3 even when Y is smooth. It is a key point of the proof of Theorem 5.7.3. Note that we do not need the assumption that Y is *embedded* in Step 1 in the proof of Theorem 5.7.3.

As a corollary of Theorem 5.7.3, we can prove the following vanishing theorem. It is the culmination of the works of several authors: Kawamata, Viehweg, Nadel, Reid, Fukuda, Ambro, Fujino, and others. To the author's best knowledge, we can not find it in the literature except [**F17**]. Note that Theorem 5.7.6 is a complete generalization of [**KMM**, Theorem 1-2-5].

THEOREM 5.7.6 (see [F17, Theorem 2.48]). *Let (X, Δ) be a log canonical pair such that Δ is a boundary $\mathbb{R}$-divisor and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L - (K_X + \Delta)$ is nef and log big over V with respect to (X, Δ), where $\pi : X \to V$ is a proper morphism. Then $R^q\pi_*\mathcal{O}_X(L) = 0$ for every $q > 0$.*

PROOF. Let $f : Y \to X$ be a log resolution of (X, Δ) such that

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

with $a_i \geq -1$ for every i. We may assume that $\sum_i E_i \cup \operatorname{Supp} f^*L$ is a simple normal crossing divisor on Y. We put

$$E = \sum_i a_i E_i$$

and

$$F = \sum_{a_j=-1} (1 - b_j)E_j,$$

where $b_j = \operatorname{mult}_{E_j}\{f^*L\}$. We note that $A = L - (K_X + \Delta)$ is nef and log big over V with respect (X, Δ) by assumption. So, we have

$$\begin{aligned} f^*A &= f^*L - f^*(K_X + \Delta) \\ &= \lceil f^*L + E + F\rceil - (K_Y + F + \{-(f^*L + E + F)\}). \end{aligned}$$

We can easily check that

$$f_*\mathcal{O}_Y(\lceil f^*L + E + F\rceil) \simeq \mathcal{O}_X(L)$$

and that $F + \{-(f^*L + E + F)\}$ has a simple normal crossing support and is a boundary $\mathbb{R}$-divisor on Y. By the above definition of F, A is nef and log big over V with respect to $f : (Y, F + \{-(f^*L + E + F)\}) \to X$. Therefore, by Theorem 5.7.3 (ii), we obtain that $\mathcal{O}_X(L)$ is π_*-acyclic. Thus, we have $R^q\pi_*\mathcal{O}_X(L) = 0$ for every $q > 0$. □

As a special case, we have the Kawamata–Viehweg vanishing theorem for klt pairs.

COROLLARY 5.7.7 (Kawamata–Viehweg vanishing theorem, see [KMM, Remark 1-2-6]). *Let (X, Δ) be a klt pair and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L - (K_X + \Delta)$ is nef and big over V, where $\pi : X \to V$ is a proper morphism. Then $R^q\pi_*\mathcal{O}_X(L) = 0$ for every $q > 0$.*

We add one example.

EXAMPLE 5.7.8. Let Y be a projective surface which has the following properties: (i) there exists a projective birational morphism $f : X \to Y$ from a smooth projective surface X, and (ii) the exceptional locus E of f is an elliptic curve with $K_X + E = f^*K_Y$. For example, Y is a cone over a smooth plane cubic curve and $f : X \to Y$ is the blow-up at the vertex of Y. We note that (X, E) is a plt pair. Let H be an ample Cartier divisor on Y. We consider a Cartier divisor $L = f^*H + K_X + E$ on X. Then $L - (K_X + E)$ is nef and big, but not log big with respect to (X, E). By the short exact sequence

$$0 \to \mathcal{O}_X(f^*H + K_X) \to \mathcal{O}_X(f^*H + K_X + E) \to \mathcal{O}_E(K_E) \to 0,$$

we obtain

$$R^1 f_*\mathcal{O}_X(f^*H + K_X + E) \simeq H^1(E, \mathcal{O}_E(K_E)) \simeq \mathbb{C}(P),$$

where $P = f(E)$. By the Leray spectral sequence, we have

$$\begin{aligned} 0 &\to H^1(Y, f_*\mathcal{O}_X(K_X + E) \otimes \mathcal{O}_Y(H)) \to H^1(X, \mathcal{O}_X(L)) \\ &\to H^0(Y, \mathbb{C}(P)) \to H^2(Y, f_*\mathcal{O}_X(K_X + E) \otimes \mathcal{O}_Y(H)) \\ &\to \cdots . \end{aligned}$$

If H is sufficiently ample, then $H^1(X, \mathcal{O}_X(L)) \simeq H^0(Y, \mathbb{C}(P)) \simeq \mathbb{C}(P)$. In particular, $H^1(X, \mathcal{O}_X(L)) \neq 0$.

REMARK 5.7.9. In Example 5.7.8, there exists an effective $\mathbb{Q}$-divisor B on X such that $L - \frac{1}{k}B$ is ample for every $k > 0$ by Kodaira's lemma (see Lemma 2.1.18). Since $L \cdot E = 0$, we have $B \cdot E < 0$. In particular,

$$(X, E + \frac{1}{k}B)$$

is not log canonical for any $k > 0$. This is the main reason why $H^1(X, \mathcal{O}_X(L)) \neq 0$. If $(X, E + \frac{1}{k}B)$ were log canonical, then the ampleness of $L - (K_X + E + \frac{1}{k}B)$ would imply $H^1(X, \mathcal{O}_X(L)) = 0$ by Theorem 5.6.4.

If Y is quasi-projective in Theorem 5.7.3, we do not need the assumption that the pair (Y, Δ) is *embedded*.

THEOREM 5.7.10 ([FF, Theorem 6.3]). *Let $f : (Y, \Delta) \to X$ be a proper morphism from a* quasi-projective *simple normal crossing pair (Y, Δ) to a scheme X such that Δ is a boundary $\mathbb{R}$-divisor. Let L be a Cartier divisor on Y and let $\pi : X \to V$ be a proper morphism between schemes. Assume that*

$$f^*H \sim_{\mathbb{R}} L - (K_Y + \Delta),$$

where H is nef and log big over V with respect to $f : (Y, \Delta) \to X$. Let q be an arbitrary non-negative integer. Then we have the following properties.

(i) *Every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ).*
(ii) *We have*
$$R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$$
for every $p > 0$.

We can easily reduce Theorem 5.7.10 to Theorem 5.7.3. For the proof of Theorem 5.7.3, see the proof of [**FF**, Theorem 6.3]. We used Theorem 5.7.10 for the proof of the main theorem of [**FF**].

5.8. From SNC pairs to NC pairs

In this section, we prove the injectivity, vanishing, and torsion-free theorems for *embedded normal crossing pairs*, although the results in this section are not necessary for the theory of quasi-log schemes discussed in Chapter 6.

Theorem 5.8.1 is a generalization of [**Am1**, Theorem 3.1].

THEOREM 5.8.1. *Let (X, Δ) be an embedded normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor and let $\pi : X \to V$ be a proper morphism between schemes. Let L be a Cartier divisor on X and let D be an effective Cartier divisor that is permissible with respect to (X, Δ). Assume the following conditions.*

(i) *$L \sim_{\mathbb{R}, \pi} K_X + \Delta + H$,*
(ii) *H is a π-semi-ample $\mathbb{R}$-divisor, and*
(iii) *$tH \sim_{\mathbb{R}, \pi} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to (X, Δ).*

Then the homomorphisms
$$R^q \pi_* \mathcal{O}_X(L) \to R^q \pi_* \mathcal{O}_X(L + D),$$
which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Theorem 5.8.2 is nothing but [**Am1**, Theorem 7.4].

THEOREM 5.8.2. *Let (Y, Δ) be an embedded normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor. Let $f : Y \to X$ be a proper morphism between schemes and let L be a Cartier divisor on Y such that $L - (K_Y + \Delta)$ is f-semi-ample. Let q be an arbitrary non-negative integer. Then we have the following properties.*

(i) *Every associated prime of* $R^q f_*\mathcal{O}_Y(L)$ *is the generic point of the* f*-image of some stratum of* (Y, Δ).
(ii) *Let* $\pi : X \to V$ *be a proper morphism between schemes. We assume that*

$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^*H$$

where H *is an* $\mathbb{R}$*-Cartier divisor on* X *which is nef and log big over* V *with respect to* $f : (Y, \Delta) \to X$. *Then we obtain that* $R^q f_*\mathcal{O}_Y(L)$ *is* π_**-acyclic, that is,*

$$R^p\pi_* R^q f_*\mathcal{O}_Y(L) = 0$$

for every $p > 0$.

Before we go to the proof, let us recall the definition of normal crossing pairs. The following definition is the same as [**Am1**, Definition 2.3] though it may look different.

Definition 5.8.3 (Normal crossing pair). A variety X has *normal crossing* singularities if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k)}$$

for some $0 \le k \le N$, where $N = \dim X$. Let X be a normal crossing variety. We say that a reduced divisor D on X is *normal crossing* if, in the above notation, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \cdots, i_l\} \subset \{k + 1, \cdots, N\}$. We say that the pair (X, Δ) is a *normal crossing pair* if the following conditions are satisfied.

(1) X is a normal crossing variety, and
(2) Δ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support is normal crossing on X.

We say that a normal crossing pair (X, Δ) is *embedded* if there exists a closed embedding $\iota : X \hookrightarrow M$, where M is a smooth variety of dimension $\dim X + 1$. We call M the *ambient space* of (X, Δ). We put

$$K_{X^\nu} + \Theta = \nu^*(K_X + \Delta),$$

where $\nu : X^\nu \to X$ is the normalization of X. A *stratum* of (X, Δ) is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ) on X.

A Cartier divisor B on a normal crossing pair (X, Δ) is called *permissible with respect to* (X, Δ) if the support of B contains no strata of the pair (X, Δ). A finite $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisor with respect to (X, Δ) is called a *permissible* $\mathbb{Q}$*-divisor* (*resp.* $\mathbb{R}$*-divisor*) *with respect to* (X, Δ).

The following definition is almost obvious.

DEFINITION 5.8.4 (Nef and log big divisors). Let $f : (Y, \Delta) \to X$ be a proper morphism from a normal crossing pair (Y, Δ) to a scheme X. Let $\pi : X \to V$ be a proper morphism between schemes and let H be an $\mathbb{R}$-Cartier divisor on X. We say that H is *nef and log big over* V *with respect to* $f : (Y, \Delta) \to X$ if and only if $H|_C$ is nef and big over V for any C, where C is the image of a stratum of (Y, Δ).

The following three lemmas are easy to check. So, we omit the proofs.

LEMMA 5.8.5. *Let X be a normal crossing divisor on a smooth variety M. Then there exists a sequence of blow-ups*

$$M_k \to M_{k-1} \to \cdots \to M_0 = M$$

with the following properties.

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of X_i for every $i \geq 0$,*
(ii) *$X_0 = X$ and X_{i+1} is the inverse image of X_i with the reduced structure for every $i \geq 0$, and*
(iii) *X_k is a simple normal crossing divisor on M_k.*

For each step σ_{i+1}, we can directly check that

$$\sigma_{i+1*}\mathcal{O}_{X_{i+1}} \simeq \mathcal{O}_{X_i}$$

and

$$R^q\sigma_{i+1*}\mathcal{O}_{X_{i+1}} = 0$$

for every $i \geq 0$ and $q \geq 1$. Let Δ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\operatorname{Supp}\Delta$ *is normal crossing. We put $\Delta_0 = \Delta$ and*

$$K_{X_{i+1}} + \Delta_{i+1} = \sigma_{i+1}^*(K_{X_i} + \Delta_i)$$

for all $i \geq 0$. Then it is obvious that Δ_i is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and $\operatorname{Supp}\Delta_i$ *is normal crossing on X_i for every $i \geq 0$. We can also check that Δ_i is a boundary $\mathbb{R}$-divisor* (*resp. a boundary $\mathbb{Q}$-divisor*) *for every $i \geq 0$ if so is Δ. If Δ is a boundary $\mathbb{R}$-divisor, then the σ_{i+1}-image of any stratum of (X_{i+1}, Δ_{i+1}) is a stratum of (X_i, Δ_i).*

REMARK 5.8.6. Each step in Lemma 5.8.5 is called *embedded log transformation* in [**Am1**, Section 2]. See also Lemma 5.7.4.

LEMMA 5.8.7. *Let X be a simple normal crossing divisor on a smooth variety M. Let $S+B$ be a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that* $\operatorname{Supp}(S+B)$ *is normal crossing, S is reduced, and $\lfloor B\rfloor = 0$. Then there exists a sequence of blow-ups*

$$M_k \to M_{k-1} \to \cdots \to M_0 = M$$

with the following properties.

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of (X_i, S_i) that is contained in S_i for every $i \geq 0$,*
(ii) *we put $X_0 = X$, $S_0 = S$, and $B_0 = B$, and X_{i+1} is the strict transform of X_i for every $i \geq 0$,*
(iii) *we define*

$$K_{X_{i+1}} + S_{i+1} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + S_i + B_i)$$

for every $i \geq 0$, where B_{i+1} is the strict transform of B_i on X_{i+1},
(iv) *the σ_{i+1}-image of any stratum of $(X_{i+1}, S_{i+1} + B_{i+1})$ is a stratum of $(X_i, S_i + B_i)$, and*
(v) *S_k is a simple normal crossing divisor on X_k.*

For each step σ_{i+1}, we can easily check that

$$\sigma_{i+1*}\mathcal{O}_{X_{i+1}} \simeq \mathcal{O}_{X_i}$$

and

$$R^q\sigma_{i+1*}\mathcal{O}_{X_{i+1}} = 0$$

for every $i \geq 0$ and $q \geq 1$. We note that X_i is simple normal crossing, $\operatorname{Supp}(S_i + B_i)$ *is normal crossing on X_i, and S_i is reduced for every $i \geq 0$.*

LEMMA 5.8.8. *Let X be a simple normal crossing divisor on a smooth variety M. Let $S+B$ be a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that* $\operatorname{Supp}(S+B)$ *is normal crossing, S is reduced and simple normal crossing, and $\lfloor B\rfloor = 0$. Then there exists a sequence of blow-ups*

$$M_k \to M_{k-1} \to \cdots \to M_0 = M$$

with the following properties.

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of $(X_i, \operatorname{Supp} B_i)$ that is contained in $\operatorname{Supp} B_i$ for every $i \geq 0$,*
(ii) *we put $X_0 = X$, $S_0 = S$, and $B_0 = B$, and X_{i+1} is the strict transform of X_i for every $i \geq 0$,*
(iii) *we define*

$$K_{X_{i+1}} + S_{i+1} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + S_i + B_i)$$

for every $i \geq 0$, where S_{i+1} is the strict transform of S_i on X_{i+1}, and

(iv) $\mathrm{Supp}(S_k + B_k)$ *is a simple normal crossing divisor on X_k.*

We note that X_i is simple normal crossing on M_i and $\mathrm{Supp}(S_i + B_i)$ *is normal crossing on X_i for every $i \geq 0$. We can easily check that $\lfloor B_i \rfloor \leq 0$ for every $i \geq 0$. The composition morphism $M_k \to M$ is denoted by σ. Let L be any Cartier divisor on X. We put $E = \lceil -B_k \rceil$. Then E is an effective σ-exceptional Cartier divisor on X_k and we obtain*

$$\sigma_* \mathcal{O}_{X_k}(\sigma^* L + E) \simeq \mathcal{O}_X(L)$$

and

$$R^q \sigma_* \mathcal{O}_{X_k}(\sigma^* L + E) = 0$$

for every $q \geq 1$ by Theorem 5.6.3 (i). We note that

$$\sigma^* L + E - (K_{X_k} + S_k + \{B_k\}) = \sigma^* L - \sigma^*(K_X + S + B)$$

is $\mathbb{R}$-linearly trivial over X and σ is an isomorphism at the generic point of any stratum of $(X_k, S_k + B_k)$.

Let us go to the proof of Theorems 5.8.1 and 5.8.2.

Proof of Theorem 5.8.1. We take a sequence of blow-ups and obtain a projective morphism $\sigma : X' \to X$ from an embedded simple normal crossing variety X' by Lemma 5.8.5. We can replace X and L with X' and $\sigma^* L$ by Leray's spectral sequence. So, we may assume that X is simple normal crossing. We put $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. Similarly, we may assume that S is simple normal crossing on X by applying Lemma 5.8.7. Finally, we use Lemma 5.8.8 and obtain a birational morphism

$$\sigma : (X', S' + B') \to (X, S + B)$$

from an embedded simple normal crossing pair $(X', S' + B')$ such that

$$K_{X'} + S' + B' = \sigma^*(K_X + S + B)$$

as in Lemma 5.8.8. By Lemma 5.8.8, we can replace $(X, S + B)$ and L with $(X', S' + \{B'\})$ and $\sigma^* L + \lceil -B' \rceil$ by Leray's spectral sequence. Then we apply Theorem 5.6.2. Thus, we obtain Theorem 5.8.1. $\square$

Proof of Theorem 5.8.2. We take a sequence of blow-ups and obtain a projective morphism $\sigma : Y' \to Y$ from an embedded simple normal crossing variety Y' by Lemma 5.8.5. We can replace Y and L with Y' and $\sigma^* L$ by Leray's spectral sequence. So, we may assume that Y is simple normal crossing. We put $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. Similarly, we may assume that S is simple normal crossing on Y by applying Lemma 5.8.7. Finally, we use Lemma 5.8.8 and obtain a birational morphism

$$\sigma : (Y', S' + B') \to (Y, S + B)$$

from an embedded simple normal crossing pair $(Y', S' + B')$ such that

$$K_{Y'} + S' + B' = \sigma^*(K_Y + S + B)$$

as in Lemma 5.8.8. By Lemma 5.8.8, we can replace $(Y, S + B)$ and L with $(Y', S' + \{B'\})$ and $\sigma^*L + \lceil -B' \rceil$ by Leray's spectral sequence. Then we apply Theorem 5.7.3. Thus, we obtain Theorem 5.8.2. ☐

5.9. Examples

In this section, we treat various supplementary examples. These examples show that some results obtained in this chapter are sharp.

5.9.1 (Hodge theoretic injectivity theorems). Let X be a smooth projective variety and let M be a Cartier divisor on X such that $N \sim mM$, where N is a reduced simple normal crossing divisor on X and $m \geq 2$. We put $\Delta = \frac{1}{m}N$ and $L = K_X + M$. In this setting, we can apply Theorem 5.4.2 since $L \sim_{\mathbb{Q}} K_X + \Delta$. If M is semi-ample, then the existence of such N and m is obvious by Bertini. Here, we give some explicit examples where M is not nef.

Example 5.9.2. We consider the $\mathbb{P}^1$-bundle

$$\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1.$$

Let E and G be the sections of π such that $E^2 = -2$ and $G^2 = 2$. We note that $E + 2F \sim G$, where F is a fiber of π. We consider $M = E + F$. Then

$$2M = 2E + 2F \sim E + G.$$

In this case, $M \cdot E = -1$. In particular, M is not nef. Unfortunately, we can easily check that

$$H^i(X, \mathcal{O}_X(K_X + M)) = 0$$

for every i. So, it is not interesting to apply Theorem 5.4.2.

Example 5.9.3. We consider the $\mathbb{P}^1$-bundle

$$\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4)) \to \mathbb{P}^1.$$

Let G (resp. E) be the positive (resp. negative) section of π, that is, the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1}(4)$ (resp. $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1}$). We put $M' = -F + 2G$, where F is a fiber of π. Then M' is not nef and

$$2M' \sim G + E + F_1 + F_2 + H,$$

where F_1 and F_2 are distinct fibers of π, and H is a general member of the free linear system $|2G|$. Note that $G+E+F_1+F_2+H$ is a reduced simple normal crossing divisor on Y. We put $X = Y \times C$, where C is an elliptic curve, and $M = p^*M'$, where $p : X \to Y$ is the projection. Then X is a smooth projective variety and M is a Cartier divisor on X. We note that M is not nef and that we can find a reduced simple normal crossing divisor N such that $N \sim 2M$. By the Künneth formula, we have

$$H^1(X, \mathcal{O}_X(K_X + M)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \simeq \mathbb{C}^2.$$

Therefore, X with $L = K_X + M$ and $\Delta = \frac{1}{2}N$ satisfies the conditions in Theorem 5.4.2. Moreover, we have $H^1(X, \mathcal{O}_X(L)) \neq 0$.

Example 5.9.2 shows that the assumptions for the Hodge theoretic injectivity theorems in Section 5.4 are geometric.

5.9.4 (Kodaira vanishing theorem for singular varieties). The following example is due to Sommese (cf. [Som, (0.2.4) Example]). It shows that the Kodaira vanishing theorem does not necessarily hold for varieties with non-lc singularities.

Proposition 5.9.5 (Sommese). *We consider the $\mathbb{P}^3$-bundle*

$$\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}) \to \mathbb{P}^1$$

over $\mathbb{P}^1$. Let $\mathcal{M} = \mathcal{O}_Y(1)$ be the tautological line bundle of $\pi : Y \to \mathbb{P}^1$. We take a general member X of the linear system $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$. Then X is a normal projective Gorenstein threefold and X is not log canonical. We put $\mathcal{L} = \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then $\mathcal{L}$ is ample. In this case, we can check that $H^2(X, \mathcal{L}^{-1}) = \mathbb{C}$. By Serre duality,

$$H^1(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}) = \mathbb{C}.$$

Therefore, the Kodaira vanishing theorem does not hold for X.

Proof. We consider the following short exact sequence

$$0 \to \mathcal{L}^{-1}(-X) \to \mathcal{L}^{-1} \to \mathcal{L}^{-1}|_X \to 0.$$

Then we have the long exact sequence

$$\begin{aligned} \cdots \to H^i(Y, \mathcal{L}^{-1}(-X)) \to H^i(Y, \mathcal{L}^{-1}) \to H^i(X, \mathcal{L}^{-1}) \\ \to H^{i+1}(Y, \mathcal{L}^{-1}(-X)) \to \cdots. \end{aligned}$$

Since $H^i(Y, \mathcal{L}^{-1}) = 0$ for $i < 4$ by the original Kodaira vanishing theorem (see Theorem 3.1.3), we obtain that

$$H^2(X, \mathcal{L}^{-1}) = H^3(Y, \mathcal{L}^{-1}(-X)).$$

Therefore, it is sufficient to prove that $H^3(Y, \mathcal{L}^{-1}(-X)) = \mathbb{C}$.

We have

$$\begin{aligned}\mathcal{L}^{-1}(-X) &= \mathcal{M}^{-1} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{M}^{-4} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(4)\\ &= \mathcal{M}^{-5} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(3).\end{aligned}$$

We note that $R^i\pi_*\mathcal{M}^{-5} = 0$ for $i \neq 3$ because $\mathcal{M} = \mathcal{O}_Y(1)$. By Grothendieck duality,

$$R\mathcal{H}om(R\pi_*\mathcal{M}^{-5}, \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1})[1]) = R\pi_* R\mathcal{H}om(\mathcal{M}^{-5}, \mathcal{O}_Y(K_Y)[4]).$$

By Grothendieck duality again,

$$\begin{aligned}R\pi_*\mathcal{M}^{-5} &= R\mathcal{H}om(R\pi_*R\mathcal{H}om(\mathcal{M}^{-5}, \mathcal{O}_Y(K_Y)[4]), \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1})[1])\\ &= R\mathcal{H}om(R\pi_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5), \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1}))[-3]\\ &= (*).\end{aligned}$$

By definition, we have

$$\begin{aligned}\mathcal{O}_Y(K_Y) &= \pi^*(\mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1}) \otimes \det(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3})) \otimes \mathcal{M}^{-4}\\ &= \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{M}^{-4}.\end{aligned}$$

By this formula, we obtain

$$\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5 = \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{M}.$$

Thus, $R^i\pi_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5) = 0$ for every $i > 0$. We note that

$$\begin{aligned}\pi_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5) &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_*\mathcal{M}\\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3})\\ &= \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(*) &= R\mathcal{H}om(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}, \mathcal{O}_{\mathbb{P}^1}(-2))[-3]\\ &= (\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)^{\oplus 3})[-3].\end{aligned}$$

So, we obtain $R^3\pi_*\mathcal{M}^{-5} = \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)^{\oplus 3}$. Thus, we have

$$R^3\pi_*\mathcal{M}^{-5} \otimes \mathcal{O}_{\mathbb{P}^1}(3) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}.$$

By the spectral sequence, we have

$$\begin{aligned}H^3(Y, \mathcal{L}^{-1}(-X)) &= H^3(Y, \mathcal{M}^{-5} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(3))\\ &= H^0(\mathbb{P}^1, R^3\pi_*(\mathcal{M}^{-5} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(3)))\\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3})\\ &= \mathbb{C}.\end{aligned}$$

Therefore, $H^2(X, \mathcal{L}^{-1}) = \mathbb{C}$.

Let us recall that X is a general member of the linear system

$$|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|.$$

Let C be the negative section of $\pi : Y \to \mathbb{P}^1$, that is, the section corresponding to the projection

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

From now, we will check that $|\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)|$ is free outside C. Once we checked it, we know that $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$ is free outside C. Then X is smooth in codimension one. Since Y is smooth, X is normal and Gorenstein by adjunction.

We take $Z \in |\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)| \neq \emptyset$. Since

$$H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0,$$

Z can not have a fiber of π as an irreducible component, that is, any irreducible component of Z is mapped onto $\mathbb{P}^1$ by $\pi : Y \to \mathbb{P}^1$. On the other hand, let l be a line in a fiber of $\pi : Y \to \mathbb{P}^1$. Then $Z \cdot l = 1$. Therefore, Z is irreducible. Let $F = \mathbb{P}^3$ be a fiber of $\pi : Y \to \mathbb{P}^1$. We consider

$$\begin{aligned} 0 = H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_Y(-F)) &\to H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\ \to H^0(F, \mathcal{O}_F(1)) \to H^1(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_Y(-F)) &\to \cdots . \end{aligned}$$

Since $(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \cdot C = -1$, every member of $|\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)|$ contains C. We put $P = F \cap C$. Then the image of

$$\alpha : H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to H^0(F, \mathcal{O}_F(1))$$

is $H^0(F, m_P \otimes \mathcal{O}_F(1))$, where m_P is the maximal ideal of P. This is because the dimension of $H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))$ is three. Thus, we know that $|\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)|$ is free outside C. In particular, $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$ is free outside C.

More explicitly, the image of the injection

$$\alpha : H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to H^0(F, \mathcal{O}_F(1))$$

is $H^0(F, m_P \otimes \mathcal{O}_F(1))$. We note that

$$H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) = \mathbb{C}^3,$$

and

$$H^0(Y, (\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}) = H^0(\mathbb{P}^1, \mathrm{Sym}^4(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3})) = \mathbb{C}^{15}.$$

We can check that the restriction of $H^0(Y, (\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4})$ to F is $\mathrm{Sym}^4 H^0(F, m_P \otimes \mathcal{O}_F(1))$. Thus, the general fiber f of $\pi : X \to \mathbb{P}^1$ is a cone in $\mathbb{P}^3$ on a smooth plane curve of degree 4 with the vertex $P = f \cap C$. Therefore, (Y, X) is not log canonical because the multiplicity of X along C is four. Thus, X is not log canonical by the inversion of adjunction. Anyway, X is the required variety. $\square$

By the same construction, we have:

EXAMPLE 5.9.6. We consider the $\mathbb{P}^{k+1}$-bundle

$$\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(k+1)}) \to \mathbb{P}^1$$

over $\mathbb{P}^1$ for $k \geq 2$. We put $\mathcal{M} = \mathcal{O}_Y(1)$ and $\mathcal{L} = \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then $\mathcal{L}$ is ample. We take a general member X of the linear system $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes(k+2)}|$. Then we can check the following properties.

(1) X is a normal projective Gorenstein $(k+1)$-fold.
(2) X is not log canonical.
(3) We can check

$$R^{k+1}\pi_*\mathcal{M}^{-(k+3)} = \mathcal{O}_{\mathbb{P}^1}(-1-k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2-k)^{\oplus(k+1)}$$

and

$$R^i\pi_*\mathcal{M}^{-(k+3)} = 0$$

for $i \neq k+1$.
(4) Since $\mathcal{L}^{-1}(-X) = \mathcal{M}^{-(k+3)} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(k+1)$, we have

$$\begin{aligned} H^{k+1}(Y, \mathcal{L}^{-1}(-X)) &= H^0(\mathbb{P}^1, R^{k+1}\pi_*\mathcal{M}^{-(k+3)} \otimes \mathcal{O}_{\mathbb{P}^1}(k+1)) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(k+1)}) \\ &= \mathbb{C}. \end{aligned}$$

Thus, $H^k(X, \mathcal{L}^{-1}) = H^{k+1}(Y, \mathcal{L}^{-1}(-X)) = \mathbb{C}$.

We note that the first cohomology group of an anti-ample line bundle on a normal variety with dimension ≥ 2 always vanishes by the following Mumford vanishing theorem.

THEOREM 5.9.7 (Mumford). *Let V be a normal complete algebraic variety and let $\mathcal{L}$ be a semi-ample line bundle on V. Assume that $\kappa(V, \mathcal{L}) \geq 2$. Then $H^1(V, \mathcal{L}^{-1}) = 0$.*

PROOF. Let $f : W \to V$ be a resolution of singularities. By Leray's spectral sequence, we obtain

$$0 \to H^1(V, f_*f^*\mathcal{L}^{-1}) \to H^1(W, f^*\mathcal{L}^{-1}) \to \cdots .$$

By the Kawamata–Viehweg vanishing theorem (see Theorem 3.3.7) and Serre duality, $H^1(W, f^*\mathcal{L}^{-1}) = 0$. Thus, we obtain $H^1(V, \mathcal{L}^{-1}) = H^1(V, f_*f^*\mathcal{L}^{-1}) = 0$. ∎

5.9.8 (On the Kawamata–Viehweg vanishing theorem). The next example shows that a naive generalization of the Kawamata–Viehweg vanishing theorem does not necessarily hold for varieties with log canonical singularities. Example 5.9.9 is also a supplement to Theorem 5.7.6.

EXAMPLE 5.9.9. We put $V = \mathbb{P}^2 \times \mathbb{P}^2$. Let $p_i : V \to \mathbb{P}^2$ be the i-th projection for $i = 1$ and 2. We define $\mathcal{L} = p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1)$ and consider the $\mathbb{P}^1$-bundle $\pi : W = \mathbb{P}_V(\mathcal{L} \oplus \mathcal{O}_V) \to V$. Let $F = \mathbb{P}^2 \times \mathbb{P}^2$ be the negative section of $\pi : W \to V$, that is, the section of π corresponding to $\mathcal{L} \oplus \mathcal{O}_V \to \mathcal{O}_V \to 0$. By using the linear system $|\mathcal{O}_W(1) \otimes \pi^* p_1^*\mathcal{O}_{\mathbb{P}^2}(1)|$, we can contract $F = \mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^2 \times \{\text{point}\}$.

Next, we consider an elliptic curve $C \subset \mathbb{P}^2$ and put $Z = C \times C \subset V = \mathbb{P}^2 \times \mathbb{P}^2$. Let $\pi : Y \to Z$ be the restriction of $\pi : W \to V$ to Z. The restriction of the above contraction morphism

$$\Phi_{|\mathcal{O}_W(1) \otimes \pi^* p_1^* \mathcal{O}_{\mathbb{P}^2}(1)|} : W \to U$$

to Y is denoted by $f : Y \to X$. Then, the exceptional locus of $f : Y \to X$ is $E = F|_Y = C \times C$ and f contracts E to $C \times \{\text{point}\}$.

Let $\mathcal{O}_W(1)$ be the tautological line bundle of the $\mathbb{P}^1$-bundle $\pi : W \to V$. By the construction, $\mathcal{O}_W(1) = \mathcal{O}_W(D)$, where D is the positive section of π, that is, the section corresponding to $\mathcal{L} \oplus \mathcal{O}_W \to \mathcal{L} \to 0$. By definition,

$$\mathcal{O}_W(K_W) = \pi^*(\mathcal{O}_V(K_V) \otimes \mathcal{L}) \otimes \mathcal{O}_W(-2).$$

By adjunction,

$$\mathcal{O}_Y(K_Y) = \pi^*(\mathcal{O}_Z(K_Z) \otimes \mathcal{L}|_Z) \otimes \mathcal{O}_Y(-2) = \pi^*(\mathcal{L}|_Z) \otimes \mathcal{O}_Y(-2).$$

Therefore,

$$\mathcal{O}_Y(K_Y + E) = \pi^*(\mathcal{L}|_Z) \otimes \mathcal{O}_Y(-2) \otimes \mathcal{O}_Y(E).$$

We note that $E = F|_Y$. Since $\mathcal{O}_Y(E) \otimes \pi^*(\mathcal{L}|_Z) \simeq \mathcal{O}_Y(D)$, we have $\mathcal{O}_Y(-(K_Y + E)) = \mathcal{O}_Y(1)$ because $\mathcal{O}_Y(1) = \mathcal{O}_Y(D)$. Thus, $-(K_Y + E)$ is nef and big.

On the other hand, it is not difficult to see that X is a normal projective Gorenstein threefold, X is log canonical but not klt along $G = f(E)$, and that X is smooth outside G. Since we can check that $f^*K_X = K_Y + E$, $-K_X$ is nef and big.

Finally, we consider the short exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0,$$

where $\mathcal{J}$ is the multiplier ideal sheaf of X. In our case, we can easily check that $\mathcal{J} = f_*\mathcal{O}_Y(-E) = \mathcal{I}_G$, where $\mathcal{I}_G$ is the defining ideal sheaf of G on X. Since $-K_X$ is nef and big, $H^i(X, \mathcal{J}) = 0$ for every $i > 0$ by Nadel's vanishing theorem (see Theorem 3.4.2). Therefore,

$$H^i(X, \mathcal{O}_X) = H^i(G, \mathcal{O}_G)$$

for every $i > 0$. Since G is an elliptic curve,

$$H^1(X, \mathcal{O}_X) = H^1(G, \mathcal{O}_G) = \mathbb{C}.$$

We note that $-K_X$ is nef and big but $-K_X$ is not log big with respect to X.

5.9.10 (On the injectivity theorem). The final example in this section supplements Theorem 5.6.2.

EXAMPLE 5.9.11. We consider the $\mathbb{P}^1$-bundle

$$\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1.$$

Let S (resp. H) be the negative (resp. positive) section of π, that is, the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}$ (resp. $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)$). Then H is semi-ample and $S+F \sim H$, where F is a fiber of π.

CLAIM. *The homomorphism*

$$H^1(X, \mathcal{O}_X(K_X + S + H)) \to H^1(X, \mathcal{O}_X(K_X + S + H + S + F))$$

induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(S+F)$ *is not injective.*

PROOF OF CLAIM. It is sufficient to see that the homomorphism

$$H^1(X, \mathcal{O}_X(K_X + S + H)) \to H^1(X, \mathcal{O}_X(K_X + S + H + F))$$

induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(F)$ is not injective. We consider the short exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_X(K_X + S + H) &\to \mathcal{O}_X(K_X + S + H + F) \\ &\to \mathcal{O}_F(K_F + (S+H)|_F) \to 0. \end{aligned}$$

We note that $F \simeq \mathbb{P}^1$ and $\mathcal{O}_F(K_F + (S+H)|_F) \simeq \mathcal{O}_{\mathbb{P}^1}$. Therefore, we obtain the following exact sequence

$$0 \to \mathbb{C} \to H^1(X, \mathcal{O}_X(K_X+S+H)) \to H^1(X, \mathcal{O}_X(K_X+S+H+F)) \to 0.$$

Thus,

$$H^1(X, \mathcal{O}_X(K_X + S + H)) \to H^1(X, \mathcal{O}_X(K_X + S + H + F))$$

is not injective. We note that $S+F$ is not *permissible* with respect to (X, S). $\square$

Therefore, the permissibility assumption is indispensable for Theorem 5.6.2.

CHAPTER 6

Fundamental theorems for quasi-log schemes

This chapter is the main part of this book. In this chapter, we introduce the notion of quasi-log schemes and establish the fundamental theorems for quasi-log schemes.

Section 6.1 is an overview of the main results of this chapter. In Section 6.2, we introduce the notion of quasi-log schemes. Note that our treatment is slightly different from Ambro's original theory of quasi-log varieties (see [**Am1**]). In Section 6.3, we discuss various basic properties, for example, adjunction and vanishing theorems, of quasi-log schemes. In Section 6.4, we show that a normal pair has a natural good quasi-log structure. By this fact, we can apply the theory of quasi-log schemes to normal pairs. We also treat toric polyhedra as examples of quasi-log schemes. Section 6.5 is devoted to the proof of the basepoint-free theorem for quasi-log schemes. In Section 6.6, we prove the rationality theorem for quasi-log schemes. In Section 6.7, we discuss the cone and contraction theorem for quasi-log schemes. Thus we establish the fundamental theorems for quasi-log schemes. In Section 6.8, we discuss some properties of quasi-log Fano schemes and related topics. Section 6.9 is devoted to the proof of the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes. Here, we prove it under some extra assumptions. For the details of the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes, see the author's recent preprint [**F40**].

6.1. Overview

In this chapter, we establish the fundamental theorems for quasi-log schemes. This means that we prove adjunction (see Theorem 6.3.4 (i)), various Kodaira type vanishing theorems (see Theorem 6.3.4 (ii)), basepoint-free theorem (see Theorem 6.5.1), rationality theorem (see Theorem 6.6.1), cone and contraction theorem (see Theorem 6.7.4), and so on, for quasi-log schemes after we introduce the notion of quasi-log schemes. Note that our formulation of the theory of quasi-log schemes is slightly different from Ambro's original one in [**Am1**].

In this book, we adopt the following definition of quasi-log schemes.

DEFINITION 6.1.1 (Quasi-log schemes, see Definition 6.2.2). A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of subvarieties $\{C\}$ coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data
$$\big(X, \omega, f : (Y, B_Y) \to X\big)$$
if there is no risk of confusion. The subvarieties C are called the *qlc strata* of $[X, \omega]$.

Once we establish the following adjunction and vanishing theorem for quasi-log schemes (see Theorem 6.3.4), the notion of quasi-log schemes becomes very useful. Therefore, Theorem 6.1.2 is a key result of the theory of quasi-log schemes. The proof of Theorem 6.1.2 heavily depends on the results discussed in Chapter 5.

THEOREM 6.1.2 (see Theorem 6.3.4). *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a (possibly empty) union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

(i) (Adjunction). *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*
(ii) (Vanishing theorem). *Assume that $\pi : X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

One of the main results of this chapter is:

THEOREM 6.1.3 (Cone and contraction theorem, Theorem 6.7.4). *Let $[X, \omega]$ be a quasi-log scheme and let $\pi : X \to S$ be a projective morphism between schemes. Then we have the following properties.*

(i) *We have:*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega \geq 0} + \overline{NE}(X/S)_{-\infty} + \sum R_j,$$

where R_j's are the ω-negative extremal rays of $\overline{NE}(X/S)$ that are rational and relatively ample at infinity. In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point.

(ii) *Let H be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(\omega + H)_{<0}$. In particular, the R_j's are discrete in the half-space $\omega_{<0}$.*

(iii) *Let F be an ω-negative extremal face of $\overline{NE}(X/S)$ that is relatively ample at infinity. Then F is a rational face. In particular, F is contractible at infinity.*

We give a proof of Theorem 6.1.3 in Section 6.7 after we establish the basepoint-free theorem for quasi-log schemes (see Theorem 6.5.1) and the rationality theorem for quasi-log schemes (see Theorem 6.6.1). Note that the proof of the basepoint-free theorem and the rationality theorem is based on Theorem 6.1.2.

In Section 6.4, we see that a normal pair has a natural quasi-log structure. By this fact, we can apply the results in this chapter to normal pairs.

As we mentioned above, our treatment is slightly different from Ambro's original one. So, if the reader wants to taste the original flavor of the theory of quasi-log varieties, then we recommend him to see [**Am1**].

6.2. On quasi-log schemes

First, let us recall the definition of globally embedded simple normal crossing pairs in order to define quasi-log schemes.

Definition 6.2.1 (Globally embedded simple normal crossing pairs, see [**F17**, Definition 2.16]). Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is a simple normal crossing divisor on M and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y).

Let us define quasi-log schemes.

Definition 6.2.2 (Quasi-log schemes). A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω

on X, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.

(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

(3) The collection of subvarieties $\{C\}$ coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$\big(X, \omega, f : (Y, B_Y) \to X\big)$$

if there is no risk of confusion. Note that a quasi-log scheme $[X, \omega]$ is the union of $\{C\}$ and $X_{-\infty}$. We also note that ω is called the *quasi-log canonical class* of $[X, \omega]$, which is defined up to $\mathbb{R}$-linear equivalence. A *relative quasi-log scheme* X/S is a quasi-log scheme X endowed with a proper morphism $\pi : X \to S$. We sometimes simply say that $[X, \omega]$ is a *quasi-log pair*. The subvarieties C are called the *qlc strata* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$, and $f : (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$.

For the details of the various definitions of quasi-log schemes, see [**F39**, Section 4 and Section 8].

Remark 6.2.3. Let $\mathrm{Div}(Y)$ be the group of Cartier divisors on Y and let $\mathrm{Pic}(Y)$ be the Picard group of Y. Let

$$\delta_Y : \mathrm{Div}(Y) \otimes \mathbb{R} \to \mathrm{Pic}(Y) \otimes \mathbb{R}$$

be the homomorphism induced by $A \mapsto \mathcal{O}_Y(A)$ where A is a Cartier divisor on Y. When ω is an $\mathbb{R}$-line bundle in Definition 6.2.2,

$$f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$$

means

$$f^*\omega = \delta_Y(K_Y + B_Y)$$

in $\mathrm{Pic}(Y) \otimes \mathbb{R}$. Even when ω is an $\mathbb{R}$-line bundle, we usually use $-\omega$ to denote the inverse of ω in $\mathrm{Pic}(X) \otimes \mathbb{R}$ if there is no risk of confusion.

We give an important remark on Definition 6.2.2.

REMARK 6.2.4 (Schemes versus varieties). A quasi-log *scheme* in Definition 6.2.2 is called a quasi-log *variety* in [**Am1**] (see also [**F17**]). However, X is not always reduced when $X_{-\infty} \neq \emptyset$ (see Example 6.2.5 below). Therefore, we will use the word *quasi-log schemes* in this paper. Note that X is reduced when $X_{-\infty} = \emptyset$ (see Remark 6.2.11 below).

EXAMPLE 6.2.5 ([**Am1**, Examples 4.3.4]). Let X be an effective Cartier divisor on a smooth variety M. Assume that Y, the reduced part of X, is non-empty. We put $\omega = (K_M + X)|_X$. Let $X_{-\infty}$ be the union of the non-reduced components of X. We put $K_Y + B_Y = (K_M + X)|_Y$. Let $f : Y \to X$ be the closed embedding. Then

$$\big(X, \omega, f : (Y, B_Y) \to X\big)$$

is a quasi-log scheme. Note that X has non-reduced irreducible components if $X_{-\infty} \neq \emptyset$. We also note that f is not surjective if $X_{-\infty} \neq \emptyset$.

REMARK 6.2.6. A *qlc stratum* of $[X, \omega]$ was originally called a *qlc center* of $[X, \omega]$ in the literature (see, [**Am1**], [**F17**], and so on). We changed the terminology.

DEFINITION 6.2.7 (Qlc centers). A closed subvariety C of X is called a *qlc center* of $[X, \omega]$ if C is a qlc stratum of $[X, \omega]$ which is not an irreducible component of X.

For various applications, the notion of qlc pairs is very useful.

DEFINITION 6.2.8 (Qlc pairs). Let $[X, \omega]$ be a quasi-log pair. We say that $[X, \omega]$ has only *quasi-log canonical singularities* (*qlc singularities*, for short) if $X_{-\infty} = \emptyset$. Assume that $[X, \omega]$ is a quasi-log pair with $X_{-\infty} = \emptyset$. Then we sometimes simply say that $[X, \omega]$ is a *qlc pair*.

We give some important remarks on the non-qlc locus $X_{-\infty}$.

REMARK 6.2.9. We put $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$. Then we obtain the following big commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{O}_Y(A-N) \arrow[r] & f_*\mathcal{O}_Y(A) \arrow[r] & f_*\mathcal{O}_N(A) & \\
0 \arrow[r] & f_*\mathcal{O}_Y(-N) \arrow[r] \arrow[u, "\alpha_1"] & f_*\mathcal{O}_Y \arrow[r] \arrow[u, "\alpha_2"] & f_*\mathcal{O}_N \arrow[u, "\alpha_3"] & \\
0 \arrow[r] & \mathcal{I}_{X_{-\infty}} \arrow[r] \arrow[u, "\beta_1"] & \mathcal{O}_X \arrow[r] \arrow[u, "\beta_2"] & \mathcal{O}_{X_{-\infty}} \arrow[r] \arrow[u, "\beta_3"] & 0
\end{tikzcd}$$

Note that α_i is a natural injection for every i. By an easy diagram chasing,

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(A-N)$$

factors through $f_*\mathcal{O}_Y(-N)$. Then we obtain β_1 and β_3. Since α_1 is injective and $\alpha_1 \circ \beta_1$ is an isomorphism, α_1 and β_1 are isomorphisms. Therefore, we obtain that $f(Y) \cap X_{-\infty} = f(N)$. Note that f is not always surjective when $X_{-\infty} \neq \emptyset$. It sometimes happens that $X_{-\infty}$ contains some irreducible components of X. See, for example, Example 6.2.5.

Remark 6.2.10 (Semi-normality). By restricting the isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(A-N)$$

to the open subset $U = X \setminus X_{-\infty}$, we obtain

$$\mathcal{O}_U \xrightarrow{\simeq} f_*\mathcal{O}_{f^{-1}(U)}(A).$$

This implies that

$$\mathcal{O}_U \xrightarrow{\simeq} f_*\mathcal{O}_{f^{-1}(U)}$$

because A is effective. Therefore, $f : f^{-1}(U) \to U$ is surjective and has connected fibers. Note that $f^{-1}(U)$ is a simple normal crossing variety. Thus, U is semi-normal. In particular, $U = X \setminus X_{-\infty}$ is reduced.

Remark 6.2.11. If the pair $[X, \omega]$ has only qlc singularities, equivalently, $X_{-\infty} = \emptyset$, then X is reduced and semi-normal by Remark 6.2.10. Note that $f(Y) \cap X_{-\infty} = \emptyset$ if and only if $B_Y = B_Y^{\leq 1}$, equivalently, $B_Y^{>1} = 0$, by the descriptions in Remark 6.2.9.

We close this section with the definition of nef and log big divisors on quasi-log schemes.

Definition 6.2.12 (Nef and log big divisors on quasi-log schemes). Let L be an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) on a quasi-log pair $[X, \omega]$ and let $\pi : X \to S$ be a proper morphism between schemes. Then L is *nef and log big over S with respect to* $[X, \omega]$ if L is π-nef and $L|_C$ is π-big for every qlc stratum C of $[X, \omega]$.

6.3. Basic properties of quasi-log schemes

In this section, we discuss some basic properties of quasi-log schemes. Theorem 6.3.4 is the main theorem of this section. Note that Theorem 6.3.4 heavily depends on the results discussed in Chapter 5.

The following proposition makes the theory of quasi-log schemes flexible.

Proposition 6.3.1 ([F17, Proposition 3.50]). *Let $f : V \to W$ be a proper birational morphism between smooth varieties and let B_W be an*

$\mathbb{R}$-divisor on W such that $\operatorname{Supp} B_W$ *is a simple normal crossing divisor on W. Assume that*

$$K_V + B_V = f^*(K_W + B_W)$$

and that $\operatorname{Supp} B_V$ *is a simple normal crossing divisor on V. Then we have*

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor).$$

Furthermore, let S be a simple normal crossing divisor on W such that $S \subset$ $\operatorname{Supp} B_W^{=1}$*. Let T be the union of the irreducible components of $B_V^{=1}$ that are mapped into S by f. Assume that* $\operatorname{Supp} f_*^{-1}B_W \cup \operatorname{Exc}(f)$ *is a simple normal crossing divisor on V. Then we have*

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1})\rceil - \lfloor B_S^{>1}\rfloor),$$

where $(K_V + B_V)|_T = K_T + B_T$ and $(K_W + B_W)|_S = K_S + B_S$.

Proof. By $K_V + B_V = f^*(K_W + B_W)$, we obtain

$$\begin{aligned} K_V =& f^*(K_W + B_W^{=1} + \{B_W\}) \\ &+ f^*(\lfloor B_W^{<1}\rfloor + \lfloor B_W^{>1}\rfloor) - (\lfloor B_V^{<1}\rfloor + \lfloor B_V^{>1}\rfloor) - B_V^{=1} - \{B_V\}. \end{aligned}$$

If $a(\nu, W, B_W^{=1} + \{B_W\}) = -1$ for a prime divisor ν over W, then we can check that $a(\nu, W, B_W) = -1$ by the same argument as in the proof of Lemma 2.3.9. Since

$$f^*(\lfloor B_W^{<1}\rfloor + \lfloor B_W^{>1}\rfloor) - (\lfloor B_V^{<1}\rfloor + \lfloor B_V^{>1}\rfloor)$$

is Cartier, we can easily see that

$$f^*(\lfloor B_W^{<1}\rfloor + \lfloor B_W^{>1}\rfloor) = \lfloor B_V^{<1}\rfloor + \lfloor B_V^{>1}\rfloor + E,$$

where E is an effective f-exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor) \simeq \mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor).$$

Next, we consider the short exact sequence:

$$\begin{aligned} 0 &\to \mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T) \\ &\to \mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor) \to \mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \to 0. \end{aligned}$$

Since $T = f^*S - F$, where F is an effective f-exceptional divisor, we can easily see that

$$f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T) \simeq \mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor - S).$$

We note that

$$\begin{aligned} (\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T) - (K_V + \{B_V\} + B_V^{=1} - T) \\ = -f^*(K_W + B_W). \end{aligned}$$

Therefore, every associated prime of $R^1f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T)$ is the generic point of the f-image of some stratum of $(V, \{B_V\}+B_V^{=1}-T)$ (see, for example, Theorem 5.6.3, Theorem 3.16.3, and [**F28**, Theorem 6.3 (i)]).

Claim. *No strata of $(V, \{B_V\}+B_V^{=1}-T)$ are mapped into S by f.*

Proof of Claim. Assume that there is a stratum C of $(V, \{B_V\}+B_V^{=1}-T)$ such that $f(C)\subset S$. Note that

$$\mathrm{Supp}\, f^*S \subset \mathrm{Supp}\, f_*^{-1}B_W \cup \mathrm{Exc}(f)$$

and

$$\mathrm{Supp}\, B_V^{=1} \subset \mathrm{Supp}\, f_*^{-1}B_W \cup \mathrm{Exc}(f).$$

Since C is also a stratum of $(V, B_V^{=1})$ and

$$C \subset \mathrm{Supp}\, f^*S,$$

there exists an irreducible component G of $B_V^{=1}$ such that

$$C \subset G \subset \mathrm{Supp}\, f^*S.$$

Therefore, by the definition of T, G is an irreducible component of T because $f(G)\subset S$ and G is an irreducible component of $B_V^{=1}$. So, C is not a stratum of $(V, \{B_V\}+B_V^{=1}-T)$. This is a contradiction. $\square$

On the other hand, $f(T)\subset S$. Therefore,

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \to R^1f_*\mathcal{O}_V(\lceil -(B_Z^{<1})\rceil - \lfloor B_Z^{>1}\rfloor - T)$$

is a zero map by Claim. Thus, we obtain

$$f_*\mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \simeq \mathcal{O}_S(\lceil -(B_S^{<1})\rceil - \lfloor B_S^{>1}\rfloor)$$

by the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor - S) \arrow[r, "\simeq"] \arrow[d] & f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor - T) \arrow[d] \\
\mathcal{O}_W(\lceil -(B_W^{<1})\rceil - \lfloor B_W^{>1}\rfloor) \arrow[r, "\simeq"] \arrow[d] & f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil - \lfloor B_V^{>1}\rfloor) \arrow[d] \\
\mathcal{O}_S(\lceil -(B_S^{<1})\rceil - \lfloor B_S^{>1}\rfloor) \arrow[r] \arrow[d] & f_*\mathcal{O}_T(\lceil -(B_T^{<1})\rceil - \lfloor B_T^{>1}\rfloor) \arrow[d] \\
0 & 0
\end{tikzcd}$$

We finish the proof. $\square$

It is easy to check:

PROPOSITION 6.3.2. *In Proposition 6.3.1, let C' be a log canonical center of (V, B_V) contained in T. Then $f(C')$ is a log canonical center of (W, B_W) contained in S or $f(C')$ is contained in $\operatorname{Supp} B_W^{>1}$. Let C be a log canonical center of (W, B_W) contained in S. Then there exists a log canonical center C' of (V, B_V) contained in T such that $f(C') = C$.*

The following important theorem is missing in [**F17**].

THEOREM 6.3.3. *In Definition 6.2.2, we may assume that the ambient space M of the globally embedded simple normal crossing pair (Y, B_Y) is quasi-projective. In particular, Y is quasi-projective.*

PROOF. In Definition 6.2.2, we may assume that $D + Y$ is an $\mathbb{R}$-divisor on a smooth variety M such that $\operatorname{Supp}(D + Y)$ is a simple normal crossing divisor on M, D and Y have no common irreducible components, and $B_Y = D|_Y$ as in Definition 6.2.1. Let $g : M' \to M$ be a projective birational morphism from a smooth quasi-projective variety M' with the following properties:

- (i) $K_{M'} + B_{M'} = g^*(K_M + D + Y)$,
- (ii) $\operatorname{Supp} B_{M'}$ is a simple normal crossing divisor on M', and
- (iii) $\operatorname{Supp} g_*^{-1}(D + Y) \cup \operatorname{Exc}(g)$ is also a simple normal crossing divisor on M'.

Let Y' be the union of the irreducible components of $B_{M'}^{=1}$ that are mapped into Y by g. We put

$$(K_{M'} + B_{M'})|_{Y'} = K_{Y'} + B_{Y'}.$$

Then

$$g_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - \lfloor B_{Y'}^{>1}\rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor)$$

by Proposition 6.3.1. This implies that

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*g_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - \lfloor B_{Y'}^{>1}\rfloor).$$

By the above construction,

$$K_{Y'} + B_{Y'} = g^*(K_Y + B_Y) \sim_{\mathbb{R}} g^*f^*\omega.$$

By Proposition 6.3.2, the collection of subvarieties $\{C\}$ in Definition 6.2.2 coincides with the images of $(Y', B_{Y'})$-strata that are not contained in $X_{-\infty}$. Therefore, by replacing M and (Y, B_Y) with M' and $(Y', B_{Y'})$, we may assume that the ambient space M is quasi-projective. □

The following theorem is a key result for the theory of quasi-log schemes. It follows from the Kollár type torsion-free and vanishing theorem for simple normal crossing varieties discussed in Chapter 5 (see Theorem 5.6.3 and Theorem 5.7.3).

THEOREM 6.3.4 (see [**Am1**, Theorems 4.4 and 7.3] and [**F17**, Theorem 3.39]). *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a (possibly empty) union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

(i) (Adjunction). *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*

(ii) (Vanishing theorem). *Assume that $\pi : X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

PROOF. By taking some blow-ups of the ambient space M of (Y, B_Y), we may assume that the union of all strata of (Y, B_Y) mapped to X', which is denoted by Y', is a union of irreducible components of Y (see Proposition 6.3.1). We put $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$ and $Y'' = Y - Y'$. We prove that $f : (Y', B_{Y'}) \to X'$ gives the desired quasi-log structure on $[X', \omega']$. By construction, we have $f^*\omega' \sim_{\mathbb{R}} K_{Y'} + B_{Y'}$ on Y'. We put $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$. We consider the following short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

By applying $\otimes \mathcal{O}_Y(A - N)$, we have

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

By applying f_*, we obtain

$$\begin{aligned} 0 \to f_*\mathcal{O}_{Y''}(A - N - Y') &\to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{Y'}(A - N) \\ &\to R^1 f_*\mathcal{O}_{Y''}(A - N - Y') \to \cdots . \end{aligned}$$

By Theorem 5.6.3, no associated prime of $R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$ is contained in $X' = f(Y')$. We note that

$$\begin{aligned} (A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''}) &= -(K_{Y''} + B_{Y''}) \\ &\sim_{\mathbb{R}} -(f^*\omega)|_{Y''}, \end{aligned}$$

where $K_{Y''}+B_{Y''}=(K_Y+B_Y)|_{Y''}$. Therefore, the connecting homomorphism $\delta : f_*\mathcal{O}_{Y'}(A-N)\to R^1f_*\mathcal{O}_{Y''}(A-N-Y')$ is zero. Thus we obtain the following short exact sequence

$$0\to f_*\mathcal{O}_{Y''}(A-N-Y')\to \mathcal{I}_{X_{-\infty}}\to f_*\mathcal{O}_{Y'}(A-N)\to 0.$$

We put $\mathcal{I}_{X'}=f_*\mathcal{O}_{Y''}(A-N-Y')$. Then $\mathcal{I}_{X'}$ defines a scheme structure on X'. We put $\mathcal{I}_{X'_{-\infty}}=\mathcal{I}_{X_{-\infty}}/\mathcal{I}_{X'}$. Then $\mathcal{I}_{X'_{-\infty}}\simeq f_*\mathcal{O}_{Y'}(A-N)$ by the above exact sequence. By the following big commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{O}_{Y''}(A-N-Y') \arrow[r] \arrow[d] & f_*\mathcal{O}_Y(A-N) \arrow[r] \arrow[d] & f_*\mathcal{O}_{Y'}(A-N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\mathcal{O}_{Y''}(A-Y') \arrow[r] & f_*\mathcal{O}_Y(A) \arrow[r] & f_*\mathcal{O}_{Y'}(A) & \\
0 \arrow[r] & \mathcal{I}_{X'} \arrow[r] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u] & \mathcal{O}_{X'} \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

we can see that $\mathcal{O}_{X'}\to f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X'_{-\infty}}\xrightarrow{\simeq} f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil-\lfloor B_{Y'}^{>1}\rfloor).$$

Therefore, $[X',\omega']$ is a quasi-log pair such that $X'_{-\infty}=X_{-\infty}$. We note the following big commutative diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_{X'} \arrow[r] \arrow[d, equal] & \mathcal{I}_{X_{-\infty}} \arrow[r] \arrow[d] & \mathcal{I}_{X'_{-\infty}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_{X'} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{X_{-\infty}} \arrow[r, "\simeq"] \arrow[d] & \mathcal{O}_{X'_{-\infty}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

By construction, the property on qlc strata is obvious. So, we obtain the desired quasi-log structure of $[X',\omega']$ in (i).

Let $f:(Y,B_Y)\to X$ be a quasi-log resolution as in the proof of (i). If $X'=X_{-\infty}$ in the above proof of (i), then we can easily see that

$$f_*\mathcal{O}_{Y''}(A-N-Y')\simeq f_*\mathcal{O}_{Y''}(A-N)\simeq \mathcal{I}_{X_{-\infty}}=\mathcal{I}_{X'}.$$

Note that

$$f^*(L-\omega) \sim_{\mathbb{R}} f^*L - (K_{Y''} + B_{Y''})$$

on Y'', where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. We also note that

$$\begin{aligned} & f^*L - (K_{Y''} + B_{Y''}) \\ & = (f^*L + A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''}) \end{aligned}$$

and that no stratum of $(Y'', \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''})$ is mapped to $X_{-\infty}$. Then, by Theorem 5.7.3, we have

$$R^i\pi_*(f_*\mathcal{O}_{Y''}(f^*L + A - N - Y')) = R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$$

for every $i > 0$. Thus, we obtain the desired vanishing theorem in (ii). $\square$

Let us recall the following well-known lemma for the reader's convenience (see [**Am1**, Proposition 4.7] and [**F17**, Proposition 3.44]).

Lemma 6.3.5. *Let $[X, \omega]$ be a quasi-log scheme with $X_{-\infty} = \emptyset$. Assume that X is the unique qlc stratum of $[X, \omega]$. Then X is normal.*

The following proof is different from Ambro's original one (see [**Am1**, Proposition 4.7]).

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Since $X_{-\infty} = \emptyset$, we have $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) \simeq \mathcal{O}_X$. This implies that $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. Let $\nu : X^\nu \to X$ be the normalization. By assumption, X is irreducible and every stratum of (Y, B_Y) is mapped onto X. Thus the indeterminacy locus of $\nu^{-1} \circ f : Y \dashrightarrow X^\nu$ contains no strata of (Y, B_Y). By modifying (Y, B_Y) suitably by Proposition 6.3.1, we may assume that $f : Y \to X$ factors through X^ν.

$$\begin{tikzcd} Y \arrow[d, "\overline{f}"'] \arrow[dr, "f"] & \\ X^\nu \arrow[r, "\nu"'] & X \end{tikzcd}$$

Note that the composition

$$\mathcal{O}_X \to \nu_*\mathcal{O}_{X^\nu} \to \nu_*\overline{f}_*\mathcal{O}_Y = f_*\mathcal{O}_Y \simeq \mathcal{O}_X$$

is an isomorphism. This implies that $\mathcal{O}_X \simeq \nu_*\mathcal{O}_{X^\nu}$. Therefore, X is normal. $\square$

We introduce $\mathrm{Nqklt}(X, \omega)$, which is a generalization of the notion of non-klt loci of normal pairs.

NOTATION 6.3.6. Let $[X, \omega]$ be a quasi-log scheme. The union of $X_{-\infty}$ with all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega)$. If $\mathrm{Nqklt}(X, \omega) \neq X_{-\infty}$, then

$$[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}]$$

is a quasi-log scheme by Theorem 6.3.4 (i). Note that $\mathrm{Nqklt}(X, \omega)$ is denoted by $\mathrm{LCS}(X)$ and is called the *LCS locus* of a quasi-log scheme $[X, \omega]$ in [**Am1**, Definition 4.6].

Theorem 6.3.7 is also a key result for the theory of quasi-log schemes.

THEOREM 6.3.7 (see [**Am1**, Proposition 4.8] and [**F17**, Theorem 3.45]). *Assume that $[X, \omega]$ is a quasi-log scheme with $X_{-\infty} = \emptyset$. Then we have the following properties.*

(i) *The intersection of two qlc strata is a union of qlc strata.*
(ii) *For any closed point $x \in X$, the set of all qlc strata passing through x has a unique minimal* (*with respect to the inclusion*) *element C_x. Moreover, C_x is normal at x.*

PROOF. Let C_1 and C_2 be two qlc strata of $[X, \omega]$. We fix $P \in C_1 \cap C_2$. It is enough to find a qlc stratum C such that $P \in C \subset C_1 \cap C_2$. The union $X' = C_1 \cup C_2$ with $\omega' = \omega|_{X'}$ is a qlc pair having two irreducible components. Hence, it is not normal at P. By Lemma 6.3.5, $P \in \mathrm{Nqklt}(X', \omega')$. Therefore, there exists a qlc stratum $C \subset C_1$ with $\dim C < \dim C_1$ such that $P \in C \cap C_2$. If $C \subset C_2$, then we are done. Otherwise, we repeat the argument with $C_1 = C$ and reach the conclusion in a finite number of steps. So, we finish the proof of (i). The uniqueness of the minimal (with respect to the inclusion) qlc stratum follows from (i) and the normality of the minimal stratum follows from Lemma 6.3.5. Thus, we have (ii). $\square$

Lemma 6.3.8 is obvious. We will sometimes use it implicitly in the theory of quasi-log schemes.

LEMMA 6.3.8. *Let $[X, \omega]$ be a quasi-log scheme. Assume that $X = V \cup X_{-\infty}$ and $V \cap X_{-\infty} = \emptyset$. Then $[V, \omega|_V]$ is a quasi-log scheme with only quasi-log canonical singularities.*

The following lemma is a slight generalization of [**F17**, Lemma 3.71], which will play a crucial role in the proof of the rationality theorem for quasi-log schemes (see Theorem 6.6.1).

LEMMA 6.3.9 (see [**F17**, Lemma 3.71] and [**F40**, Lemma 3.16]). *Let $[X, \omega]$ be a quasi-log scheme with $X_{-\infty} = \emptyset$ and let $x \in X$ be a closed point. Let $D_1, D_2, \cdots, D_k$ be effective Cartier divisors on X such that $x \in \mathrm{Supp}\, D_i$ for every i. Let $f : (Y, B_Y) \to X$ be a quasi-log* *resolution. Assume that the normalization of* $(Y, B_Y + \sum_{i=1}^k f^*D_i)$ *is sub log canonical. This means that* (Y^ν, Ξ) *is sub log canonical, where* $\nu : Y^\nu \to Y$ *is the normalization and*

$$K_{Y^\nu} + \Xi = \nu^*(K_Y + B_Y + \sum_{i=1}^k f^*D_i).$$

Note that it requires that no irreducible component of Y *is mapped into* $\bigcup_{i=1}^k \operatorname{Supp} D_i$. *Then* $k \leq \dim_x X$. *More precisely,* $k \leq \dim_x C_x$, *where* C_x *is the minimal qlc stratum of* $[X, \omega]$ *passing through* x.

Proof. We prove this lemma by induction on the dimension.

Step 1. By Proposition 6.3.1, we may assume that

$$(Y, \operatorname{Supp} B_Y + \sum_{i=1}^k f^*D_i)$$

is a globally embedded simple normal crossing pair. Note that

$$f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) \simeq \mathcal{O}_X.$$

Therefore, there is a stratum S_i of $(Y, B_Y + f^*D_i)$ mapped onto D_i for every i. Note that $f : (Y, B_Y + \sum_{i=1}^k f^*D_i) \to X$ gives a natural quasi-log structure on $[X, \omega + \sum_{i=1}^k D_i]$ with only quasi-log canonical singularities.

Step 2. In this step, we assume that $\dim_x X = 1$. If x is a qlc stratum of $[X, \omega]$, then we have $k = 0$. Therefore, we may assume that x is not a qlc stratum of $[X, \omega]$. By shrinking X around x, we may assume that every stratum of (Y, B_Y) is mapped onto X. Then X is irreducible and normal (see Lemma 6.3.5), and $f : Y \to X$ is flat. In this case, we can easily check that $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) \simeq \mathcal{O}_X$ implies $k \leq 1 = \dim_x X$.

Step 3. We assume that $\dim_x X \geq 2$. If x is a qlc stratum of $[X, \omega]$, then $k = 0$. So we may assume that x is not a qlc stratum of $[X, \omega]$. Let C be the minimal qlc stratum of $[X, \omega]$ passing through x. By shrinking X around x, we may assume that C is normal (see Theorem 6.3.7). By Proposition 6.3.1, we may assume that the union of all strata of (Y, B_Y) mapped to C, which is denoted by Y', is a union of some irreducible components of Y. Then $f : (Y', B_{Y'}) \to C$ gives a natural quasi-log structure induced by the original quasi-log structure $f : (Y, B_Y) \to X$ (see Theorem 3.2.7). Therefore, by the induction on the dimension, we have $k \leq \dim_x C \leq \dim_x X$ when $\dim_x C < \dim_x X$. Thus we may assume that X is the unique qlc stratum of $[X, \omega]$. Note that f :

$(Y, B_Y + f^*D_1) \to X$ gives a natural quasi-log structure on $[X, \omega + D_1]$ with only quasi-log canonical singularities. Let X' be the union of qlc strata of $[X, \omega + D_1]$ contained in $\operatorname{Supp} D_1$. Then $[X', (\omega + D_1)|_{X'}]$ is a qlc pair with $\dim_x X' < \dim_x X$. Note that $[X', (\omega + D_1)|_{X'}]$ with $D_2|_{X'}, \cdots, D_k|_{X'}$ satisfies the condition similar to the original one for $[X, \omega]$ with $D_1, \cdots, D_k$. Therefore, $k - 1 \le \dim_x X' < \dim_x X$. This implies $k \le \dim_x X$.

Anyway, we obtained the desired inequality $k \le \dim_x C_x$, where C_x is the minimal qlc stratum of $[X, \omega]$ passing through x. □

6.4. On quasi-log structures of normal pairs

In this section, we see that a normal pair has a natural quasi-log structure. By this fact, we can use the theory of quasi-log schemes for the study of normal pairs. Moreover, we discuss toric varieties and toric polyhedra as examples of quasi-log schemes.

6.4.1 (Quasi-log structures for normal pairs). Let (X, B) be a normal pair, that is, X is a normal variety and B is an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. We put $\omega = K_X + B$. Let $f : Y \to X$ be a resolution such that $\operatorname{Supp} f_*^{-1}B \cup \operatorname{Exc}(f)$ is a simple normal crossing divisor on Y. We put

$$K_Y + B_Y = f^*(K_X + B).$$

Since B is effective, $\lceil -(B_Y^{<1}) \rceil$ is effective and f-exceptional. Therefore, $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil) \simeq \mathcal{O}_X$. We put

$$\mathcal{I}_{X_{-\infty}} = f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor).$$

Then $\mathcal{I}_{X_{-\infty}}$ is an ideal sheaf on X. By Proposition 6.3.1, $\mathcal{I}_{X_{-\infty}}$ is independent of the resolution $f : Y \to X$. It is nothing but the non-lc ideal sheaf $\mathcal{J}_{\mathrm{NLC}}(X, B)$ of the pair (X, B). By this ideal sheaf $\mathcal{I}_{X_{-\infty}}$, we define a proper closed subscheme $X_{-\infty}$ of X. Let $\{C\}$ be the set of log canonical strata of (X, B). Then, by definition, the set $\{C\}$ coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$. We put $M = Y \times \mathbb{C}$ and $D = B \times \mathbb{C}$. Then $(Y, B_Y) \simeq (Y \times \{0\}, B_Y \times \{0\})$ is a globally embedded simple normal crossing pair with the ambient space M. Therefore, the data

$$\big(X, \omega, f : (Y, B_Y) \to X\big)$$

gives a natural quasi-log structure, which is compatible with the original structure of the normal pair (X, B). By the above description, (X, B) is log canonical if and only if $\big(X, \omega, f : (Y, B_Y) \to X\big)$ has only qlc singularities. Therefore, the notion of qlc pairs is a generalization of that of log canonical pairs.

6.4.2. We use the same notation as in 6.4.1. Let X' be the union of $X_{-\infty}$ with a union of some log canonical centers of (X, B). Then, by Theorem 6.3.4 (i), X' has a natural quasi-log structure with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the log canonical centers of (X, B) that are contained in X'. By the proof of Theorem 6.3.4 and Proposition 6.3.1, we can easily check that the quasi-log structure of $[X', \omega']$ is essentially unique, that is, it is independent of the resolution $f : Y \to X$.

By the descriptions in 6.4.1 and 6.4.2, Theorem 3.16.4 and Theorem 3.16.5 become special cases of Theorem 6.3.4 (ii).

We can treat toric varieties and toric polyhedra as examples of quasi-log schemes.

6.4.3 (Toric polyhedron). Here, we freely use the basic notation of the toric geometry (see, for example, [**Ful**]).

DEFINITION 6.4.4. For a subset Φ of a fan Δ, we say that Φ is *star closed* if $\sigma \in \Phi, \tau \in \Delta$ and $\sigma \prec \tau$ imply $\tau \in \Phi$.

DEFINITION 6.4.5 (Toric polyhedron). For a star closed subset Φ of a fan Δ, we denote by $Y = Y(\Phi)$ the subscheme $\bigcup_{\sigma \in \Phi} V(\sigma)$ of $X = X(\Delta)$, and we call it the *toric polyhedron associated to* Φ.

Let $X = X(\Delta)$ be a toric variety and let D be the complement of the big torus. Then the following property is well known and is easy to check.

PROPOSITION 6.4.6. *The pair (X, D) is log canonical and $K_X + D \sim 0$. Let W be a closed subvariety of X. Then, W is a log canonical center of (X, D) if and only if $W = V(\sigma)$ for some $\sigma \in \Delta \setminus \{0\}$.*

Therefore, we have the following theorem by adjunction (see Theorem 6.3.4 (i)).

THEOREM 6.4.7. *Let $Y = Y(\Phi)$ be a toric polyhedron on $X = X(\Delta)$. Then, the log canonical pair (X, D) induces a natural quasi-log structure on $[Y, 0]$. Note that $[Y, 0]$ has only qlc singularities. Let W be a closed subvariety of Y. Then, W is a qlc stratum of $[Y, 0]$ if and only if $W = V(\sigma)$ for some $\sigma \in \Phi$.*

Thus, we can use the theory of quasi-log schemes to investigate toric varieties and toric polyhedra. For example, we have the following result as a special case of Theorem 6.3.4 (ii).

COROLLARY 6.4.8. *We use the same notation as in Theorem 6.4.7. Assume that X is projective and L is an ample Cartier divisor on X. Then $H^i(X, \mathcal{I}_Y \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_Y$ is the defining ideal sheaf of Y on X. In particular, the restriction map*

$$H^0(X, \mathcal{O}_X(L)) \to H^0(Y, \mathcal{O}_Y(L))$$

is surjective.

We can prove various vanishing theorems for toric varieties and toric polyhedra without appealing the results in Chapter 5. For the details, see [**F11**] and [**F16**].

6.5. Basepoint-free theorem for quasi-log schemes

Theorem 6.5.1 is the main theorem of this section. It is [**Am1**, Theorem 5.1].

THEOREM 6.5.1 (Basepoint-free theorem). *Let $[X, \omega]$ be a quasi-log scheme and let $\pi : X \to S$ be a projective morphism between schemes. Let L be a π-nef Cartier divisor on X. Assume that*

(i) *$qL - \omega$ is π-ample for some real number $q > 0$, and*
(ii) *$\mathcal{O}_{X_{-\infty}}(mL)$ is $\pi|_{X_{-\infty}}$-generated for every $m \gg 0$.*

Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$, that is, there exists a positive number m_0 such that $\mathcal{O}_X(mL)$ is π-generated for every $m \geq m_0$.

Before we prove Theorem 6.5.1, let us prepare some lemmas. Lemma 6.5.2 is a variant of Shokurov's concentration method (see, for example, [**Sh1**] and [**KoMo**, Section 3.5]).

LEMMA 6.5.2 (see [**F28**, Lemma 12.2]). *Let $f : Y \to Z$ be a projective morphism from a normal variety Y onto an affine variety Z. Let V be a general closed subvariety of Y such that $f : V \to Z$ is generically finite. Let M be an f-ample $\mathbb{R}$-divisor on Y. Assume that*

$$(M|_F)^d > km^d,$$

where F is a general fiber of $f : Y \to Z$, $d = \dim F$, and k is the mapping degree of $f : V \to Z$. Then we can find an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on Y such that

$$D \sim_{\mathbb{R}} M$$

and that $\mathrm{mult}_V D > m$. If M is a $\mathbb{Q}$-divisor, then we can make D a $\mathbb{Q}$-divisor with $D \sim_{\mathbb{Q}} M$.

PROOF. We can write

$$M = M_1 + a_2 M_2 + \cdots + a_l M_l,$$

where M_1 is an f-ample $\mathbb{Q}$-divisor such that $(M_1|_F)^d > km^d$, a_i is a positive real number, and M_i is an f-ample Cartier divisor for every i. If M is a $\mathbb{Q}$-divisor, then we may assume that $l = 2$ and a_2 is rational. Let $\mathcal{I}_V$ be the defining ideal sheaf of V on Y. We consider the following exact sequence

$$\begin{aligned} 0 &\to f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm}) \to f_*\mathcal{O}_Y(pM_1) \\ &\to f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{O}_Y/\mathcal{I}_V^{pm}) \to \cdots \end{aligned}$$

for a sufficiently large and divisible positive integer p. By restricting the above sequence to a sufficiently general fiber F of f, we can check that the rank of $f_*\mathcal{O}_Y(pM_1)$ is greater than that of $f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{O}_Y/\mathcal{I}_V^{pm})$ by the usual estimates (see Lemma 6.5.3 below). Therefore, $f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm}) \neq 0$. Let D_1 be a member of

$$H^0(Z, f_*(\mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm})) = H^0(Y, \mathcal{O}_Y(pM_1) \otimes \mathcal{I}_V^{pm})$$

and let D_i be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $D_i \sim_{\mathbb{Q}} M_i$ for $i \geq 2$. We can take D_2 with $\mathrm{mult}_V D_2 > 0$. Then $D = (1/p)D_1 + a_2 D_2 + \cdots + a_l D_l$ satisfies the desired properties. $\square$

We note the following well-known lemma. The proof is obvious.

LEMMA 6.5.3. *Let X be a normal projective variety with* $\dim X = d$ *and let A be an ample $\mathbb{Q}$-divisor on X such that aA is Cartier for some positive integer a. Then*

$$\begin{aligned} h^0(X, \mathcal{O}_X(taA)) &= \chi(X, \mathcal{O}_X(taA)) \\ &= \frac{(taA)^d}{d!} + (\textit{lower terms in } t) \end{aligned}$$

by the Riemann–Roch formula and Serre's vanishing theorem for every $t \gg 0$.

Let $P \in X$ be a smooth point. Then

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{O}_X/m_P^{\alpha} &= \binom{\alpha - 1 + d}{d} \\ &= \frac{\alpha^d}{d!} + (\textit{lower terms in } \alpha) \end{aligned}$$

for all $\alpha \geq 1$, where m_P is the maximal ideal associated to P.

Let us start the proof of Theorem 6.5.1.

Proof of Theorem 6.5.1. Without loss of generality, we may assume that S is affine. We use induction on the dimension of $\dim X \setminus X_{-\infty}$. Theorem 6.5.1 is obviously true when $\dim X \setminus X_{-\infty} = 0$.

Claim 1. *$\mathcal{O}_X(mL)$ is π-generated around $\mathrm{Nqklt}(X, \omega)$ for every $m \gg 0$.*

Proof of Claim. We put $X' = \mathrm{Nqklt}(X, \omega)$. Then $[X', \omega']$, where $\omega' = \omega|_{X'}$, is a quasi-log scheme by adjunction when $X' \neq X_{-\infty}$ (see Theorem 6.3.4 (i)). If $X' = X_{-\infty}$, then $\mathcal{O}_{X'}(mL)$ is π-generated for every $m \gg 0$ by the assumption (ii). If $X' \neq X_{-\infty}$, then $\mathcal{O}_{X'}(mL)$ is π-generated for every $m \gg 0$ by induction on the dimension of $X \setminus X_{-\infty}$. By Theorem 6.3.4 (ii), $R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for every $m \geq q$. Therefore, the restriction map $\pi_*\mathcal{O}_X(mL)) \to \pi_*\mathcal{O}_{X'}(mL)$ is surjective for every $m \geq q$. By the following commutative diagram:

$$\begin{tikzcd}
\pi^*\pi_*\mathcal{O}_X(mL) \arrow[r, "\alpha"] \arrow[d] & \pi^*\pi_*\mathcal{O}_{X'}(mL) \arrow[r] \arrow[d] & 0 \\
\mathcal{O}_X(mL) \arrow[r] & \mathcal{O}_{X'}(mL) \arrow[r] & 0,
\end{tikzcd}$$

we know that $\mathcal{O}_X(mL)$ is π-generated around X' for every $m \gg 0$. $\square$

Claim 2. *$\mathcal{O}_X(mL)$ is π-generated on a non-empty Zariski open set for every $m \gg 0$.*

Proof of Claim. By Claim 1, we may assume that $\mathrm{Nqklt}(X, \omega)$ is empty. Without loss of generality, we may assume that X is connected. Then, by Theorem 6.3.7 and Lemma 6.3.5, X is irreducible and normal. Therefore, we may further assume that S is an irreducible variety.

If L is π-numerically trivial, then $\pi_*\mathcal{O}_X(L)$ is not zero. This is because

$$\begin{aligned}
h^0(X_\eta, \mathcal{O}_{X_\eta}(L)) &= \chi(X_\eta, \mathcal{O}_{X_\eta}(L)) \\
&= \chi(X_\eta, \mathcal{O}_{X_\eta}) = h^0(X_\eta, \mathcal{O}_{X_\eta}) > 0
\end{aligned}$$

by Theorem 6.3.4 (ii) and by [Kle, Chapter II §2 Theorem 1], where X_η is the generic fiber of $\pi : X \to S$. Let D be a general member of $|L|$. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution and let M be the ambient space of (Y, B_Y). By taking blow-ups of M, we may assume that $(Y, \mathrm{Supp}\, B_Y + f^*D)$ is a globally embedded simple normal crossing pair by Proposition 6.3.1. We note that every stratum of (Y, B_Y) is mapped onto X by assumption. We can take a positive real number $c \leq 1$ such that $B_Y + cf^*D$ is a subboundary and some stratum of

$(Y, B_Y + cf^*D)$ does not dominate X. Note that $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) \simeq \mathcal{O}_X$. Then $f : (Y, B_Y + cf^*D) \to X$ gives a natural quasi-log structure on the pair $[X, \omega + cD]$ with only qlc singularities. We note that $qL - (\omega + cD)$ is π-ample. By Claim 1, $\mathcal{O}_X(mL)$ is π-generated around $\mathrm{Nqklt}(X, \omega + cD) \neq \emptyset$ for every $m \gg 0$. So, we may assume that L is not π-numerically trivial.

We take a general closed subvariety V of X such that $\pi : V \to S$ is generically finite. Then we can take an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X such that $\mathrm{mult}_V D > k \cdot \mathrm{codim}_X V$, where k is the mapping degree of $\pi : V \to S$, and that $D \sim_{\mathbb{R}} (q + r)L - \omega$ for some $r > 0$ by Lemma 6.5.2. By taking blow-ups of M, we may assume that $(Y, \mathrm{Supp}\, B_Y + f^*D)$ is a globally embedded simple normal crossing pair by Proposition 6.3.1. By the construction of D, we can find a positive real number $c < 1$ such that $B_Y + cf^*D$ is a subboundary and some stratum of $(Y, B_Y + cf^*D)$ does not dominate X. Note that $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil) \simeq \mathcal{O}_X$. Then $f : (Y, B_Y + cf^*D) \to X$ gives a natural quasi-log structure on the pair $[X, \omega + cD]$ with only qlc singularities. We note that $q'L - (\omega + cD)$ is π-ample by $c < 1$, where $q' = q + cr$. By construction, $\mathrm{Nqklt}(X, \omega + cD)$ is non-empty. Therefore, by applying Claim 1 to $[X, \omega + cD]$, $\mathcal{O}_X(mL)$ is π-generated around $\mathrm{Nqklt}(X, \omega + cD)$ for every $m \gg 0$.

So, we finish the proof of Claim 2. $\square$

Let p be a prime number and let l be a large integer. Then we have that $\pi_*\mathcal{O}_X(p^l L) \neq 0$ by Claim 2 and that $\mathcal{O}_X(p^l L)$ is π-generated around $\mathrm{Nqklt}(X, \omega)$ by Claim 1.

Claim 3. *If the relative base locus* $\mathrm{Bs}_\pi |p^l L|$ (*with the reduced scheme structure*) *is not empty, then there exists a positive integer a such that* $\mathrm{Bs}_\pi |p^{al} L|$ *is strictly smaller than* $\mathrm{Bs}_\pi |p^l L|$.

Proof of Claim. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. We take a general member $D \in |p^l L|$. We note that S is affine and $|p^l L|$ is free around $\mathrm{Nqklt}(X, \omega)$. Thus, f^*D intersects any strata of $(Y, \mathrm{Supp}\, B_Y)$ transversally over $X \setminus \mathrm{Bs}_\pi |p^l L|$ by Bertini and f^*D contains no strata of (Y, B_Y). By taking blow-ups of M suitably, we may assume that $(Y, \mathrm{Supp}\, B_Y + f^*D)$ is a global embedded simple normal crossing pair by Proposition 6.3.1. We take the maximal positive real number c such that $B_Y + cf^*D$ is a subboundary over $X \setminus X_{-\infty}$. We note that $c \leq 1$. Here, we used the fact that $\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ over $X \setminus X_{-\infty}$. Then $f : (Y, B_Y + cf^*D) \to X$ gives a natural quasi-log structure on the pair $[X, \omega' = \omega + cD]$. Note that $\mathrm{Nqlc}(X, \omega) = \mathrm{Nqlc}(X, \omega')$ by construction. We also note that

$[X, \omega']$ has a qlc center C that intersects $\mathrm{Bs}_\pi |p^l L|$ by construction. By induction on the dimension, $\mathcal{O}_{C\cup X_{-\infty}}(mL)$ is π-generated for every $m \gg 0$ since $(q+cp^l)L-(\omega+cD)$ is π-ample. We can lift the sections of $\mathcal{O}_{C\cup X_{-\infty}}(mL)$ to X for $m \geq q+cp^l$ by Theorem 6.3.4 (ii). Then we obtain that $\mathcal{O}_X(mL)$ is π-generated around C for every $m \gg 0$. Therefore, $\mathrm{Bs}_\pi |p^{al}L|$ is strictly smaller than $\mathrm{Bs}_\pi |p^l L|$ for some positive integer a $\square$

CLAIM 4. *$\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

PROOF OF CLAIM. By Claim 3 and the noetherian induction, we obtain that $\mathcal{O}_X(p^l L)$ and $\mathcal{O}_X(p'^{l'} L)$ are π-generated for large l and l', where p and p' are prime numbers and they are relatively prime. So, there exists a positive number m_0 such that $\mathcal{O}_X(mL)$ is π-generated for every $m \geq m_0$. $\square$

Thus we obtained the desired result. $\square$

Corollary 6.5.4 is a special case of Theorem 6.5.1. Note that a log canonical pair has a natural quasi-log structure with only qlc singularities (see 6.4.1).

COROLLARY 6.5.4 (Basepoint-free theorem for log canonical pairs, see [**F27**] and [**F28**]). *Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a projective morphism. Let L be a π-nef Cartier divisor on X. Assume that $qL-(K_X+\Delta)$ is π-ample for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

We strongly recommend the reader to see [**F28**, Section 2], where we describe the difference between the approach discussed in this section and the framework established in [**F28**].

6.6. Rationality theorem for quasi-log schemes

In this section, we prove the following rationality theorem for quasi-log schemes (see [**Am1**, Theorem 5.9]).

THEOREM 6.6.1 (Rationality theorem). *Assume that $[X, \omega]$ is a quasi-log scheme such that ω is $\mathbb{Q}$-Cartier. This means that ω is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier divisor (or a $\mathbb{Q}$-line bundle) on X. Let $\pi : X \to S$ be a projective morphism between schemes and let H be a π-ample Cartier divisor on X. Assume that r is a positive number such that*

(1) *$H + r\omega$ is π-nef but not π-ample, and*
(2) *$(H + r\omega)|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample.*

Then r is a rational number, and in reduced form, r has denominator at most $a(\dim X+1)$, where $a\omega$ is $\mathbb{R}$-linearly equivalent to a Cartier divisor (or a line bundle) on X.

Before the proof of Theorem 6.6.1, we recall the following lemmas.

LEMMA 6.6.2 (see [KoMo, Lemma 3.19]). *Let $P(x,y)$ be a nontrivial polynomial of degree $\leq n$ and assume that P vanishes for all sufficiently large integral solutions of $0 < ay - rx < \varepsilon$ for some fixed positive integer a and positive ε for some $r \in \mathbb{R}$. Then r is rational, and in reduced form, r has denominator $\leq a(n+1)/\varepsilon$.*

PROOF. We assume that r is irrational. Then an infinite number of integral points in the (x,y)-plane on each side of the line $ay-rx=0$ are closer than $\varepsilon/(n+2)$ to that line. So there is a large integral solution (x', y') with $0 < ay' - rx' < \varepsilon/(n+2)$. In this case, we see that

$$(2x', 2y'), \cdots, ((n+1)x', (n+1)y')$$

are also solutions by hypothesis. So $(y'x - x'y)$ divides P, since P and $(y'x - x'y)$ have $(n+1)$ common zeroes. We choose a smaller ε and repeat the argument. We do this $n+1$ times to get a contradiction.

Now we assume that $r = u/v$ in lowest terms. For given j, let (x', y') be a solution of $ay - rx = aj/v$. Note that an integral solution exists for every j. Then we have $a(y'+ku) - r(x'+akv) = aj/v$ for all k. So, as above, if $aj/v < \varepsilon$, $(ay - rx) - (aj/v)$ must divide P. Therefore, we can have at most n such values of j. Thus $a(n+1)/v \geq \varepsilon$. □

LEMMA 6.6.3. *Let C be a projective variety and let D_1 and D_2 be Cartier divisors on X. Consider the Hilbert polynomial*

$$P(u_1, u_2) = \chi(C, \mathcal{O}_C(u_1 D_1 + u_2 D_2)).$$

If D_1 is ample, then $P(u_1, u_2)$ is a nontrivial polynomial of total degree $\leq \dim C$. This is because $P(u_1, 0) = h^0(C, \mathcal{O}_C(u_1 D_1)) \not\equiv 0$ if u_1 is sufficiently large.

PROOF OF THEOREM 6.6.1. Without loss of generality, we may assume that $a\omega$ itself is Cartier. Let m be a positive integer such that $H' = mH$ is π-very ample. If $H' + r'\omega$ is π-nef but not π-ample, and $(H' + r'\omega)|_{\mathrm{Nqlc}(X,\omega)}$ is $\pi|_{\mathrm{Nqlc}(X,\omega)}$-ample, then we have

$$H + r\omega = \frac{1}{m}(H' + r'\omega).$$

This gives $r = \frac{1}{m}r'$. Thus, r is rational if and only if r' is rational. Assume furthermore that r' has denominator v. Then r has denominator dividing mv. Since m can be arbitrary sufficiently large integer, this implies that r has denominator dividing v. Therefore, by replacing H with mH, we may assume that H is very ample over S.

For each $(p,q) \in \mathbb{Z}^2$, let $L(p,q)$ denote the relative base locus of the linear system associated to $M(p,q) = pH + qa\omega$ on X (with the reduced scheme structure), that is,

$$L(p,q) = \mathrm{Supp}(\mathrm{Coker}(\pi^*\pi_*\mathcal{O}_X(M(p,q)) \to \mathcal{O}_X(M(p,q)))).$$

By definition, $L(p,q) = X$ if and only if $\pi_*\mathcal{O}_X(M(p,q)) = 0$.

Claim 1 (cf. [KoMo, Claim 3.20]). *Let ε be a positive real number with $\varepsilon \le 1$. For (p,q) sufficiently large and $0 < aq - rp < \varepsilon$, $L(p,q)$ is the same subset of X. We call this subset L_0. Let $I \subset \mathbb{Z}^2$ be the set of (p,q) for which $0 < aq - rp < 1$ and $L(p,q) = L_0$. Then I contains all sufficiently large (p,q) with $0 < aq - rp < 1$.*

Proof. We fix $(p_0, q_0) \in \mathbb{Z}^2$ such that $p_0 > 0$ and $0 < aq_0 - rp_0 < 1$. Since H is π-very ample, there exists a positive integer m_0 such that $\mathcal{O}_X(mH + ja\omega)$ is π-generated for every $m > m_0$ and every $0 \le j \le q_0 - 1$. Let M be the round-up of

$$\left(m_0 + \frac{1}{r}\right) \Big/ \left(\frac{a}{r} - \frac{p_0}{q_0}\right).$$

If $(p', q') \in \mathbb{Z}^2$ such that $0 < aq' - rp' < 1$ and $q' \ge M + q_0 - 1$, then we can write

$$p'H + q'a\omega = k(p_0 H + q_0 a\omega) + (lH + ja\omega)$$

for some $k \ge 0$, $0 \le j \le q_0 - 1$ with $l > m_0$. This is because we can uniquely write $q' = kq_0 + j$ with $0 \le j \le q_0 - 1$. Thus, we have $kq_0 \ge M$. So, we obtain

$$l = p' - kp_0 > \frac{a}{r}q' - \frac{1}{r} - (kq_0)\frac{p_0}{q_0} \ge \left(\frac{a}{r} - \frac{p_0}{q_0}\right)M - \frac{1}{r} \ge m_0.$$

Therefore, $L(p', q') \subset L(p_0, q_0)$. By the noetherian induction, we obtain the desired closed subset $L_0 \subset X$ and $I \subset \mathbb{Z}^2$. $\square$

Claim 2. *We have $L_0 \cap \mathrm{Nqlc}(X, \omega) = \emptyset$.*

Proof. We take $(\alpha, \beta) \in \mathbb{Q}^2$ such that $\alpha > 0$, $\beta > 0$, and $\beta a/\alpha > r$ is sufficiently close to r. Then $(\alpha H + \beta a\omega)|_{\mathrm{Nqlc}(X,\omega)}$ is $\pi|_{\mathrm{Nqlc}(X,\omega)}$-ample because $(H + r\omega)|_{\mathrm{Nqlc}(X,\omega)}$ is $\pi|_{\mathrm{Nqlc}(X,\omega)}$-ample. If $0 < aq - rp < 1$ and $(p,q) \in \mathbb{Z}^2$ is sufficiently large, then

$$M(p,q) = mM(\alpha, \beta) + (M(p,q) - mM(\alpha, \beta))$$

such that $M(p,q) - mM(\alpha, \beta)$ is π-very ample and that

$$m(\alpha H + \beta a\omega)|_{\mathrm{Nqlc}(X,\omega)}$$

is also $\pi|_{\mathrm{Nqlc}(X,\omega)}$-very ample. It can be checked by the same argument as in the proof of Claim 1. Therefore, $\mathcal{O}_{\mathrm{Nqlc}(X,\omega)}(M(p,q))$ is π-very ample. Since

$$\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_{\mathrm{Nlc}(X,B)}(M(p,q))$$

is surjective by the vanishing theorem: Theorem 6.3.4 (ii), we obtain $L(p,q) \cap \mathrm{Nqlc}(X,\omega) = \emptyset$. We note that

$$M(p,q) - \omega = pH + (qa-1)\omega$$

is π-ample because (p,q) is sufficiently large and $aq - rp < 1$. By Claim 1, we have $L_0 \cap \mathrm{Nqlc}(X,\omega) = \emptyset$. $\square$

Claim 3. *We assume that r is not rational or that r is rational and has denominator $> a(n+1)$ in reduced form, where $n = \dim X$. Then, for (p,q) sufficiently large and $0 < aq - rp < 1$, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of every qlc stratum of $[X,\omega]$.*

Proof of Claim. We note that

$$M(p,q) - \omega = pH + (qa-1)\omega.$$

If $aq - rp < 1$ and (p,q) is sufficiently large, then $M(p,q) - \omega$ is π-ample. Let C be a qlc stratum of $[X,\omega]$. We note that we may assume $C \cap \mathrm{Nqlc}(X,\omega) = \emptyset$ by Claim 2. Then

$$P_{C_\eta}(p,q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$$

is a non-zero polynomial of degree at most $\dim C_\eta \leq \dim X$ by Lemma 6.6.3. Note that C_η is the generic fiber of $C \to \pi(C)$. By Lemma 6.6.2, there exists (p,q) such that $P_{C_\eta}(p,q) \neq 0$, (p,q) sufficiently large, and $0 < aq - rp < 1$. By the π-ampleness of $M(p,q) - \omega$,

$$P_{C_\eta}(p,q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$$

and

$$\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_C(M(p,q))$$

is surjective by Theorem 6.3.4 (ii). We note that

$$R^1\pi_*(\mathcal{I}_{C \cup \mathrm{Nqlc}(X,\omega)} \otimes \mathcal{O}_X(M(p,q))) = 0$$

by the vanishing theorem (see Theorem 6.3.4 (ii)) and that $\mathrm{Nqlc}(X,\omega) \cap C = \emptyset$. Therefore, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of C. By combining this with Claim 1, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of every qlc stratum of $[X,\omega]$ if (p,q) is sufficiently large with $0 < aq - rp < 1$. So, we obtain Claim 3. $\square$

Note that $\mathcal{O}_X(M(p,q))$ is not π-generated for $(p,q) \in I$ because $M(p,q)$ is not π-nef. Therefore, $L_0 \neq \emptyset$. We shrink S to an affine open subset intersecting $\pi(L_0)$. Let $D_1, \cdots, D_{n+1}$ be general members of

$$\pi_*\mathcal{O}_X(M(p_0,q_0)) = H^0(X, \mathcal{O}_X(M(p_0,q_0)))$$

with $(p_0,q_0) \in I$. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. We consider $f : (Y, B_Y + \sum_{i=1}^{n+1} f^*D_i) \to X$. By taking blow-ups of the ambient space M of (Y, B_Y), we may assume that $(Y, \operatorname{Supp} B_Y + \sum_{i=1}^{n+1} f^*D_i)$ is a globally embedded simple normal crossing pair by Proposition 6.3.1. We can take the maximal positive real number c such that $B_Y + c\sum_{i=1}^{n+1} f^*D_i$ is subboundary over $X \setminus \operatorname{Nqlc}(X, \omega)$ by Claim 3. Note that f^*D_i contains no strata of (Y, B_Y) for every i since D_i is a general member of $H^0(X, \mathcal{O}_X(M(p,q)))$ for every i (see Claim 3). On the other hand, $c < 1$ by Lemma 6.3.9. Then $f : (Y, B_Y + c\sum_{i=1}^{n+1} f^*D_i) \to X$ gives a natural quasi-log structure on the pair $[X, \omega + cD]$, where $D = \sum_{i=1}^{n+1} D_i$. By construction, the pair $[X, \omega + cD]$ has some qlc centers contained in L_0. Let C be a qlc center contained in L_0. We note that $\operatorname{Nqlc}(X, \omega) = \operatorname{Nqlc}(X, \omega + cD)$ by construction. In particular, $C \cap \operatorname{Nqlc}(X, \omega + cD) = C \cap \operatorname{Nqlc}(X, \omega) = \emptyset$. We consider

$$\omega + cD = c(n+1)p_0H + (1 + c(n+1)q_0a)\omega.$$

Thus we have

$$\begin{aligned} &pH + qa\omega - (\omega + cD) \\ &= (p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0a))\omega. \end{aligned}$$

If p and q are large enough and $0 < aq - rp \leq aq_0 - rp_0$, then

$$pH + qa\omega - (\omega + cD)$$

is π-ample. This is because

$$\begin{aligned} &(p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0a))\omega \\ &= (p - (1 + c(n+1))p_0)H + (qa - (1 + c(n+1))q_0a)\omega \\ &\quad + p_0H + (q_0a - 1)\omega. \end{aligned}$$

Suppose that r is not rational. There must be arbitrarily large (p,q) such that $0 < aq - rp < \varepsilon = aq_0 - rp_0$ and

$$\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) \neq 0$$

by Lemma 6.6.2 because

$$P_{C_\eta}(p,q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$$

is a nontrivial polynomial of degree at most $\dim C_\eta$ by Lemma 6.6.3. Since $M(p,q)-(\omega+cD)$ is π-ample by $0<aq-rp<aq_0-rp_0$, we have

$$h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) \neq 0$$

by the vanishing theorem: Theorem 6.3.4 (ii). By the vanishing theorem: Theorem 6.3.4 (ii),

$$\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_C(M(p,q))$$

is surjective because $M(p,q)-(\omega+cD)$ is π-ample. We note that

$$R^1\pi_*(\mathcal{I}_{C\cup \mathrm{Nqlc}(X,\omega+cD)} \otimes \mathcal{O}_X(M(p,q))) = 0$$

by Theorem 6.3.4 (ii) and that $C \cap \mathrm{Nqlc}(X,\omega+cD)=\emptyset$. Thus C is not contained in $L(p,q)$. Therefore, $L(p,q)$ is a proper subset of $L(p_0,q_0)=L_0$, giving the desired contradiction. So now we know that r is rational.

We next suppose that the assertion of the theorem concerning the denominator of r is false. We choose $(p_0,q_0)\in I$ such that aq_0-rp_0 is the maximum, say it is equal to d/v. If $0<aq-rp\le d/v$ and (p,q) is sufficiently large, then

$$\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$$

since $M(p,q)-(\omega+cD)$ is π-ample. There exists sufficiently large (p,q) in the strip $0<aq-rp<1$ with $\varepsilon=1$ for which

$$h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) \neq 0$$

by Lemma 6.6.2 since $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$ is a nontrivial polynomial of degree at most $\dim C_\eta$ by Lemma 6.6.3. Note that $aq-rp\le d/v = aq_0-rp_0$ holds automatically for $(p,q)\in I$. Since

$$\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_C(M(p,q))$$

is surjective by the π-ampleness of $M(p,q)-(\omega+cD)$, we obtain the desired contradiction by the same reason as above. So, we finish the proof of the rationality theorem. □

6.7. Cone theorem for quasi-log schemes

The main theorem of this section is the cone theorem for quasi-log schemes (see [**Am1**, Theorem 5.10]). Before we state the main theorem, let us fix the notation.

Definition 6.7.1 (see [**Am1**, Definition 5.2]). Let $[X,\omega]$ be a quasi-log scheme with the non-qlc locus $X_{-\infty}$. Let $\pi: X\to S$ be a projective morphism between schemes. We put

$$\overline{NE}(X/S)_{-\infty} = \mathrm{Im}(\overline{NE}(X_{-\infty}/S) \to \overline{NE}(X/S)).$$

We sometimes use $\overline{NE}(X/S)_{\mathrm{Nqlc}(X,\omega)}$ to denote $\overline{NE}(X/S)_{-\infty}$. For an $\mathbb{R}$-Cartier divisor D, we define

$$D_{\geq 0} = \{z \in N_1(X/S) \mid D \cdot z \geq 0\}.$$

Similarly, we can define $D_{>0}$, $D_{\leq 0}$, and $D_{<0}$. We also define

$$D^{\perp} = \{z \in N_1(X/S) \mid D \cdot z = 0\}.$$

We use the following notation

$$\overline{NE}(X/S)_{D\geq 0} = \overline{NE}(X/S) \cap D_{\geq 0},$$

and similarly for >0, ≤ 0, and <0.

Definition 6.7.2 (see [Am1, Definition 5.3]). An *extremal face* of $\overline{NE}(X/S)$ is a non-zero subcone $F \subset \overline{NE}(X/S)$ such that $z, z' \in F$ and $z+z' \in F$ imply that $z, z' \in F$. Equivalently, $F = \overline{NE}(X/S) \cap H^{\perp}$ for some π-nef $\mathbb{R}$-divisor H, which is called a *supporting function* of F. An *extremal ray* is a one-dimensional extremal face.

(1) An extremal face F is called *ω-negative* if $F \cap \overline{NE}(X/S)_{\omega \geq 0} = \{0\}$.
(2) An extremal face F is called *rational* if we can choose a π-nef $\mathbb{Q}$-divisor H as a support function of F.
(3) An extremal face F is called *relatively ample at infinity* if $F \cap \overline{NE}(X/S)_{-\infty} = \{0\}$. Equivalently, $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample for any supporting function H of F.
(4) An extremal face F is called *contractible at infinity* if it has a rational supporting function H such that $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-semi-ample.

The following theorem is a direct consequence of Theorem 6.5.1.

Theorem 6.7.3 (Contraction theorem). *Let $[X, \omega]$ be a quasi-log scheme and let $\pi : X \to S$ be a projective morphism between schemes. Let H be a π-nef Cartier divisor such that $F = H^{\perp} \cap \overline{NE}(X/S)$ is ω-negative and contractible at infinity. Then there exists a projective morphism $\varphi_F : X \to Y$ over S with the following properties.*

(1) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*
(2) $\mathcal{O}_Y \simeq (\varphi_F)_*\mathcal{O}_X$.
(3) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Assume that $L^{\otimes m}|_{X_{-\infty}}$ is $\varphi_F|_{X_{-\infty}}$-generated for every $m \gg 0$. Then there is a line bundle L_Y on Y such that $L \simeq \varphi_F^* L_Y$.*

PROOF. By assumption, $qH-\omega$ is π-ample for some positive integer q and $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-semi-ample. By Theorem 6.5.1, $\mathcal{O}_X(mH)$ is π-generated for some large m. We take the Stein factorization of the associated morphism. Then, we have the contraction morphism $\varphi_F : X \to Y$ with the properties (1) and (2).

We consider $\varphi_F : X \to Y$ and $\overline{NE}(X/Y)$. Then $\overline{NE}(X/Y) = F$, L is numerically trivial over Y, and $-\omega$ is φ_F-ample. Applying the basepoint-free theorem (see Theorem 6.5.1) over Y, both $L^{\otimes m}$ and $L^{\otimes(m+1)}$ are pull-backs of line bundles on Y. Their difference gives a line bundle L_Y such that $L \simeq \varphi_F^* L_Y$. □

One of the main purposes of this book is to establish the following theorem.

THEOREM 6.7.4 (Cone theorem). *Let $[X, \omega]$ be a quasi-log scheme and let $\pi : X \to S$ be a projective morphism between schemes. Then we have the following properties.*

(1) *$\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum R_j$, where R_j's are the ω-negative extremal rays of $\overline{NE}(X/S)$ that are rational and relatively ample at infinity. In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point.*
(2) *Let H be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(\omega + H)_{<0}$. In particular, the R_j's are discrete in the half-space $\omega_{<0}$.*
(3) *Let F be an ω-negative extremal face of $\overline{NE}(X/S)$ that is relatively ample at infinity. Then F is a rational face. In particular, F is contractible at infinity.*

We closely follow the proof of the cone theorem in [KMM].

PROOF. First, we assume that ω is $\mathbb{Q}$-Cartier. This means that ω is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier divisor. We may assume that $\dim_{\mathbb{R}} N_1(X/S) \geq 2$ and ω is not π-nef. Otherwise, the theorem is obvious.

STEP 1. In this step, we prove:

CLAIM. *We have*

$$\overline{NE}(X/S) = \overline{\overline{NE}(X/S)_{\omega\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_F F},$$

where F's vary among all rational proper ω-negative extremal faces that are relatively ample at infinity and $\overline{}$ denotes the closure with respect to the real topology.

PROOF OF CLAIM. We put

$$B = \overline{\overline{NE}(X/S)_{\omega \geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_F F}.$$

It is clear that $\overline{NE}(X/S) \supset B$. We note that each F is spanned by curves on X mapped to points on S by Theorem 6.7.3 (1). Supposing $\overline{NE}(X/S) \neq B$, we shall derive a contradiction. There is a separating function M which is Cartier and is not a multiple of ω in $N^1(X/S)$ such that $M > 0$ on $B \setminus \{0\}$ and $M \cdot z_0 < 0$ for some $z_0 \in \overline{NE}(X/S)$. Let C be the dual cone of $\overline{NE}(X/S)_{\omega \geq 0}$, that is,

$$C = \{D \in N^1(X/S) \mid D \cdot z \geq 0 \text{ for } z \in \overline{NE}(X/S)_{\omega \geq 0}\}.$$

Then C is generated by π-nef divisors and ω. Since M is positive on $\overline{NE}(X/S)_{\omega \geq 0} \setminus \{0\}$, M is in the interior of C, and hence there exists a π-ample $\mathbb{Q}$-divisor A such that $M - A = L' + p\omega$ in $N^1(X/S)$, where L' is a π-nef $\mathbb{Q}$-divisor on X and p is a non-negative rational number. Therefore, M is expressed in the form $M = H + p\omega$ in $N^1(X/S)$, where $H = A + L'$ is a π-ample $\mathbb{Q}$-divisor. The rationality theorem (see Theorem 6.6.1) implies that there exists a positive rational number $r < p$ such that $L = H + r\omega$ is π-nef but not π-ample, and $L|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample. Note that $L \neq 0$ in $N^1(X/S)$, since M is not a multiple of ω. Thus the extremal face F_L associated to the supporting function L is contained in B, which implies $M > 0$ on F_L. Therefore, $p < r$. This is a contradiction. This completes the proof of Claim. □

STEP 2. In this step, we prove:

CLAIM. *In the equality of Step 1, it is sufficient to assume that F's vary among all rational ω-negative extremal rays that are relatively ample at infinity.*

PROOF OF CLAIM. Let F be a rational proper ω-negative extremal face that is relatively ample at infinity, and assume that $\dim F \geq 2$. Let $\varphi_F : X \to W$ be the associated contraction. Note that $-\omega$ is φ_F-ample. By Step 1, we obtain

$$F = \overline{NE}(X/W) = \overline{\sum_G G},$$

where the G's are the rational proper ω-negative extremal faces of $\overline{NE}(X/W)$. We note that $\overline{NE}(X/W)_{-\infty} = 0$ because φ_F embeds $X_{-\infty}$ into W. The G's are also ω-negative extremal faces of $\overline{NE}(X/S)$ that are ample at infinity, and $\dim G < \dim F$. By induction, we obtain

$$(\heartsuit) \qquad \overline{NE}(X/S) = \overline{\overline{NE}(X/S)_{\omega \geq 0} + \overline{NE}(X/S)_{-\infty} + \sum R_j},$$

where the R_j's are ω-negative rational extremal rays. Note that each R_j does not intersect $\overline{NE}(X/S)_{-\infty}$. $\square$

STEP 3. The contraction theorem (see Theorem 6.7.3) guarantees that for each extremal ray R_j there exists a reduced irreducible curve C_j on X such that $[C_j] \in R_j$. Let $\psi_j : X \to W_j$ be the contraction morphism of R_j, and let A be a π-ample Cartier divisor. We set

$$r_j = -\frac{A \cdot C_j}{\omega \cdot C_j}.$$

Then $A+r_j\omega$ is ψ_j-nef but not ψ_j-ample, and $(A+r_j\omega)|_{X_{-\infty}}$ is $\psi_j|_{X_{-\infty}}$-ample. By the rationality theorem (see Theorem 6.6.1), expressing $r_j = u_j/v_j$ with $u_j, v_j \in \mathbb{Z}_{>0}$ and $(u_j, v_j) = 1$, we have the inequality $v_j \leq a(\dim X + 1)$.

STEP 4. Now take π-ample Cartier divisors $H_1, H_2, \cdots, H_{\rho-1}$ such that ω and the H_i's form a basis of $N^1(X/S)$, where $\rho = \dim_{\mathbb{R}} N^1(X/S)$. By Step 3, the intersection of the extremal rays R_j with the hyperplane

$$\{z \in N_1(X/S) \mid a\omega \cdot z = -1\}$$

in $N_1(X/S)$ lie on the lattice

$$\Lambda = \{z \in N_1(X/S) \mid a\omega \cdot z = -1, H_i \cdot z \in (a(a(\dim X + 1))!)^{-1}\mathbb{Z}\}.$$

This implies that the extremal rays are discrete in the half space

$$\{z \in N_1(X/S) \mid \omega \cdot z < 0\}.$$

Thus we can omit the closure sign $\overline{}$ from the formula ($\heartsuit$) and this completes the proof of (1) when ω is $\mathbb{Q}$-Cartier.

STEP 5. Let H be a π-ample $\mathbb{R}$-divisor on X. We choose $0 < \varepsilon_i \ll 1$ for $1 \leq i \leq \rho - 1$ such that $H - \sum_{i=1}^{\rho-1} \varepsilon_i H_i$ is π-ample. Then the R_j's included in $(\omega + H)_{<0}$ correspond to some elements of the above lattice Λ for which $\sum_{i=1}^{\rho-1} \varepsilon_i H_i \cdot z < 1/a$. Therefore, we obtain (2).

STEP 6. The vector space $V = F^{\perp} \subset N^1(X/S)$ is defined over $\mathbb{Q}$ because F is generated by some of the R_j's. There exists a π-ample $\mathbb{R}$-divisor H such that F is contained in $(\omega + H)_{<0}$. Let $\langle F \rangle$ be the vector space spanned by F. We put

$$W_F = \overline{NE}(X/S)_{\omega+H\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_{R_j \not\subset F} R_j.$$

Then W_F is a closed cone, $\overline{NE}(X/S) = W_F + F$, and $W_F \cap \langle F \rangle = \{0\}$. The supporting functions of F are the elements of V that are positive on $W_F \setminus \{0\}$. This is a non-empty open set and thus it contains a rational element that, after scaling, gives a π-nef Cartier divisor L such that $F = L^{\perp} \cap \overline{NE}(X/S)$. Therefore, F is rational. So, we have (3).

From now on, we assume that ω is $\mathbb{R}$-Cartier.

STEP 7. Let H be a π-ample $\mathbb{R}$-divisor on X. We shall prove (2). We assume that there are infinitely many R_j's in $(\omega + H)_{<0}$ and get a contradiction. There exists an affine open subset U of S such that $\overline{NE}(\pi^{-1}(U)/U)$ has infinitely many $(\omega + H)$-negative extremal rays. So, we shrink S and may assume that S is affine. We can write $H = E + H'$, where H' is π-ample, $[X, \omega+E]$ is a quasi-log pair with the same qlc centers and non-qlc locus as $[X, \omega]$, and $\omega + E$ is $\mathbb{Q}$-Cartier. Since $\omega + H = \omega + E + H'$, we have

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega+H\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_{\text{finite}} R_j.$$

This is a contradiction. Thus, we obtain (2). The statement (1) is a direct consequence of (2). Of course, (3) holds by Step 6 once we obtain (1) and (2).

So, we finish the proof of the cone theorem. □

By combining Theorem 6.7.4 and the main result of [F33] (see Theorem 4.11.9), we obtain the cone and contraction theorem for semi log canonical pairs in full generality. For the details, see [F33].

We close this section with the following nontrivial example.

EXAMPLE 6.7.5 ([F17, Example 3.76]). We consider the first projection $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. We take a blow-up $\mu : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ at $(0, \infty)$. Let A_∞ (resp. A_0) be the strict transform of $\mathbb{P}^1 \times \{\infty\}$ (resp. $\mathbb{P}^1 \times \{0\}$) on Z. We define $M = \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0))$ and X is the restriction of M on $(p \circ \mu)^{-1}(0)$. Then X is a simple normal crossing divisor on M. More explicitly, X is a $\mathbb{P}^1$-bundle over $(p \circ \mu)^{-1}(0)$ and is obtained by gluing $X_1 = \mathbb{P}^1 \times \mathbb{P}^1$ and $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ along a fiber. In particular, $(X, 0)$ is a semi log canonical surface. By construction, $M \to Z$ has two sections. Let D^+ (resp. D^-) be the restriction of the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$ (resp. $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z \to 0$). Then it is easy to see that D^+ is a nef Cartier divisor on X and that the linear system $|mD^+|$ is free for every $m > 0$. Note that M is a projective toric variety. Let E be the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$. Then, it is easy to see that E is a nef Cartier divisor on M. Therefore, the linear system $|E|$ is free. In particular, $|D^+|$ is free on X since $D^+ = E|_X$. So, $|mD^+|$ is free for every $m > 0$. We take a general member $B_0 \in |mD^+|$ with $m \geq 2$. We consider $K_X + B$ with $B = D^- + B_0 + B_1 + B_2$, where

B_1 and B_2 are general fibers of $X_1 = \mathbb{P}^1 \times \mathbb{P}^1 \subset X$. We note that B_0 does not intersect D^-. Then (X, B) is an embedded simple normal crossing pair. In particular, (X, B) is a semi log canonical surface. Of course, $[X, K_X + B]$ has a natural quasi-log structure with only qlc singularities (see also Theorem 4.11.9). It is easy to see that there exists only one integral curve C on $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \subset X$ such that $C \cdot (K_X + B) < 0$. Note that C is nothing but the negative section of $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$. We also note that $(K_X + B)|_{X_1}$ is ample on X_1. By the cone theorem (see Theorem 6.7.4), we obtain

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B)\geq 0} + \mathbb{R}_{\geq 0}[C].$$

By Theorem 6.7.4, we have $\varphi : X \to W$ which contracts C. We can easily see that $K_W + B_W$, where $B_W = \varphi_* B$, is not $\mathbb{Q}$-Cartier because C is not $\mathbb{Q}$-Cartier on X. Therefore, we can not always run the minimal model program for semi log canonical surfaces.

The above example implies that the cone and contraction theorem for quasi-log schemes do not directly produce the minimal model program for quasi-log schemes or semi log canonical pairs. For some related examples, see [**Ko11**]. However, Kento Fujita ([**Fk2**]) established a variant of the minimal model program for *semi-terminal pairs* in order to construct *semi-terminal modifications* for quasi-projective *demi-normal pairs*. His arguments use not only the cone and contraction theorem for semi log canonical pairs (see Theorem 6.7.4), but also Kollár's gluing theory (see [**Ko13**, Section 5]). For the details, see [**Fk2**].

6.8. On quasi-log Fano schemes

In this section, we discuss quasi-log Fano schemes and some related results.

Let us introduce the notion of quasi-log Fano schemes.

Definition 6.8.1 (Quasi-log Fano schemes). Let $[X, \omega]$ be a quasi-log scheme and let $\pi : X \to S$ be a projective morphism between schemes. If $-\omega$ is π-ample, then $[X, \omega]$ is called a *relative quasi-log Fano scheme over S*. When S is a point, we simply say that $[X, \omega]$ is a *quasi-log Fano scheme*.

The following result is an easy consequence of the adjunction and the vanishing theorem for quasi-log schemes: Theorem 6.3.4.

Theorem 6.8.2 (see [**Am1**, Theorem 6.6]). *Let $[X, \omega]$ be a quasi-log scheme and let $\pi : X \to S$ be a proper morphism between schemes* *such that* $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$ *and that* $-\omega$ *is nef and log big over* S *with respect to* $[X, \omega]$. *Let* $P \in S$ *be a closed point.*

(i) *Assume that* $X_{-\infty} \cap \pi^{-1}(P) \neq \emptyset$ *and* C *is a qlc stratum such that* $C \cap \pi^{-1}(P) \neq \emptyset$. *Then* $C \cap X_{-\infty} \cap \pi^{-1}(P) \neq \emptyset$.

(ii) *Assume that* $[X, \omega]$ *has only qlc singularities, that is,* $X_{-\infty} = \emptyset$. *Then the set of all qlc strata intersecting* $\pi^{-1}(P)$ *has a unique minimal element with respect to the inclusion.*

Proof. Let C be a qlc stratum of $[X, \omega]$ such that $P \in \pi(C) \cap \pi(X_{-\infty})$. Then $X' = C \cup X_{-\infty}$ with $\omega' = \omega|_{X'}$ is a quasi-log scheme and the restriction map $\pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'}$ is surjective by Theorem 6.3.4. Since $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$, $X_{-\infty}$ and C intersect over a neighborhood of P. So, we have (i).

Assume that $[X, \omega]$ has only qlc singularities, that is, $\mathrm{Nqlc}(X, \omega) = \emptyset$. Let C_1 and C_2 be two qlc strata of $[X, \omega]$ such that $P \in \pi(C_1) \cap \pi(C_2)$. The union $X' = C_1 \cup C_2$ with $\omega' = \omega|_{X'}$ is a qlc pair and the restriction map $\pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'}$ is surjective. Therefore, C_1 and C_2 intersect over P. Furthermore, the intersection $C_1 \cap C_2$ is a union of qlc strata by Theorem 6.3.7. Therefore, there exists a unique qlc stratum C_P over a neighborhood of P such that $C_P \subset C$ for every qlc stratum C with $P \in \pi(C)$. So, we finish the proof of (ii). $\square$

The following corollary is obvious by Theorem 6.8.2.

Corollary 6.8.3. *Let* (X, Δ) *be a proper log canonical pair. Assume that* $-(K_X + \Delta)$ *is nef and log big with respect to* (X, Δ) *and that* (X, Δ) *is not klt. Then there exists a unique minimal log canonical center* C_0 *such that every log canonical center contains* C_0. *In particular,* $\mathrm{Nklt}(X, \Delta)$ *is connected.*

For some related results, see [**F36**, Section 5]. In [**F39**], we obtain:

Theorem 6.8.4 (see [**F39**, Corollary 1.3]). *Let* $[X, \omega]$ *be a quasi-log Fano scheme with only qlc singularities. Then the algebraic fundamental group of* X *is trivial, equivalently,* X *has nontrivial finite étale covers.*

We think that Theorem 6.8.4 is not so obvious. For the details of Theorem 6.8.4 and some related results and conjectures, see [**F39**].

6.9. Basepoint-free theorem of Reid–Fukuda type

In this section, we explain the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes. For the details, see [**F40**].

In [**F40**], we obtain:

THEOREM 6.9.1 (Basepoint-free theorem of Reid–Fukuda type for quasi-log schemes). *Let $[X, \omega]$ be a quasi-log scheme, let $\pi : X \to S$ be a projective morphism between schemes, and let L be a π-nef Cartier divisor on X such that $qL - \omega$ is nef and log big over S with respect to $[X, \omega]$ for some positive real number q. Assume that $\mathcal{O}_{X_{-\infty}}(mL)$ is π-generated for every $m \gg 0$. Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

REMARK 6.9.2. Theorem 6.9.1 was stated in [**Am1**, Theorem 7.2] without proof. Although Ambro wrote that the proof of [**Am1**, Theorem 7.2] is parallel to that of [**Am1**, Theorem 5.1], it does not seem to be true. For the details, see [**F40**, Remark 1.4].

By applying Theorem 6.9.1 to normal pairs, we obtain:

THEOREM 6.9.3. *Let X be a normal variety, let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier, and let $\pi : X \to S$ be a projective morphism between schemes. Let L be a π-nef Cartier divisor on X such that $qL - (K_X + \Delta)$ is nef and log big over S with respect to (X, Δ) for some positive real number q. Assume that $\mathcal{O}_{\mathrm{Nlc}(X,\Delta)}(mL)$ is π-generated for every $m \gg 0$. Note that $\mathrm{Nlc}(X, \Delta)$ denotes the non-lc locus of (X, Δ) and is defined by the non-lc ideal sheaf $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ of (X, Δ). Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

As a special case, we have:

COROLLARY 6.9.4 (see [**F17**, Theorem 4.4]). *Let (X, Δ) be a log canonical pair and let $\pi : X \to S$ be a projective morphism onto a variety S. Let L be a π-nef Cartier divisor on X such that $qL - (K_X + \Delta)$ is nef and log big over S with respect to (X, Δ) for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.*

In this section, we prove Theorem 6.9.1 under the extra assumption that $X_{-\infty} = \emptyset$, which is sufficient to Corollary 6.9.4.

First, let us prepare an easy lemma.

LEMMA 6.9.5 (cf. [**F40**, Lemma 3.15]). *Let $[X, \omega]$ be a quasi-log scheme with $\mathrm{Nqlc}(X, \omega) = \emptyset$ and let E be a finite $\mathbb{R}_{>0}$-linear combination of effective Cartier divisors on X. We put*

$$\widetilde{\omega} = \omega + \varepsilon E$$

with $0 < \varepsilon \ll 1$. Then $[X, \widetilde{\omega}]$ has a natural quasi-log structure with the following properties.

(i) *Let $\{C_i\}_{i \in I}$ be the set of qlc strata of $[X, \omega]$ contained in $\mathrm{Supp}\, E$. We put*

$$X^{\spadesuit} = (\bigcup_{i \in I} C_i)$$

as in Theorem 6.3.4. Then $\mathrm{Nqlc}(X, \widetilde{\omega})$ *coincides with* $X^\spadesuit$ *scheme theoretically.*

(ii) C *is a qlc stratum of* $[X, \widetilde{\omega}]$ *if and only if* C *is a qlc stratum of* $[X, \omega]$ *with* $C \not\subset \operatorname{Supp} E$.

PROOF. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in Definition 6.2.2. By Proposition 6.3.1, the union of all strata of (Y, B_Y) mapped to $X^\spadesuit$, which is denoted by Y'', is a union of some irreducible components of Y. We put $Y' = Y - Y''$ and

$$K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}.$$

We may further assume that $(Y', B_{Y'} + f^*E)$ is a globally embedded simple normal crossing pair by Proposition 6.3.1. We consider $f : (Y', B_{Y'} + \varepsilon f^*E) \to X$ with $0 < \varepsilon \ll 1$. We put $A = \lceil -(B_Y^{<1}) \rceil$. Then $X^\spadesuit$ is defined by the ideal sheaf $f_*\mathcal{O}_{Y'}(A - Y'')$ (see the proof of Theorem 6.3.4 (i)). Note that

$$\begin{aligned}(A - Y'')|_{Y'} &= -\lfloor B_{Y'} + \varepsilon f^*E \rfloor + (B_{Y'} + \varepsilon f^*E)^{=1} \\ &= \lceil -(B_{Y'} + \varepsilon f^*E)^{<1} \rceil - \lfloor (B_{Y'} + \varepsilon f^*E)^{>1} \rfloor.\end{aligned}$$

Therefore, if we define $\mathrm{Nqlc}(X, \widetilde{\omega})$ by the ideal sheaf

$$f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'} + \varepsilon f^*E)^{<1} \rceil - \lfloor (B_{Y'} + \varepsilon f^*E)^{>1} \rfloor) = f_*\mathcal{O}_{Y'}(A - Y''),$$

then $f : (Y', B_{Y'} + \varepsilon f^*E) \to X$ gives the desired quasi-log structure on $[X, \widetilde{\omega}]$. $\square$

Let us prove Theorem 6.9.1 under the extra assumption that $X_{-\infty} = \emptyset$.

PROOF OF THEOREM 6.9.1 WHEN $X_{-\infty} = \emptyset$. We divide the proof into several steps.

STEP 1. If $\dim X = 0$, then Theorem 6.9.1 obviously holds true. From now on, we assume that Theorem 6.9.1 holds for any quasi-log scheme Z with $Z_{-\infty} = \emptyset$ and $\dim Z < \dim X$.

STEP 2. We take a qlc stratum C of $[X, \omega]$. We put $X' = C$. Then X' has a natural quasi-log structure induced by $[X, \omega]$ (see Theorem 6.3.4 (i)). By the vanishing theorem (see Theorem 6.3.4 (ii)), we have $R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for every $m \geq q$. Therefore, we obtain that $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{X'}(mL)$ is surjective for every $m \geq q$. Thus, we may assume that X is irreducible for the proof of Theorem 6.9.1 by the following commutative diagram.

$$\begin{array}{ccccc} \pi^*\pi_*\mathcal{O}_X(mL) & \longrightarrow & \pi^*\pi_*\mathcal{O}_{X'}(mL) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0 \end{array}$$

STEP 3. We may further assume that S is affine for the proof of Theorem 6.9.1. Of course, we may assume that X is connected.

STEP 4. In this step, we assume that X is the unique qlc stratum of $[X, \omega]$. By Lemma 6.3.5, we have that X is normal. By Kodaira's lemma (see Lemma 2.1.18), we can write $qL - \omega \sim_{\mathbb{R}} A + E$ on X such that A is a π-ample $\mathbb{Q}$-divisor on X and E is a finite $\mathbb{R}_{>0}$-linear combination of effective Cartier divisors on X. We put $\widetilde{\omega} = \omega + \varepsilon E$ with $0 < \varepsilon \ll 1$. Then $[X, \widetilde{\omega}]$ is a quasi-log scheme with $\mathrm{Nqlc}(X, \widetilde{\omega}) = \emptyset$ (see Lemma 6.9.5). Note that

$$qL - \widetilde{\omega} \sim_{\mathbb{R}} (1-\varepsilon)(qL - \omega) + \varepsilon A$$

is π-ample. Therefore, by the basepoint-free theorem for quasi-log schemes (see Theorem 6.5.1), we obtain that $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$.

STEP 5. From now on, by Step 4, we may assume that there is a qlc center C' of $[X, \omega]$ (see Theorem 6.3.7 (i)). We put

$$X' = \big(\bigcup_{i \in I} C_i\big)$$

as in Theorem 6.3.4, where $\{C_i\}_{i \in I}$ is the set of all qlc centers of $[X, \omega]$, equivalently, $X' = \mathrm{Nqklt}(X, \omega)$. Then, by induction on the dimension, $\mathcal{O}_{X'}(mL)$ is π-generated for every $m \gg 0$. By the same arguments as in Step 2, that is, the surjectivity of the restriction map $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{X'}(mL)$ for every $m \geq q$, $\mathcal{O}_X(mL)$ is π-generated in a neighborhood of X' for every large and positive integer m. In particular, for every prime number p and every large positive integer l, $\mathcal{O}_X(p^l L)$ is π-generated in a neighborhood of $X' = \mathrm{Nqklt}(X, \omega)$.

STEP 6. In this step, we prove the following claim.

CLAIM. *If the relative base locus $\mathrm{Bs}_\pi |p^l L|$ (with the reduced scheme structure) is not empty, then there is a positive integer a such that $\mathrm{Bs}_\pi |p^{al} L|$ is strictly smaller than $\mathrm{Bs}_\pi |p^l L|$.*

PROOF OF CLAIM. Note that $\mathrm{Bs}_\pi |p^{al} L| \subseteq \mathrm{Bs}_\pi |p^l L|$ for every positive integer a. Since $qL - \omega$ is nef and big over S, we can write

$$qL - \omega \sim_{\mathbb{R}} A + E$$

on X by Kodaira's lemma (see, for example, Lemma 2.1.18, [**F33**, Lemma A.10], and so on) where A is a π-ample $\mathbb{Q}$-divisor on X and E is a finite $\mathbb{R}_{>0}$-linear combination of effective Cartier divisors on X. We note that X is projective over S and that X is not necessarily normal. By Lemma 6.9.5, we have a new quasi-log structure on $[X, \widetilde{\omega}]$, where $\widetilde{\omega} = \omega + \varepsilon E$ with $0 < \varepsilon \ll 1$, such that

$$\mathrm{Nqlc}(X, \widetilde{\omega}) = (\bigcup_{i \in I} C_i)$$

where $\{C_i\}_{i \in I}$ is the set of qlc centers of $[X, \omega]$ contained in $\mathrm{Supp}\, E$.

We put $n = \dim X$. Let $D_1, \cdots, D_{n+1}$ be general members of $|p^l L|$. Note that $\mathcal{O}_X(p^l L)$ is π-generated in a neighborhood of $\mathrm{Nqklt}(X, \omega)$. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \widetilde{\omega}]$. We consider $f : (Y, B_Y + \sum_{i=1}^{n+1} f^* D_i) \to X$. We may assume that $(Y, \mathrm{Supp}\, B_Y + \sum_{i=1}^{n+1} f^* D_i)$ is a globally embedded simple normal crossing pair by Proposition 6.3.1. By Step 5, we can take the minimal positive real number c such that $B_Y + c \sum_{i=1}^{n+1} f^* D_i$ is a subboundary $\mathbb{R}$-divisor over $X \setminus \mathrm{Nqlc}(X, \widetilde{\omega})$. Note that we have $c < 1$ by Lemma 6.3.9. Thus,

$$f : (Y, B_Y + c \sum_{i=1}^{n+1} f^* D_i) \to X$$

gives a natural quasi-log structure on $[X, \widetilde{\omega} + c \sum_{i=1}^{n+1} D_i]$. Note that $[X, \widetilde{\omega} + c \sum_{i=1}^{n+1} D_i]$ has only quasi-log canonical singularities on $X \setminus \mathrm{Nqlc}(X, \widetilde{\omega})$. We also note that D_i is a general member of $|p^l L|$ for every i. By construction, there is a qlc center C_0 of $[X, \widetilde{\omega} + c \sum_{i=1}^{n+1} D_i]$ contained in $\mathrm{Bs}_\pi |p^l L|$. We put $\widetilde{\omega} + c \sum_{i=1}^{n+1} D_i = \overline{\omega}$. Then

$$C_0 \cap \mathrm{Nqlc}(X, \overline{\omega}) = \emptyset$$

because

$$\mathrm{Bs}_\pi |p^l L| \cap \mathrm{Nqklt}(X, \omega) = \emptyset.$$

Note that $\mathrm{Nqlc}(X, \overline{\omega}) = \mathrm{Nqlc}(X, \widetilde{\omega})$ by construction. We also note that

$$(q + cp^l)L - \overline{\omega} \sim_{\mathbb{R}} (1 - \varepsilon)(qL - \omega) + \varepsilon A$$

is ample over S. Therefore,

$$\pi_* \mathcal{O}_X(mL) \to \pi_* \mathcal{O}_{C_0}(mL) \oplus \pi_* \mathcal{O}_{\mathrm{Nqlc}(X, \overline{\omega})}(mL)$$

is surjective for every $m \geq q + cp^l$ since

$$R^1 \pi_* (\mathcal{I}_{C_0 \cup \mathrm{Nqklt}(X, \overline{\omega})} \otimes \mathcal{O}_X(mL)) = 0$$

for every $m \geq q + cp^l$ by Theorem 6.3.4 (ii). Moreover, $\mathcal{O}_{C_0}(mL)$ is π-generated for every $m \gg 0$ by the basepoint-free theorem for quasi-log schemes (see Theorem 6.5.1). Note that $[C_0, \overline{\omega}|_{C_0}]$ is a quasi-log scheme with only quasi-log canonical singularities by Theorem 6.3.4 (i) and Lemma 6.3.8. Therefore, we can construct a section s of $\mathcal{O}_X(p^{al}L)$ for some positive integer a such that $s|_{C_0}$ is not zero and s is zero on $\mathrm{Nqlc}(X, \overline{\omega})$. Thus, $\mathrm{Bs}_\pi |p^{al}L|$ is strictly smaller than $\mathrm{Bs}_\pi |p^l L|$. We complete the proof of Claim. □

STEP 7. By Step 6 and the noetherian induction, $\mathcal{O}_X(p^l L)$ and $\mathcal{O}_X(p'^{l'} L)$ are both π-generated for large l and l', where p and p' are distinct prime numbers. So, there exists a positive integer m_0 such that $\mathcal{O}_X(mL)$ is π-generated for every $m \geq m_0$.

Thus we obtain the desired basepoint-free theorem. □

For the proof of Theorem 6.9.1 with $X_{-\infty} \neq \emptyset$, we need various new operations on quasi-log schemes. The proof of Theorem 6.9.1 in [**F40**] is much harder than the proof given in this section. For the details, see [**F40**].

CHAPTER 7

Some supplementary topics

In this chapter, we treat some related results and supplementary topics.

In Section 7.1, we discuss Alexeev's criterion for Serre's S_3 condition (see [**Ale3**]) with slight generalizations. Note that Alexeev's criterion is a clever application of our new torsion-free theorem (see Theorem 5.6.3 (i) or Theorem 3.16.3 (i)). Although we have already obtained various related results and several generalizations (see, for example, [**AH**], [**Ko12**], [**Kv6**], and so on), we only treat Alexeev's criterion here. Note that log canonical singularities are not necessarily Cohen–Macaulay. In Section 7.2, we collect some basic properties of cone singularities for the reader's convenience. The results in Section 7.2 are useful when we construct various examples. We have already used them several times in this book. In Section 7.3, we give some examples of threefolds. They show that we need flips even when we run the minimal model program for a smooth projective threefold with the unique smooth projective minimal model. This means that we necessarily have singular varieties in the intermediate step of the above minimal model program. In Section 7.4, we describe an explicit example of threefold toric log flip. It may help us understand the proof of the special termination theorem in [**F13**]. In Section 7.5, we explicitly construct a three-dimensional non-$\mathbb{Q}$-factorial canonical Gorenstein toric flip. It may help us understand the non-$\mathbb{Q}$-factorial minimal model program explained in Section 4.9. In this example, the flipped variety is smooth and the Picard number increases by a flip.

7.1. Alexeev's criterion for S_3 condition

In this section, we explain Alexeev's criterion for Serre's S_3 condition (see Theorem 7.1.1). It is a clever application of Theorem 5.6.3 (i) (see also Theorem 3.16.3 (i)). In general, log canonical singularities are not Cohen–Macaulay. So, the results in this section will be useful for the study of log canonical pairs.

THEOREM 7.1.1 (cf. [**Ale3**, Lemma 3.2]). *Let (X, B) be a log canonical pair with $\dim X = n \geq 3$ and let $P \in X$ be a scheme theoretic point* *such that* $\dim \overline{\{P\}} \leq n-3$. *Assume that* $\overline{\{P\}}$ *is not a log canonical center of* (X, B). *Then the local ring* $\mathcal{O}_{X,P}$ *satisfies Serre's* S_3 *condition.*

We slightly changed the original formulation. The following proof is essentially the same as Alexeev's. We use local cohomologies to calculate depths.

PROOF. We note that $\mathcal{O}_{X,P}$ satisfies Serre's S_2 condition because X is normal. Since the assertion is local, we may assume that X is affine. Let $f : Y \to X$ be a resolution of X such that $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}B$ is a simple normal crossing divisor on Y. Then we can write

$$K_Y + B_Y = f^*(K_X + B)$$

such that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor on Y. We put $A = \lceil -(B_Y^{<1}) \rceil \geq 0$. Then we obtain

$$A = K_Y + B_Y^{=1} + \{B_Y\} - f^*(K_X + B).$$

Therefore, by Theorem 5.6.3 (i) or Theorem 3.16.3 (i), every associated prime of $R^1 f_* \mathcal{O}_Y(A)$ is the generic point of some log canonical center of (X, B). Thus, P is not an associated prime of $R^1 f_* \mathcal{O}_Y(A)$ by assumption.

We put $X_P = \operatorname{Spec} \mathcal{O}_{X,P}$ and $Y_P = Y \times_X X_P$. Then P is a closed point of X_P and it is sufficient to prove that $H^2_P(X_P, \mathcal{O}_{X_P}) = 0$. We put $F = f^{-1}(P)$, where $f : Y_P \to X_P$. Then we have the following vanishing theorem. It is nothing but Lemma 3.15.2 when P is a closed point of X (see [Har4, Chapter III, Exercise 2.5]).

LEMMA 7.1.2 (cf. Lemma 3.15.2). *We have* $H^i_F(Y_P, \mathcal{O}_{Y_P}) = 0$ *for* $i < n - \dim \overline{\{P\}}$.

PROOF OF LEMMA 7.1.2. Let I denote an injective hull of $\mathcal{O}_{X_P}/m_P$ as an $\mathcal{O}_{X_P}$-module, where m_P is the maximal ideal corresponding to P. We have

$$\begin{aligned} R\Gamma_F \mathcal{O}_{Y_P} &\simeq R\Gamma_P(Rf_* \mathcal{O}_{Y_P}) \\ &\simeq \operatorname{Hom}(R\mathcal{H}om(Rf_*\mathcal{O}_{Y_P}, \omega^\bullet_{X_P}), I) \\ &\simeq \operatorname{Hom}(Rf_*\mathcal{O}_Y(K_Y) \otimes \mathcal{O}_{X_P}[n-m], I), \end{aligned}$$

where $m = \dim \overline{\{P\}}$, by the local duality theorem ([Har1, Chapter V, Theorem 6.2]) and Grothendieck duality ([Har1, Chapter VII, Theorem 3.3]). We note the shift that normalizes the dualizing complex $\omega^\bullet_{X_P}$. Therefore, we obtain $H^i_F(Y_P, \mathcal{O}_{Y_P}) = 0$ for $i < n - m$ because $R^j f_* \mathcal{O}_Y(K_Y) = 0$ for every $j > 0$ by the Grauert–Riemenschneider vanishing theorem (see Theorem 3.2.7). $\square$

Let us go back to the proof of the theorem. We use the method of two spectral sequences discussed in Section 3.15. We consider the following spectral sequences

$$E_2^{p,q} = H^p_P(X_P, R^q f_*\mathcal{O}_{Y_P}(A)) \Rightarrow H^{p+q}_F(Y_P, \mathcal{O}_{Y_P}(A)),$$

and

$${}'E_2^{p,q} = H^p_P(X_P, R^q f_*\mathcal{O}_{Y_P}) \Rightarrow H^{p+q}_F(Y_P, \mathcal{O}_{Y_P}).$$

By the above spectral sequences, we have the commutative diagram.

$$\begin{tikzcd}
H^2_F(Y_P, \mathcal{O}_{Y_P}) \arrow[r] & H^2_F(Y_P, \mathcal{O}_{Y_P}(A)) \\
H^2_P(X_P, f_*\mathcal{O}_{Y_P}) \arrow[u] \arrow[d, equal] & H^2_P(X_P, f_*\mathcal{O}_{Y_P}(A)) \arrow[u, "\phi"'] \arrow[d, equal] \\
H^2_P(X_P, \mathcal{O}_{X_P}) \arrow[r, equal] & H^2_P(X_P, \mathcal{O}_{X_P})
\end{tikzcd}$$

Since P is not an associated prime of $R^1 f_*\mathcal{O}_Y(A)$, we have

$$E_2^{0,1} = H^0_P(X_P, R^1 f_*\mathcal{O}_{Y_P}(A)) = 0.$$

By the edge sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \xrightarrow{\phi} E^2 \to \cdots,$$

we know that $\phi : E_2^{2,0} \to E^2$ is injective. Therefore, $H^2_P(X_P, \mathcal{O}_{X_P}) \to H^2_F(Y_P, \mathcal{O}_{Y_P})$ is injective by the above big commutative diagram. Thus, we obtain $H^2_P(X_P, \mathcal{O}_{X_P}) = 0$ since $H^2_F(Y_P, \mathcal{O}_{Y_P}) = 0$ by Lemma 7.1.2. □

REMARK 7.1.3. The original argument in the proof of [**Ale3**, Lemma 3.2] has some compactification problems when X is not projective. Our proof does not need any compactifications of X.

As an easy application of Theorem 7.1.1, we have the following result. It is [**Ale3**, Theorem 3.4].

THEOREM 7.1.4 (see [**Ale3**, Theorem 3.4]). *Let (X, B) be a log canonical pair and let D be an effective Cartier divisor. Assume that the pair $(X, B + \varepsilon D)$ is log canonical for some $\varepsilon > 0$. Then D is S_2.*

PROOF. Without loss of generality, we may assume that $\dim X = n \geq 3$. Let $P \in D \subset X$ be a scheme theoretic point such that $\dim \overline{\{P\}} \leq n-3$. We localize X at P and assume that $X = \operatorname{Spec} \mathcal{O}_{X,P}$.

By assumption, $\overline{\{P\}}$ is not a log canonical center of (X, B). By Theorem 7.1.1, we obtain that $H^i_P(X, \mathcal{O}_X) = 0$ for $i < 3$. Therefore, $H^i_P(D, \mathcal{O}_D) = 0$ for $i < 2$ by the long exact sequence

$$\cdots \to H^i_P(X, \mathcal{O}_X(-D)) \to H^i_P(X, \mathcal{O}_X) \to H^i_P(D, \mathcal{O}_D) \to \cdots .$$

We note that $H^i_P(X, \mathcal{O}_X(-D)) \simeq H^i_P(X, \mathcal{O}_X) = 0$ for $i < 3$. Thus, D satisfies Serre's S_2 condition. $\square$

We give a supplement to adjunction (see Theorem 6.3.4 (i)). It may be useful for the study of limits of stable pairs (see [**Ale3**]).

Theorem 7.1.5 (Adjunction for Cartier divisors on log canonical pairs). *Let (X, B) be a log canonical pair and let D be an effective Cartier divisor on X such that $(X, B + D)$ is log canonical. Let V be a union of log canonical centers of (X, B). We consider V as a reduced closed subscheme of X. We define a scheme structure on $V \cap D$ by the following short exact sequence*

$$0 \to \mathcal{O}_V(-D) \to \mathcal{O}_V \to \mathcal{O}_{V\cap D} \to 0.$$

Then, $\mathcal{O}_{V\cap D}$ is reduced and semi-normal.

Proof. First, we note that $V \cap D$ is a union of log canonical centers of $(X, B + D)$ (see Theorem 6.3.7). We also note that D contains no log canonical centers of (X, B) since $(X, B + D)$ is log canonical. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f^{-1}_*(B + D)$ is a simple normal crossing divisor on Y. We can write

$$K_Y + B_Y = f^*(K_X + B + D)$$

such that $\mathrm{Supp}\, B_Y$ is a simple normal crossing divisor on Y. We take more blow-ups and may assume that $f^{-1}(V \cap D)$ and $f^{-1}(V)$ are simple normal crossing divisors. Then the union of all strata of $B^{=1}_Y$ mapped to $V \cap D$ (resp. V), which is denoted by W (resp. $U+W$), is a divisor on Y. We put $A = \lceil -(B^{<1}_Y)\rceil \geq 0$ and consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Y(A-U-W) \arrow[r] \arrow[d] & \mathcal{O}_Y(A) \arrow[r] \arrow[d, equal] & \mathcal{O}_{U+W}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_Y(A-W) \arrow[r] & \mathcal{O}_Y(A) \arrow[r] & \mathcal{O}_W(A) \arrow[r] & 0
\end{tikzcd}$$

By applying f_*, we obtain the big commutative diagram by Theorem 5.6.3 (i) and Theorem 6.3.4 (i).

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & f_*\mathcal{O}_U(A-W) \arrow[d] & \\
0 \arrow[r] & f_*\mathcal{O}_Y(A-U-W) \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\mathcal{O}_Y(A-W) \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] & \mathcal{O}_{V\cap D} \arrow[r] \arrow[d] & 0 \\
 & f_*\mathcal{O}_U(A-W) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

A key point is that the connecting homomorphism

$$f_*\mathcal{O}_U(A-W) \to R^1f_*\mathcal{O}_Y(A-U-W)$$

is a zero map by Theorem 5.6.3 (i). We note that $\mathcal{O}_V$ and $\mathcal{O}_{V\cap D}$ in the above diagram are the structure sheaves of qlc pairs $[V, (K_X+B+D)|_V]$ and $[V\cap D, (K_X+B+D)|_{V\cap D}]$ induced by $(X, B+D)$ (see Theorem 6.3.4 (i)). In particular, $\mathcal{O}_V \simeq f_*\mathcal{O}_{U+W}$ and $\mathcal{O}_{V\cap D} \simeq f_*\mathcal{O}_W$. So, $\mathcal{O}_V$ and $\mathcal{O}_{V\cap D}$ are reduced and semi-normal since W and $U+W$ are simple normal crossing divisors on Y.

Therefore, to prove this theorem, it is sufficient to see that $f_*\mathcal{O}_U(A-W) \simeq \mathcal{O}_V(-D)$. We can write

$$A = K_Y + B_Y^{=1} + \{B_Y\} - f^*(K_X+B+D)$$

and

$$f^*D = W + E + f_*^{-1}D,$$

where E is an effective f-exceptional divisor. We note that $f_*^{-1}D \cap U = \emptyset$. Since $A - W = A - f^*D + E + f_*^{-1}D$, it is enough to see that $f_*\mathcal{O}_U(A+E+f_*^{-1}D) \simeq f_*\mathcal{O}_U(A+E) \simeq \mathcal{O}_V$. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A+E-U) \to \mathcal{O}_Y(A+E) \to \mathcal{O}_U(A+E) \to 0.$$

Note that

$$A + E - U = K_Y + B_Y^{=1} - f_*^{-1}D - U - W + \{B_Y\} - f^*(K_X+B).$$

Thus, the connecting homomorphism $f_*\mathcal{O}_U(A+E) \to R^1f_*\mathcal{O}_Y(A+E-U)$ is a zero map by Theorem 5.6.3 (i). Therefore, we obtain that

$$0 \to f_*\mathcal{O}_Y(A+E-U) \to \mathcal{O}_X \to f_*\mathcal{O}_U(A+E) \to 0.$$

We can easily check that $f_*\mathcal{O}_Y(A+E-U) = \mathcal{I}_V$, the defining ideal sheaf of V. So, we have $f_*\mathcal{O}_U(A+E) \simeq \mathcal{O}_V$. We finish the proof of this theorem. $\square$

The next corollary is one of the main results in [**Ale3**]. The original proof in [**Ale3**] depends on the S_2-fication. Our proof uses adjunction (see Theorem 7.1.5). As a consequence, we obtain the semi-normality of $\lfloor B\rfloor \cap D$.

Corollary 7.1.6 (cf. [**Ale3**, Theorem 4.1]). *Let (X, B) be a log canonical pair and let D be an effective Cartier divisor on D such that $(X, B+D)$ is log canonical. Then D is S_2 and the scheme $\lfloor B\rfloor \cap D$ is reduced and semi-normal.*

Proof. By Theorem 7.1.4, D satisfies Serre's S_2 condition. By Theorem 7.1.5, $\lfloor B\rfloor \cap D$ is reduced and semi-normal. $\square$

The following proposition may be useful. So, we include it here. It is [**Ale3**, Lemma 3.1] with slight modifications as Theorem 7.1.1.

Proposition 7.1.7 (cf. [**Ale3**, Lemma 3.1]). *Let X be a normal variety with $\dim X = n \geq 3$ and let $f : Y \to X$ be a resolution of singularities. Let $P \in X$ be a scheme theoretic point such that $\dim \overline{\{P\}} \leq n-3$. Then the local ring $\mathcal{O}_{X,P}$ is S_3 if and only if P is not an associated prime of $R^1f_*\mathcal{O}_Y$.*

Proof. We put $X_P = \operatorname{Spec} \mathcal{O}_{X,P}$, $Y_P = Y \times_X X_P$, and $F = f^{-1}(P)$, where $f : Y_P \to X_P$. We consider the following spectral sequence

$$E_2^{i,j} = H^i_P(X, R^jf_*\mathcal{O}_{Y_P}) \Rightarrow H^{i+j}_F(Y_P, \mathcal{O}_{Y_P}).$$

Since $H^1_F(Y_P, \mathcal{O}_{Y_P}) = H^2_F(Y_P, \mathcal{O}_{Y_P}) = 0$ by Lemma 7.1.2, we have an isomorphism $H^0_P(X_P, R^1f_*\mathcal{O}_{Y_P}) \simeq H^2_P(X_P, \mathcal{O}_{X_P})$. Therefore, the depth of $\mathcal{O}_{X,P}$ is ≥ 3 if and only if $H^2_P(X_P, \mathcal{O}_{X_P}) = H^0_P(X_P, R^1f_*\mathcal{O}_{Y_P}) = 0$. It is equivalent to the condition that P is not an associated prime of $R^1f_*\mathcal{O}_Y$. $\square$

7.1.8 (Supplements). Here, we give a slight generalization of [**Ale3**, Theorem 3.5]. We can prove it by a similar method to the proof of Theorem 7.1.1.

Theorem 7.1.9 (cf. [**Ale3**, Theorem 3.5]). *Let (X, B) be a log canonical pair and let D be an effective Cartier divisor on X such* *that $(X, B+\varepsilon D)$ is log canonical for some $\varepsilon > 0$. Let V be a union of some log canonical centers of (X, B). We consider V as a reduced closed subscheme of X. We can define a scheme structure on $V \cap D$ by the following exact sequence*

$$0 \to \mathcal{O}_V(-D) \to \mathcal{O}_V \to \mathcal{O}_{V\cap D} \to 0.$$

Then the scheme $V \cap D$ satisfies Serre's S_1 condition. In particular, $\lfloor B \rfloor \cap D$ has no embedded point.

Proof. Without loss of generality, we may assume that X is affine. We take a resolution $f : Y \to X$ such that $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}B$ is a simple normal crossing divisor on Y. Then we can write

$$K_Y + B_Y = f^*(K_X + B)$$

such that $\mathrm{Supp}\, B_Y$ is a simple normal crossing divisor on Y. We take more blow-ups and may assume that the union of all strata of $B_Y^{=1}$ mapped to V, which is denoted by W, is a divisor on Y. Moreover, for any log canonical center C of (X, B) contained in V, we may assume that $f^{-1}(C)$ is a divisor on Y. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A-W) \to \mathcal{O}_Y(A) \to \mathcal{O}_W(A) \to 0,$$

where $A = \lceil -(B_Y^{<1}) \rceil \geq 0$. By taking higher direct images, we obtain

$$0 \to f_*\mathcal{O}_Y(A-W) \to \mathcal{O}_X \to f_*\mathcal{O}_W(A) \to R^1 f_*\mathcal{O}_Y(A-W) \to \cdots .$$

By Theorem 5.6.3 (i) or Theorem 3.16.3 (i), we have that $f_*\mathcal{O}_W(A) \to R^1 f_*\mathcal{O}_Y(A-W)$ is a zero map, $f_*\mathcal{O}_W(A) \simeq \mathcal{O}_V$, and $f_*\mathcal{O}_Y(A-W) \simeq \mathcal{I}_V$, the defining ideal sheaf of V on X. We note that $f_*\mathcal{O}_W \simeq \mathcal{O}_V$. In particular, $\mathcal{O}_V$ is reduced and semi-normal. For the details, see Theorem 6.3.4 (i).

Let $P \in V \cap D$ be a scheme theoretic point such that the height of P in $\mathcal{O}_{V\cap D}$ is ≥ 1. We may assume that $\dim V \geq 2$ around P. Otherwise, the theorem is trivial. We put $V_P = \mathrm{Spec}\, \mathcal{O}_{V,P}$, $W_P = W \times_V V_P$, and $F = f^{-1}(P)$, where $f : W_P \to V_P$. The pull-back of D on V_P is denoted by D for simplicity. To check this theorem, it is sufficient to see that $H^0_P(V_P \cap D, \mathcal{O}_{V_P\cap D}) = 0$. First, we note that $H^0_P(V_P, \mathcal{O}_{V_P}) = H^0_F(W_P, \mathcal{O}_{W_P}) = 0$ by $f_*\mathcal{O}_W \simeq \mathcal{O}_V$. Next, as in the proof of Lemma 7.1.2, we have

$$\begin{aligned} R\Gamma_F\mathcal{O}_{W_P} &\simeq R\Gamma_P(Rf_*\mathcal{O}_{W_P}) \\ &\simeq \mathrm{Hom}(R\mathcal{H}om(Rf_*\mathcal{O}_{W_P}, \omega^\bullet_{V_P}), I) \\ &\simeq \mathrm{Hom}(Rf_*\mathcal{O}_W(K_W) \otimes \mathcal{O}_{V_P}[n-1-m], I), \end{aligned}$$

where $n = \dim X$, $m = \dim \overline{\{P\}}$, and I is an injective hull of $\mathcal{O}_{V_P}/m_P$ as an $\mathcal{O}_{V_P}$-module such that m_P is the maximal ideal corresponding to P. Once we obtain $R^{n-m-2}f_*\mathcal{O}_W(K_W)\otimes\mathcal{O}_{V_P} = 0$, then $H^1_F(W_P, \mathcal{O}_{W_P}) = 0$. It implies that $H^1_P(V_P, \mathcal{O}_{V_P}) = 0$ since $H^1_P(V_P, \mathcal{O}_{V_P}) \subset H^1_F(W_P, \mathcal{O}_{W_P})$. By the long exact sequence

$$\begin{aligned}\cdots \to H^0_P(V_P, \mathcal{O}_{V_P}) &\to H^0_P(V_P \cap D, \mathcal{O}_{V_P\cap D})\\ &\to H^1_P(V_P, \mathcal{O}_{V_P}(-D)) \to \cdots,\end{aligned}$$

we obtain $H^0_P(V_P \cap D, \mathcal{O}_{V_P\cap D}) = 0$. This is because $H^0_P(V_P, \mathcal{O}_{V_P}) = 0$ and $H^1_P(V_P, \mathcal{O}_{V_P}(-D)) \simeq H^1_P(V_P, \mathcal{O}_{V_P}) = 0$. So, it is sufficient to see that $R^{n-m-2}f_*\mathcal{O}_W(K_W) \otimes \mathcal{O}_{V_P} = 0$.

To check the vanishing of $R^{n-m-2}f_*\mathcal{O}_W(K_W)\otimes\mathcal{O}_{V_P}$, by taking general hyperplane cuts m times, we may assume that $m = 0$ and $P \in X$ is a closed point. We note that the dimension of any irreducible component of V passing through P is ≥ 2 since P is not a log canonical center of (X, B) (see Theorem 6.3.7).

On the other hand, we can write $W = U_1 + U_2$ such that U_2 is the union of all the irreducible components of W whose images by f have dimensions ≥ 2 and $U_1 = W - U_2$. We note that the dimension of the image of any stratum of U_2 by f is ≥ 2 by the construction of $f : Y \to X$. We consider the following exact sequence

$$\begin{aligned}\cdots \to R^{n-2}f_*\mathcal{O}_{U_2}(K_{U_2}) &\to R^{n-2}f_*\mathcal{O}_W(K_W)\\ &\to R^{n-2}f_*\mathcal{O}_{U_1}(K_{U_1} + U_2|_{U_1}) \to R^{n-1}f_*\mathcal{O}_{U_2}(K_{U_2}) \to \cdots.\end{aligned}$$

We have $R^{n-2}f_*\mathcal{O}_{U_2}(K_{U_2}) = R^{n-1}f_*\mathcal{O}_{U_2}(K_{U_2}) = 0$ around P since the dimension of general fibers of $f : U_2 \to f(U_2)$ is $\leq n-3$. Thus, we obtain $R^{n-2}f_*\mathcal{O}_W(K_W) \simeq R^{n-2}f_*\mathcal{O}_{U_1}(K_{U_1} + U_2|_{U_1})$ around P. Therefore, the support of $R^{n-2}f_*\mathcal{O}_W(K_W)$ around P is contained in one-dimensional log canonical centers of (X, B) in V and $R^{n-2}f_*\mathcal{O}_W(K_W)$ has no zero-dimensional associated prime around P by Theorem 5.6.3 (i). Note that, by the above argument, we have $R^{n-2}f_*\mathcal{O}_W(K_W) = 0$ when $U_1 = 0$. When $U_1 \neq 0$, by taking a general hyperplane cut of X again, we have the vanishing of $R^{n-2}f_*\mathcal{O}_W(K_W)$ around P by Lemma 7.1.10 below. So, we finish the proof. □

We have already used the following lemma in the proof of Theorem 7.1.9.

LEMMA 7.1.10. *Let (Z, Δ) be a d-dimensional log canonical pair and let $x \in Z$ be a closed point such that x is a log canonical center of (Z, Δ). Let V be a union of some log canonical centers of (Z, Δ) such that $\dim V > 0$, $x \in V$, and x is not isolated in V. Let $f : Y \to Z$ be* *a resolution such that $f^{-1}(x)$ and $f^{-1}(V)$ are divisors on Y and that* $\mathrm{Exc}(f)\cup\mathrm{Supp}\, f_*^{-1}\Delta$ *is a simple normal crossing divisor on Y. We can write*

$$K_Y + B_Y = f^*(K_Z + \Delta)$$

such that $\mathrm{Supp}\, B_Y$ *is a simple normal crossing divisor on Y. Let W be the union of all the irreducible components of $B_Y^{=1}$ mapped to V. Then* $R^{d-1}f_*\mathcal{O}_W(K_W) = 0$ *around x.*

PROOF. We can write $W = W_1 + W_2$, where W_2 is the union of all the irreducible components of W mapped to x by f and $W_1 = W - W_2$. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(K_Y) \to \mathcal{O}_Y(K_Y + W) \to \mathcal{O}_W(K_W) \to 0.$$

By the Grauert–Riemenschneider vanishing theorem (see Theorem 3.2.7), we obtain that

$$R^{d-1}f_*\mathcal{O}_Y(K_Y + W) \simeq R^{d-1}f_*\mathcal{O}_W(K_W).$$

Next, we consider the short exact sequence

$$0 \to \mathcal{O}_Y(K_Y + W_1) \to \mathcal{O}_Y(K_Y + W) \to \mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}) \to 0.$$

Around x, the image of any irreducible component of W_1 by f is positive dimensional. Therefore, $R^{d-1}f_*\mathcal{O}_Y(K_Y + W_1) = 0$ near x. It can be checked by induction on the number of irreducible components using the following exact sequence

$$\begin{aligned}\cdots \to R^{d-1}f_*\mathcal{O}_Y(K_Y + W_1 - S) &\to R^{d-1}f_*\mathcal{O}_Y(K_Y + W_1)\\ &\to R^{d-1}f_*\mathcal{O}_S(K_S + (W_1 - S)|_S) \to \cdots,\end{aligned}$$

where S is an irreducible component of W_1. On the other hand, we have

$$R^{d-1}f_*\mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}) \simeq H^{d-1}(W_2, \mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}))$$

and $H^{d-1}(W_2, \mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}))$ is dual to $H^0(W_2, \mathcal{O}_{W_2}(-W_1|_{W_2}))$. Note that $f_*\mathcal{O}_{W_2} \simeq \mathcal{O}_x$ and $f_*\mathcal{O}_W \simeq \mathcal{O}_V$ by the usual argument on adjunction (see Theorem 6.3.4 (i)). Since W_2 and $W = W_1 + W_2$ are connected over x, $H^0(W_2, \mathcal{O}_{W_2}(-W_1|_{W_2})) = 0$. We note that $W_1|_{W_2} \neq 0$ since x is not isolated in V. This means that $R^{d-1}f_*\mathcal{O}_W(K_W) = 0$ around x by the above arguments. □

7.2. Cone singularities

In this section, we collect some basic facts on cone singularities for the reader's convenience. They are useful when we construct examples.

First, let us give two useful lemmas.

Lemma 7.2.1. *Let X be an n-dimensional normal variety and let $f : Y \to X$ be a resolution of singularities. Assume that $R^i f_*\mathcal{O}_Y = 0$ for $1 \le i \le n-2$. Then X is Cohen–Macaulay.*

Proof. We may assume that $n \ge 3$. Since $\operatorname{Supp} R^{n-1}f_*\mathcal{O}_Y$ is zero-dimensional, we may assume that there exists a closed point $x \in X$ such that X has only rational singularities outside x by shrinking X around x. Therefore, it is sufficient to see that the depth of $\mathcal{O}_{X,x}$ is $\ge n = \dim X$. We consider the following spectral sequence

$$E_2^{i,j} = H^i_x(X, R^j f_*\mathcal{O}_Y) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y),$$

where $F = f^{-1}(x)$. Then $H^i_x(X, \mathcal{O}_X) = E_2^{i,0} \simeq E_\infty^{i,0} = 0$ for $i \le n-1$. This is because $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i \le n-1$ by Lemma 3.15.2. This means that the depth of $\mathcal{O}_{X,x}$ is $\ge n$. So, we obtain that X is Cohen–Macaulay. $\square$

Lemma 7.2.2. *Let X be an n-dimensional normal variety and let $f : Y \to X$ be a resolution of singularities. Let $x \in X$ be a closed point. Assume that X is Cohen–Macaulay and that X has only rational singularities outside x. Then $R^i f_*\mathcal{O}_Y = 0$ for $1 \le i \le n-2$.*

Proof. We may assume that $n \ge 3$. By assumption, we have $\operatorname{Supp} R^i f_*\mathcal{O}_Y \subset \{x\}$ for $1 \le i \le n-1$. We consider the following spectral sequence

$$E_2^{i,j} = H^i_x(X, R^j f_*\mathcal{O}_Y) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y),$$

where $F = f^{-1}(x)$. Then $H^0_x(X, R^j f_*\mathcal{O}_Y) = E_2^{0,j} \simeq E_\infty^{0,j} = 0$ for $j \le n-2$ since $E_2^{i,j} = 0$ for $i > 0$ and $j > 0$, $E_2^{i,0} = 0$ for $i \le n-1$, and $H^j_F(Y, \mathcal{O}_Y) = 0$ for $j < n$ by Lemma 3.15.2. Therefore, we obtain $R^i f_*\mathcal{O}_Y = 0$ for $1 \le i \le n-2$. $\square$

We point out the following fact explicitly for the reader's convenience. It is [**Ko8**, 11.2 Theorem. (11.2.5)].

Lemma 7.2.3. *Let $f : Y \to X$ be a proper morphism, let $x \in X$ be a closed point, and let G be a sheaf on Y. If $\operatorname{Supp} R^j f_*G \subset \{x\}$ for $1 \le i < k$ and $H^i_F(Y, G) = 0$ for $i \le k$ where $F = f^{-1}(x)$, then $R^j f_*G \simeq H^{j+1}_x(X, f_*G)$ for $j = 1, \cdots, k-1$.*

The assumptions in Lemma 7.2.2 are satisfied for n-dimensional isolated Cohen–Macaulay singularities. Therefore, we have the following corollary of Lemmas 7.2.1 and 7.2.2.

Corollary 7.2.4. *Let $x \in X$ be an n-dimensional normal isolated singularity. Then $x \in X$ is Cohen–Macaulay if and only if $R^i f_*\mathcal{O}_Y = 0$ for $1 \le i \le n-2$, where $f : Y \to X$ is a resolution of singularities.*

We note the following easy example.

EXAMPLE 7.2.5. Let V be a cone over a smooth plane cubic curve and let $\varphi : W \to V$ be the blow-up at the vertex. Then W is smooth and $K_W = \varphi^* K_V - E$, where E is an elliptic curve. In particular, V is log canonical. Let C be a smooth curve. We put $Y = W \times C$, $X = V \times C$, and $f = \varphi \times \mathrm{id}_C : Y \to X$, where id_C is the identity map of C. By construction, X is a log canonical threefold. We can easily check that X is Cohen–Macaulay (see also Theorem 7.1.1 and Proposition 7.1.7). We note that $R^1 f_* \mathcal{O}_Y \neq 0$ and that $R^1 f_* \mathcal{O}_Y$ has no zero-dimensional associated components. Therefore, the Cohen–Macaulayness of X does not necessarily imply the vanishing of $R^1 f_* \mathcal{O}_Y$.

Let us go to cone singularities (see also [**Ko8**, 3.8 Example] and [**Ko10**, Exercises 70, 71]).

LEMMA 7.2.6 (Projective normality). *Let $X \subset \mathbb{P}^N$ be a normal projective irreducible variety and let $V \subset \mathbb{A}^{N+1}$ be the cone over X. Then V is normal if and only if*

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m))$$

is surjective for every $m \geq 0$. In this case, $X \subset \mathbb{P}^N$ is said to be projectively normal.

PROOF. Without loss of generality, we may assume that $\dim X \geq 1$. Let $P \in V$ be the vertex of V. By construction, we have $H^0_P(V, \mathcal{O}_V) = 0$. We consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{A}^{N+1}, \mathcal{O}_{\mathbb{A}^{N+1}}) \arrow[r] \arrow[d] & H^0(\mathbb{A}^{N+1} \setminus P, \mathcal{O}_{\mathbb{A}^{N+1}}) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & H^0(V, \mathcal{O}_V) \arrow[r] \arrow[d] & H^0(V \setminus P, \mathcal{O}_V) \arrow[r] & H^1_P(V, \mathcal{O}_V) \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

We note that $H^i(V, \mathcal{O}_V) = 0$ for every $i > 0$ since V is affine. By the above commutative diagram, it is easy to see that the following conditions are equivalent.

(a) V is normal.
(b) the depth of $\mathcal{O}_{V,P}$ is ≥ 2.
(c) $H^1_P(V, \mathcal{O}_V) = 0$.
(d) $H^0(\mathbb{A}^{N+1} \setminus P, \mathcal{O}_{\mathbb{A}^{N+1}}) \to H^0(V \setminus P, \mathcal{O}_V)$ is surjective.

The condition (d) is equivalent to the condition that $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m))$ is surjective for every $m \geq 0$. We note that

$$H^0(\mathbb{A}^{N+1} \setminus P, \mathcal{O}_{\mathbb{A}^{N+1}}) \simeq \bigoplus_{m \geq 0} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$$

and

$$H^0(V \setminus P, \mathcal{O}_V) \simeq \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m)).$$

So, we finish the proof. □

The next lemma is more or less well known to the experts.

Lemma 7.2.7. *Let $X \subset \mathbb{P}^N$ be a normal projective irreducible variety and let $V \subset \mathbb{A}^{N+1}$ be the cone over X. Assume that X is projectively normal and that X has only rational singularities. Then we have the following properties.*

(1) *V is Cohen–Macaulay if and only if $H^i(X, \mathcal{O}_X(m)) = 0$ for every $0 < i < \dim X$ and $m \geq 0$.*
(2) *V has only rational singularities if and only if $H^i(X, \mathcal{O}_X(m)) = 0$ for every $i > 0$ and $m \geq 0$.*

Proof. We put $n = \dim X$ and may assume $n \geq 1$. For (1), it is sufficient to prove that $H^i_P(V, \mathcal{O}_V) = 0$ for $2 \leq i \leq n$ if and only if $H^i(X, \mathcal{O}_X(m)) = 0$ for every $0 < i < n$ and $m \geq 0$ since V is normal, where $P \in V$ is the vertex of V. Let $f : W \to V$ be the blow-up at P and $E \simeq X$ the exceptional divisor of f. We note that W is the total space of $\mathcal{O}_X(-1)$ over $E \simeq X$ and that W has only rational singularities. Since V is affine, we obtain $H^i(V \setminus P, \mathcal{O}_V) \simeq H^{i+1}_P(V, \mathcal{O}_V)$ for every $i \geq 1$. Since W has only rational singularities, we have $H^i_E(W, \mathcal{O}_W) = 0$ for $i < n+1$ (see Lemma 3.15.2 and Remark 3.15.3). Therefore,

$$H^i(V \setminus P, \mathcal{O}_V) \simeq H^i(W \setminus E, \mathcal{O}_W) \simeq H^i(W, \mathcal{O}_W)$$

for $i \leq n-1$. Thus,

$$H^i_P(V, \mathcal{O}_V) \simeq H^{i-1}(V \setminus P, \mathcal{O}_V) \simeq H^{i-1}(W, \mathcal{O}_W) \simeq \bigoplus_{m \geq 0} H^{i-1}(X, \mathcal{O}_X(m))$$

for $2 \leq i \leq n$. So, we obtain the desired equivalence.

For (2), we consider the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[r] & H^n(V \setminus P, \mathcal{O}_V) \arrow[r] \arrow[d, "\simeq"] & H^{n+1}_P(V, \mathcal{O}_V) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H^n(W, \mathcal{O}_W) \arrow[r] & H^n(W \setminus E, \mathcal{O}_W) \arrow[r] & H^{n+1}_E(W, \mathcal{O}_W) &
\end{tikzcd}$$

We note that V is Cohen–Macaulay if and only if $R^i f_*\mathcal{O}_W = 0$ for $1 \le i \le n-1$ (see Lemmas 7.2.1 and 7.2.2) since W has only rational singularities. From now on, we assume that V is Cohen–Macaulay. Then, V has only rational singularities if and only if $R^n f_*\mathcal{O}_W = 0$. By the same argument as in the proof of Theorem 3.15.1, the kernel of α is $H^0_P(V, R^n f_*\mathcal{O}_W)$. Thus, $R^n f_*\mathcal{O}_W = 0$ if and only if $H^n(W, \mathcal{O}_W) \simeq \bigoplus_{m\ge 0} H^n(X, \mathcal{O}_X(m)) = 0$ by the above commutative diagram. So, we obtain the statement (2) with the aid of (1). $\square$

The following proposition is very useful when we construct various examples. We have already used it several times in this book.

PROPOSITION 7.2.8. *Let $X \subset \mathbb{P}^N$ be a normal projective irreducible variety and let $V \subset \mathbb{A}^{N+1}$ be the cone over X. Assume that X is projectively normal. Let Δ be an effective $\mathbb{R}$-divisor on X and let B be the cone over Δ. Then, we have the following properties.*

(1) *$K_V + B$ is $\mathbb{R}$-Cartier if and only if $K_X + \Delta \sim_{\mathbb{R}} rH$ for some $r \in \mathbb{R}$, where $H \subset X$ is the hyperplane divisor on $X \subset \mathbb{P}^N$.*
(2) *If $K_X + \Delta \sim_{\mathbb{R}} rH$, then (V, B) is*
    (a) *terminal if and only if $r < -1$ and (X, Δ) is terminal,*
    (b) *canonical if and only if $r \le -1$ and (X, Δ) is canonical,*
    (c) *klt if and only if $r < 0$ and (X, Δ) is klt, and*
    (d) *lc if and only if $r \le 0$ and (X, Δ) is lc.*

PROOF. Let $f : W \to V$ be the blow-up at the vertex $P \in V$ and $E \simeq X$ the exceptional divisor of f. If $K_V + B$ is $\mathbb{R}$-Cartier, then $K_W + f_*^{-1}B \sim_{\mathbb{R}} f^*(K_V + B) + aE$ for some $a \in \mathbb{R}$. By restricting it to E, we obtain that $K_X + \Delta \sim_{\mathbb{R}} -(a+1)H$. On the other hand, if $K_X + \Delta \sim_{\mathbb{R}} rH$, then $K_W + f_*^{-1}B \sim_{\mathbb{R}, f} -(r+1)E$. Therefore, $K_V + B$ is $\mathbb{R}$-Cartier on V. Thus, we have the statement (1). For (2), it is sufficient to note that

$$K_W + f_*^{-1}B = f^*(K_V + B) - (r+1)E$$

and that W is the total space of $\mathcal{O}_X(-1)$ over $E \simeq X$. $\square$

7.3. Francia's flip revisited

In this section, we treat some explicit examples of threefolds. These examples show that the notion of flips is indispensable for the study of higher-dimensional algebraic varieties.

In Example 7.3.1, we construct Francia's flip on a projective toric variety explicitly. Francia's flip is a monumental example (see [Fra]). So, we include it here. Our description may look slightly different from the usual one because we use the toric geometry.

EXAMPLE 7.3.1. We fix a lattice $N \simeq \mathbb{Z}^3$ and consider the lattice points $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, $e_4 = (1,1,-2)$, and $e_5 = (-1,-1,1)$. First, we consider the complete fan Δ_1 spanned by e_1, e_2, e_4, and e_5. Since $e_1 + e_2 + e_4 + 2e_5 = 0$, $X_1 = X(\Delta_1)$ is $\mathbb{P}(1,1,1,2)$. Next, we take the blow-up $f : X_2 = X(\Delta_2) \to X_1$ along the ray $e_3 = (0,0,1)$. Then X_2 is a projective $\mathbb{Q}$-factorial toric variety with only one $\frac{1}{2}(1,1,1)$-singular point. Since $\rho(X_2) = 2$, we have one more contraction morphism $\varphi : X_2 \to X_3 = X(\Delta_3)$. This contraction morphism φ corresponds to the removal of the wall $\langle e_1, e_2\rangle$ from Δ_2. We can easily check that φ is a flipping contraction. By adding the wall $\langle e_3, e_4\rangle$ to Δ_3, we obtain the following flipping diagram.

$$\begin{tikzcd} X_2 \arrow[rr, dashed] \arrow[dr, "\varphi"'] & & X_4 \arrow[dl, "\psi"] \\ & X_3 & \end{tikzcd}$$

It is an example of Francia's flip. Note that $e_3 + e_4 + e_5 = 0$ and $e_1 + e_2 = 2e_3 + 1e_4 + 0e_5$. Therefore, we can easily check that $X_4 \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ (see, for example, [**Ful**, Exercise in Section 2.4]). We can also check that the flipped curve $C \simeq \mathbb{P}^1$ is the section of $\pi : \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1$ defined by the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1} \to 0$.

By taking double covers, we have an interesting example of smooth projective threefolds (cf. [**Fra**]).

EXAMPLE 7.3.2. We use the same notation as in Example 7.3.1. Let $g : X_5 \to X_2$ be the blow-up along the ray $e_6 = (1,1,-1)$. Then X_5 is a smooth projective toric variety. Let $\mathcal{O}_{X_4}(1)$ be the tautological line bundle of the $\mathbb{P}^2$-bundle $\pi : X_4 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1$. It is easy to see that $\mathcal{O}_{X_4}(1)$ is nef and $\mathcal{O}_{X_4}(1) \cdot C = 0$, where C is the flipped curve. Therefore, there exists a line bundle $\mathcal{L}$ on X_3 such that $\mathcal{O}_{X_4}(1) \simeq \psi^*\mathcal{L}$, where $\psi : X_4 \to X_3$. We take a general member $D \in |\mathcal{L}^{\otimes 8}|$. We note that $|\mathcal{L}|$ is free since $\mathcal{L}$ is nef. We take a double cover $X \to X_4$ (resp. $Y \to X_5$) ramifying along $\operatorname{Supp} \psi^{-1}D$ (resp. $\operatorname{Supp}(\varphi \circ g)^{-1}D$). Then X is a smooth projective variety such that K_X is ample. It is obvious that Y is a smooth projective variety and is birational to X. So, X is a smooth projective threefold with ample canonical divisor and is the unique minimal model of a smooth projective threefold Y. It is easy to see that we need flips to obtain the minimal model X from Y by running a minimal model program. In particular, the minimal model program from Y to X must pass through singular varieties by Theorem 1.1.4. Note that $Y \dashrightarrow X$ is not a morphism.

Example 7.3.2 clarifies the difference between the minimal model theory for smooth projective surfaces and the minimal model program for higher-dimensional algebraic varieties.

7.4. A sample computation of a log flip

In this section, we treat an example of threefold toric log flips. In general, it is difficult to know what happens around the flipping curve. Therefore, the following nontrivial example is valuable because we can see the behavior of the flip explicitly. It helps us understand the proof of the special termination theorem in [**F13**].

Example 7.4.1. We fix a lattice $N = \mathbb{Z}^3$. We put $e_1 = (1, 0, 0)$, $e_2 = (-1, 2, 0)$, $e_3 = (0, 0, 1)$, and $e_4 = (-1, 3, -3)$. We consider the fan

$$\Delta = \{\langle e_1, e_3, e_4\rangle, \langle e_2, e_3, e_4\rangle, \text{and their faces}\}.$$

We put $X = X(\Delta)$, that is, X is the toric variety associated to the fan Δ. We define torus invariant prime divisors $D_i = V(e_i)$ for $1 \le i \le 4$. We can easily check the following claim.

Claim. *The pair $(X, D_1 + D_3)$ is a $\mathbb{Q}$-factorial dlt pair.*

We put $C = V(\langle e_3, e_4\rangle) \simeq \mathbb{P}^1$, which is a torus invariant irreducible curve on X. Since $\langle e_2, e_3, e_4\rangle$ is a non-singular cone, the intersection number $D_2 \cdot C = 1$. Therefore,

$$C \cdot D_4 = -\frac{2}{3}$$

and

$$-(K_X + D_1 + D_3) \cdot C = \frac{1}{3}.$$

We note the linear relation $e_1 + 3e_2 - 6e_3 - 2e_4 = 0$. We put $Y = X(\langle e_1, e_2, e_3, e_4\rangle)$, that is, Y is the affine toric variety associated to the cone $\langle e_1, e_2, e_3, e_4\rangle$. Then we have the next claim.

Claim. *The birational map $f : X \to Y$ is a flipping contraction with respect to $K_X + D_1 + D_3$ such that $-D_3$ is f-ample.*

Note that $f : (X, D_1 + D_3) \to Y$ is a pl flipping contraction in the sense of [**F13**, Definition 4.3.1]. We note the intersection numbers $C \cdot D_1 = \frac{1}{3}$ and $D_3 \cdot C = -2$. Let $\varphi : X \dashrightarrow X^+$ be the flip of f. We note that the flip φ is an isomorphism around any generic points of log canonical centers of $(X, D_1 + D_3)$. We restrict the flipping diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

to D_3. Then we have the following diagram.

$$\begin{tikzcd} D_3 \arrow[rr, dashed] \arrow[dr] & & D_3^+ \arrow[dl] \\ & f(D_3) & \end{tikzcd}$$

It is not difficult to see that $D_3^+ \to f(D_3)$ is an isomorphism. We put $(K_X + D_1 + D_3)|_{D_3} = K_{D_3} + B$. Then $f : D_3 \to f(D_3)$ is an extremal divisorial contraction with respect to $K_{D_3} + B$. We note that $B = D_1|_{D_3}$.

Claim. *The birational morphism $f : D_3 \to f(D_3)$ contracts $E \simeq \mathbb{P}^1$ to a point Q on $D_3^+ \simeq f(D_3)$ and Q is a $\frac{1}{2}(1,1)$-singular point on $D_3^+ \simeq f(D_3)$. The surface D_3 has a $\frac{1}{3}(1,1)$-singular point P, which is the intersection of E and B. We also have the adjunction formula $(K_{D_3} + B)|_B = K_B + \frac{2}{3}P$.*

Let D_i^+ be the torus invariant prime divisor $V(e_i)$ on X^+ for all i and let B^+ be the strict transform of B on D_3^+.

Claim. *We have*

$$(K_{X^+} + D_1^+ + D_3^+)|_{D_3^+} = K_{D_3^+} + B^+$$

and

$$(K_{D_3^+} + B^+)|_{B^+} = K_{B^+} + \frac{1}{2}Q.$$

We note that $f^+ : D_3^+ \to f(D_3)$ is an isomorphism. In particular,

$$\begin{tikzcd} D_3 \arrow[rr, dashed] \arrow[dr] & & D_3^+ \arrow[dl] \\ & f(D_3) & \end{tikzcd}$$

is of type (DS) in the sense of [**F13**, Definition 4.2.6]. Moreover, $f : B \to B^+$ is an isomorphism but $f : (B, \frac{2}{3}P) \to (B^+, \frac{1}{2}Q)$ is not an isomorphism of pairs (see [**F13**, Definition 4.2.5]). We note that B is a log canonical center of $(X, D_1 + D_3)$. So, we need [**F13**, Lemma

4.2.15]. Next, we restrict the flipping diagram to D_1. Then we obtain the diagram.

$$\begin{tikzcd} D_1 \arrow[rr, dashed] \arrow[dr] & & D_1^+ \arrow[dl] \\ & f(D_1) & \end{tikzcd}$$

In this case, $f : D_1 \to f(D_1)$ is an isomorphism.

CLAIM. *The surfaces D_1 and D_1^+ are smooth.*

It can be directly checked. Moreover, we obtain the following adjunction formulas.

CLAIM. *We have*

$$(K_X + D_1 + D_3)|_{D_1} = K_{D_1} + B + \frac{2}{3}B',$$

where B (resp. B') comes from the intersection of D_1 and D_3 (resp. D_4). We also obtain

$$(K_{X^+} + D_1^+ + D_3^+)|_{D_1^+} = K_{D_1^+} + B^+ + \frac{2}{3}B'^+ + \frac{1}{2}F,$$

where B^+ (resp. B'^+) is the strict transform of B (resp. B') and F is the exceptional curve of $f^+ : D_1^+ \to f(D_1)$.

CLAIM. *The birational morphism $f^+ : D_1^+ \to f(D_1) \simeq D_1$ is the blow-up at $P = B \cap B'$.*

We can easily check that

$$K_{D_1^+} + B^+ + \frac{2}{3}B'^+ + \frac{1}{2}F = f^{+*}(K_{D_1} + B + \frac{2}{3}B') - \frac{1}{6}F.$$

It is obvious that $K_{D_1^+} + B^+ + \frac{2}{3}B'^+ + \frac{1}{2}F$ is f^+-ample. Note that F comes from the intersection of D_1^+ and D_2^+. Note that the diagram

$$\begin{tikzcd} D_1 \arrow[rr, dashed] \arrow[dr] & & D_1^+ \arrow[dl] \\ & f(D_1) & \end{tikzcd}$$

is of type (SD) in the sense of [**F13**, Definition 4.2.6].

7.5. A non-$\mathbb{Q}$-factorial flip

The author apologizes for the mistake in [**F12**, Example 4.4.2]. For the details on three-dimensional terminal toric flips, see [**FSTU**]. In this section, we explicitly construct a three-dimensional non-$\mathbb{Q}$-factorial canonical Gorenstein toric flip. We think that it is not so easy to construct such examples without using the toric geometry.

Example 7.5.1 (Non-$\mathbb{Q}$-factorial canonical Gorenstein toric flip). We fix a lattice $N = \mathbb{Z}^3$. Let n be a positive integer with $n \geq 2$. We take lattice points $e_0 = (0, -1, 0)$,

$$e_i = (n + 1 - i, \sum_{k=n+1-i}^{n-1} k, 1)$$

for $1 \leq i \leq n + 1$, and $e_{n+2} = (-1, 0, 1)$. Note that $e_1 = (n, 0, 1)$. We consider the following fans.

$$\begin{aligned}
\Delta_X &= \{\langle e_0, e_1, e_{n+2}\rangle, \langle e_1, e_2, \cdots, e_{n+1}, e_{n+2}\rangle, \text{and their faces}\}, \\
\Delta_W &= \{\langle e_0, e_1, \cdots, e_{n+1}, e_{n+2}\rangle, \text{and its faces}\}, \text{and} \\
\Delta_{X^+} &= \{\langle e_0, e_i, e_{i+1}\rangle, \text{for } i = 1, \cdots, n+1, \text{and their faces}\}.
\end{aligned}$$

We define $X = X(\Delta_X), X^+ = X(\Delta_{X^+})$, and $W = X(\Delta_W)$. Then we have a diagram of toric varieties.

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\
& W &
\end{tikzcd}$$

We can easily check the following properties.

(i) X has only canonical Gorenstein singularities.
(ii) X is not $\mathbb{Q}$-factorial.
(iii) X^+ is smooth.
(iv) $\varphi : X \to W$ and $\varphi^+ : X^+ \to W$ are small projective toric morphisms.
(v) $-K_X$ is φ-ample and K_{X^+} is φ^+-ample.
(vi) $\rho(X/W) = 1$ and $\rho(X^+/W) = n$.

Therefore, the above diagram is a desired flipping diagram. We note that

$$e_i + e_{i+2} = 2e_{i+1} + e_0$$

for $i = 1, \cdots, n-1$ and

$$e_n + e_{n+2} = 2e_{n+1} + \frac{n(n-1)}{2} e_0.$$

We recommend the reader to draw pictures of Δ_X and Δ_{X^+} when $n = 2$.

By this example, we see that a flip sometimes increases the Picard number when the flipping variety X is not $\mathbb{Q}$-factorial. Moreover, the flipped variety X^+ sometimes becomes $\mathbb{Q}$-factorial even when the flipping variety X is not $\mathbb{Q}$-factorial.

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